

Complexité avancée

TD 6

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November 5, 2014

1 The Polynomial Hierarchy

Exercise 1. Complete problems for levels of PH Show that the following problem $\Sigma_k\text{QBF}$ is Σ_k^P -complete (under polynomial time reductions).

INPUT : A quantified boolean formula $\exists X_1 \forall X_2 \exists \dots Q_k X_k \phi$, where $X_1, \dots, X_k$ are k sets of variables, Q_k is the quantifier $\forall$ if k is even, and the quantifier $\exists$ if k is odd, ϕ is a boolean formula over variables $\bigcup_{i=1..k} X_i$;

QUESTION : is the input formula true?

Show also that the problem $\Pi_k\text{QBF}$, defined as above except that $\exists$ and $\forall$ quantifiers are swapped, is Π_k^P -complete.

Exercise 2. Collapse of PH

- Prove that if at some level $k \geq 1$ of the polynomial hierarchy $\Sigma_k^P = \Pi_k^P$, then the polynomial hierarchy collapses to the k -th level, that is $\mathbf{PH} = \Sigma_k^P$.
- Prove that if there exists a **PH**-complete language (complete under polynomial time reductions), then there exists k such that the polynomial hierarchy collapses to its k -th level (i.e. $\mathbf{PH} = \Sigma_k^P$).

Exercise 3. P, NP and oracles Show that there exists an oracle A such that $\mathbf{P}^A = \mathbf{NP}^A$.

Exercise 4. class DP **DP** is the class of languages of the form $L_1 \cap L_2$, where $L_1 \in \mathbf{NP}$ and $L_2 \in \mathbf{coNP}$. (In other words a language in **DP** is the difference of two **NP** languages.)

EXACT INDEPENDENT SET is the problem of checking, for an input graph G , and integer k , whether the maximum size of an independent set of G is k (that is whether G has an independent set of size k , and all other independent sets of G have size at most k . Recall that an independent set of a graph is a set I of vertices such that no two vertices of I are connected by an edge).

- Show that EXACT INDEPENDENT SET is in $\Sigma_2^P \cap \Pi_2^P$;
- Show that EXACT INDEPENDENT SET is in **DP**;
- Show that $\mathbf{DP} \subseteq \Sigma_2^P \cap \Pi_2^P$;
- Show that EXACT INDEPENDENT SET is **DP**-complete under polynomial time reductions.