Complexité avancée
TD 2

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Exercise 1. Turing machine with certificate tape A Turing machine with "certificate tape" is a deterministic Turing machine with a read-only input tape, a write-only output tape, a fixed number of working tapes, and an extra input tape, called certificate tape, which is "read-once" (i.e, it is a read-only tape and moreover, at each transition of the machine, the head on the certificate tape must either stay or move right). We will denote by $M(x, u)$ the computation of such a machine M starting with x on the input tape and u on the certificate tape.

We say that M runs in space $f(n)$ if for each input x of size n, and certificate u, the machine halts and uses, on its working tapes, a space bounded by $f(n)$.

Show that the following is an alternative definition of **NL** : A language L is in **NL** iff there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a Turing machine with certificate tape M, running in logarithmic space, such that :

$$x \in L \text{ iff } \exists u, |u| \leq p(|x|) \text{ and } M(x, u) \text{ accepts.} \tag{1}$$

Exercise 2. Two-way certificate tapes If in Exercise 1 we remove the read-once restriction and we allow the Turing machine to move back and forth on the certificate tape, languages L satisfying (1) define another complexity class. Which one? Give a proof of your answer.

Exercise 3. Closure under complement of context-sensitive languages A context-senstitive language is a language generated by a grammar of the form $\langle V_T, V_N, S, R\rangle$ where :

- V_T is a set of terminal symbols;
- V_N a set of non-terminal symbols;
- $S \in V_N$ is the axiom;
- R is a set of production rules of the form :

$$\alpha A \beta \to \alpha \omega \beta$$

with $\alpha, \beta \in (V_T \cup V_N)^*$, the symbol $A \in V_N$ and $\omega \in (V_T \cup V_N)^+$.

A *linear bounded automaton* (LBA) is a single-tape, nondeterministic Turing machine where the input is written between special end-markers and the computation can never leave the space between these markers (nor overwrite them). Thus, over input $x_1, \dots x_n$ the initial content of that tape is $\# x_1 x_2 \dots x_n \#$ and the tape head can never leave this block of consecutive cells.

1. Show that a language is context-sensitive if and only if it is the language accepted by an LBA. (You can also use the following equivalent form for context-sensitive

grammars due to Kuroda. Production rules must all be of the form :

$$A \to BC \text{ or } AB \to CD \text{ or } A \to a$$

where A, B, C are non-terminals and a is a terminal symbol.)

2. Show that a language is accepted by an LBA if and only if it is in **NSPACE**$(O(n))$.
3. Prove that context-sensitive languages are closed under complement (i.e. the complement of a context-sensitive language is context-sensitive).

Exercise 4. FORMULA-GAME FORMULA-GAME is the following game. There are two players, Player 1 and Player 2 which alternatively make moves on a given board. The board is a boolean formula $\phi(x_1, \ldots x_{2n})$, and the moves of the players consist in picking truth values for the variables $x_1, \ldots x_{2n}$ in this order. Specifically, Player 1 choses the value of x_1, then Player 2 choses the value of x_2, then Player 1 choses the value of x_3, and so on. Player 1 wins the game if ϕ is true under the variable assignment produced in the game. Player 1 has a winning strategy if he has a way of choosing his moves so that he wins the game no matter the moves of Player 2.

Show that the following problem is **PSPACE**-complete :
INPUT : a boolean formula ϕ
QUESTION : does Player 1 have a winning strategy for FORMULA-GAME on board ϕ?