Decidable Characterization of $FO^2(<, +1)$ and locality of DA

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Abstract

Several years ago Thérien and Wilke exhibited a decidable characterization of the languages of words that are definable in $FO^2(<, +1)$ [7]. Their proof relies on three separate ingredients. The first one is the characterization of the languages that are definable in $FO^2(<)$ as those whose syntactic semigroup belongs to the variety **DA**. Then, this result is combined with a *wreath product argument* showing that being definable in $FO^2(<, +1)$ corresponds to having a syntactic semigroup in $\mathbf{DA} * \mathbf{D}$. Finally, proving that membership of a semigroup in $\mathbf{DA} * \mathbf{D}$ is decidable requires a third ingredient: the "locality" of **DA**, a result proved in [1]. In this note we present a new self-contained and simple proof that definability in $FO^2(<, +1)$ is decidable. We obtain the locality of **DA** as a corollary.

1 Introduction

Regular languages form a robust class of languages characterized by completely different equivalent formalisms such as automata, finite semigroups or monadic second-order logic, MSO(<). In particular, the connection between MSO(<) definability and recognizability by semigroups has been used to investigate the expressive power of fragments of MSO(<). For this purpose, finding *decidable characterizations* of such fragments often serves as a yardstick. A *decidable characterization* is an algorithm which, given as input a regular language, decides whether it can be defined in the fragment under investigation. More than the algorithm itself, the main motivation is the insight given by its proof. Indeed, in order to prove a decidable characterization, one needs to consider and understand *all* properties that can be expressed in the fragment.

Usually a decidable characterization is presented by exhibiting a variety of semigroups **V** such that a language is definable in the fragment iff its syntactic semigroup is in **V**. Ideally, membership of a semigroup in **V** is defined as a finite set of equations that need to be satisfied by all elements of the semigroup. Since the syntactic semigroup of a language is a finite canonical object that can effectively be computed from any representation of the language, this yields decidability. The most striking example, known as McNaughton-Papert-Schützenberger's Theorem [5, 4], is the characterization of first-order logic equipped with a predicate "<" denoting the linear-order over words, FO(<). The result states that a regular language is definable in FO(<) iff its syntactic semigroup is aperiodic (i.e. satisfies the identity $s^\omega = s^{\omega+1}$ where ω is the size of the syntactic semigroup).

Another successful story is the two-variable fragment of FO(<). Actually two fragments are of interest: $FO^2(<)$ and $FO^2(<, +1)$. $FO^2(<)$ is a restriction of FO(<) where only two variables may be used (and reused). $FO^2(<, +1)$ is then obtained by adding a predicate "+1" for the successor relation. Note that in full first-order logic, "+1" can be defined from the order "<." However, this requires more than two variables and therefore $FO^2(<, +1)$ is strictly more expressive than $FO^2(<)$.

In [7], Thérien and Wilke proved characterizations for both $\mathrm{FO}^2(<)$ and $\mathrm{FO}^2(<,+1)$. They show that a language is definable in $\mathrm{FO}^2(<)$ (resp. $\mathrm{FO}^2(<,+1)$) iff its syntactic semigroup is in the variety $\mathbf{DA}$ (resp. $\mathbf{DA} * \mathbf{D}$). However, the arguments used for proving that these two characterizations are decidable, are very different. For $\mathrm{FO}^2(<)$, this is immediate as $\mathbf{DA}$ is defined by an equation: a semigroup belongs to $\mathbf{DA}$ if it satisfies $(st)^\omega t(st)^\omega = (st)^\omega$[1].

On the other hand, the variety $\mathbf{DA} * \mathbf{D}$ is constructed from the varieties $\mathbf{DA}$ and $\mathbf{D}$ using an agebraic product called the *wreath product* ("$*$"). The advantage of this definition is that Thérien and Wilke are able to obtain their characterization of $\mathrm{FO}^2(<,+1)$ (with $\mathbf{DA} * \mathbf{D}$) as a consequence of their characterization of $\mathrm{FO}^2(<)$ (with $\mathbf{DA}$) using a an algebraic argument known as the *wreath product principle*. The downside is that $\mathbf{DA} * \mathbf{D}$ is not defined using identities and decidability of its membership is not immediate. In fact there exist varieties $\mathbf{V}$ with decidable membership such that membership in $\mathbf{V} * \mathbf{D}$ is undecidable[2]. The special case of $\mathbf{DA} * \mathbf{D}$ is solved using the *locality* of $\mathbf{DA}$, established in [1]. It follows from the locality of $\mathbf{DA}$ that $\mathbf{DA} * \mathbf{D} = \mathbf{LDA}$ where $\mathbf{LDA}$ is the variety of semigroups S such that for all idempotents e of S, eSe is a semigroup in $\mathbf{DA}$. From this definition, identities characterizing $\mathbf{LDA}$ can be derived from those of $\mathbf{DA}$: $(esete)^\omega t(esete)^\omega = (esete)^\omega$ (where e is an idempotent) and the decidability of its membership follows.

In this paper we present a new proof of the characterization of $\mathrm{FO}^2(<,+1)$ by taking a different approach. We directly show that a language is definable in $\mathrm{FO}^2(<,+1)$ iff its syntactic semigroup satisfies the identity $(esete)^\omega s(esete)^\omega = (esete)^\omega$. Our proof remains simple and relies only on elementary combinatorial arguments. We essentially show that when the equation holds one can reduce the problem of constructing an $\mathrm{FO}^2(<,+1)$ formula for the language to constructing an $\mathrm{FO}^2(<)$ formula for another language over a modified alphabet.

The paper is organized as follows. We start with the necessary notations. The key part is Section 3 where we prove that the identity ensures definability in $\mathrm{FO}^2(<,+1)$. In Section 4 we give a standard game argument showing that the equation is implied by definability in $\mathrm{FO}^2(<,+1)$.

2 Notations

Words and Languages. We fix a finite alphabet A. We denote by A^+ the set of all nonempty finite words and by A^* the set of all finite words over A. We denote the empty word by ε. If u, v are words, we denote by $u \cdot v$ or by uv the word obtained from the concatenation of u and v.

For convenience, we only consider languages that do not contain the empty word. That is, a *language* is a subset of A^+ (this does not affect the generality of the argument). In this paper, we consider regular languages, i.e., languages that can be defined by a *nondeterministic finite automata* (NFA). In the paper, we work with the algebraic representation of regular languages in terms of semigroups.

Semigroups and Monoids. A *semigroup* is a set S equipped with an associative operation $s \cdot t$ (often written st). A *monoid* is a semigroup M having a neutral element 1_M, i.e., such that $s \cdot 1_M = 1_M \cdot s = s$ for all $s \in M$. In particular, given a semigroup S, we denote by S^1 the monoid $S \cup \{1_S\}$ constructed from S by adding a neutral element 1_S. Note that even is S already a monoid, the neutral element 1_S added in S^1 is a new element, i.e. $1_S \notin S$.

An element e of a semigroup is *idempotent* if $e^2 = e$. Given a *finite* semigroup S, it is folklore and easy to see that there is an integer $\omega(S)$ (denoted by ω when S is understood) such that for all s of S, s^ω is idempotent.

Observe that the set A^+ equipped with the concatenation operation is a semigroup. We say that a language L *is recognized by a semigroup* S if there exists a semigroup morphism $\alpha : A^+ \to S$ and a subset $F \subseteq S$ such that $L = \alpha^{-1}(F)$. It is well known that a language is regular if and only if it can be recognized by a *finite* semigroup. Moreover, from any NFA recognizing some language

[1]The authors of [7] actually use the identity $(str)^\omega t(str)^\omega = (str)^\omega$ as the definition of $\mathbf{DA}$. We use here a simpler identity that is equivalent to it, see for instance[3].

L, one can compute a canonical semigroup recognizing L, called the *syntactic semigroup* of L (the transition semigroup of the minimal deterministic automata recognizing it).

Logic. As usual a word can be seen as a logical structure whose domain is the sequence of positions in the word. We work with unary predicates P_a for all $a \in A$ denoting positions carrying the letter a and two binary predicates $+1$ and $<$ denoting the successor relation and the order relation among positions. First-order logic is then defined as usual and we denote by $\mathrm{FO}^2(<)$ the two variable restriction of $\mathrm{FO}(<)$ and by $\mathrm{FO}^2(<,+1)$ the two variable restriction of $\mathrm{FO}(<,+1)$.

3 Characterization of $\mathbf{FO}^2(<,+1)$

In this section we prove the characterization of $\mathrm{FO}^2(<,+1)$:

Theorem 1. *A regular word language L is definable in* $\mathrm{FO}^2(<,+1)$ *iff its syntactic semigroup S satisfies, for all $s,t,e \in S$ with e idempotent:*

$$(esete)^\omega = (esete)^\omega t(esete)^\omega \tag{1}$$

There are two directions to prove. That (1) is necessary follows from a classical Ehrenfeucht-Fraïssé argument. We state it in the next proposition whose proof is is postponed to Section 4.

Proposition 2. *If a language L is definable in* $\mathrm{FO}^2(<,+1)$, *its syntactic semigroup satisfies* (1).

The remainder of this section is devoted to the proof of the other direction:

Proposition 3. *If a language L has a syntactic semigroup satisfying* (1) *then it is definable in* $\mathrm{FO}^2(<,+1)$.

Let L be a regular language whose syntactic semigroup S satisfies (1) and let $\alpha : A^+ \to S$ be the syntactic morphism of L. We need to construct an $\mathrm{FO}^2(<,+1)$ formula that defines L. This is done in two steps. First, we construct an $\mathrm{FO}^2(<)$ formula over an alphabet B built from A and S. We then derive the desired formula from it.

We begin with the definition of the new alphabet B. Recall that S^1 is the monoid obtained from S by adding an artificial neutral element 1_S. We denote by $E(S^1)$ the set of idempotents of S^1 and fix an arbitrary linear order over it. Consider the new alphabet

$$B = \{(e,s,f) \mid e,f \in E(S^1), s \in S\},$$

Observe that the morphism α can be generalized as a monoid morphism $\beta : B^* \to S^1$ by setting $\beta((e,s,f)) = esf$. We say that a word $u = (e_0,s_0,f_0)\cdots(e_n,s_n,f_n) \in B^*$ is *well-formed* iff for all $i < n-1$, $f_i = e_{i+1} \neq 1_S$ (note that e_0 and f_n can have value 1_S).

To every word $w \in A^+$, we associate a well-formed companion word $\lfloor w \rfloor \in B^+$ such that $\alpha(w) = \beta(\lfloor w \rfloor)$ (the construction is inspired by [6]). If w has length smaller than $|S|$ then $\lfloor w \rfloor$ is $(1_S, \alpha(w_0), 1_S)$.

Otherwise, assume that $w = a_1 \cdots a_\ell$ with $\ell > |S|$. Fix k such that $1 \leq k \leq \ell - |S|$.

It follows from a pigeon-hole principle argument that there exist $k \leq i < j \leq k + |S|$ such that: $\alpha(a_k \cdots a_i) = \alpha(a_k \cdots a_j)$. We then have $\alpha(a_k \cdots a_i) = \alpha(a_k \cdots a_i)(\alpha(a_{i+1} \cdots a_j))^\omega$. This implies that there is an idempotent e such that $\alpha(a_k \cdots a_i) = \alpha(a_k \cdots a_i)e$. We set i_k as the smallest such $i \geq k$ and e_k as smallest such idempotent for i_k. Doing this for all k yields a set $\{i_1, \dots, i_{\ell-|S|}\}$ of indices together with associated idempotents: $e_1, \dots, e_{\ell-S}$. Observe that it may happen that $i_k = i_{k+1}$. For this reason we rename the set of indices as $\{j_1, \dots, j_h\} = \{i_1, \dots, i_{\ell-|S|}\}$ with associated idempotents $f_1, \dots, f_h$ and such that for all k, $j_k < j_{k+1}$.

We then decompose w as $w = w_1 \cdots w_{h+1}$ where: $w_1 = a_1 \dots a_{j_1} \in A^+$, for all $k \in \{1, \dots, h\}$, $w_k = a_{j_{k-1}+1} \cdots a_{j_k} \in A^+$ and $w_{h+1} = a_{j_h+1} \cdots a_\ell \in A^+$. Observe that by construction, for all k, w_k has length smaller than $|S|$ and

$$\alpha(w) = \alpha(w_1) f_1 \alpha(w_1) \cdots f_h \alpha(w_{h+1}) \tag{2}$$

We define $\lfloor w \rfloor = b_1 \cdots b_{h+1} \in B^*$ with $b_k = (f_{k-1}, \alpha(w_k), f_k)$ (we set $f_0 = 1_S$ and $f_{h+1} = 1_S$). Notice that by (2) we have $\beta(\lfloor w \rfloor) = \alpha(w)$.

We say that a position x in w is *distinguished* if it corresponds to the leftmost position of one of the factors w_k of w. To any distinguished position x in w, one can associate the corresponding position $\widehat{x}$ in $\lfloor w \rfloor$. Reciprocally to any position y of $\lfloor w \rfloor$ one associates the position $\check{y}$ in w such that $\widehat{\check{y}} = y$.

The following observation will be crucial in the proof. It essentially states that one can test in $\mathrm{FO}^2(<,+1)$ whether a position x of a word in A^+ is distinguished as well as the label of the corresponding position $\hat{x}$ in $\lfloor w \rfloor$.

Claim 4. *For any $b \in B$ there exists a formula $\alpha_b(x)$ of $\mathrm{FO}^2(<,+1)$ such that for any $w \in A^+$ and any position x of w we have*
$w \models \alpha_b(x)$ iff x is a distinguished position of w such that $\widehat{x}$ has label b in $\lfloor w \rfloor$.

Proof sketch. This is because by construction the neighborhood of x of size $|S|$ determines whether x is distinguished and the label of $\widehat{x}$. □

We can now prove Proposition 3 as an immediate consequence of the following two lemmas.

Lemma 5. *For all $\varphi \in \mathrm{FO}^2(<)$ over the alphabet B there exists a formula $\psi \in \mathrm{FO}^2(<,+1)$ over the alphabet A such that for all $w \in A^+$*

$$w \models \psi \text{ iff } \lfloor w \rfloor \models \varphi.$$

Lemma 6. *For all $s \in S$ there exists a formula $\varphi_s \in \mathrm{FO}^2(<)$ over the alphabet B such that for all well-formed words w:*

$$w \models \varphi_s \text{ iff } \beta(w) = s.$$

Before proving the lemmas, we use them to construct an $\mathrm{FO}^2(<,+1)$ formula that defines L and conclude the proof of Proposition 3. Let F be the subset of S such that $L = \alpha^{-1}(F)$. Let φ be the disjunction of all the formula of $\mathrm{FO}^2(<)$ given by Lemma 6 for all $s \in F$. Finally, let $\psi \in \mathrm{FO}^2(<,+1)$ be the formula constructed from φ as given by Lemma 5. From the lemmas it follows that for all $w \in A^*$:

$$w \models \psi \text{ iff } \lfloor w \rfloor \models \varphi \text{ iff } \beta(\lfloor w \rfloor) \in F \text{ iff } \alpha(w) \in F \text{ iff } w \in L$$

It remains to prove Lemma 5 and Lemma 6. We devote a subsection to each proof.

3.1 Proof of Lemma 5

Fix a formula $\varphi \in \mathrm{FO}^2(<)$ over the alphabet B. The construction of $\psi \in \mathrm{FO}^2(<,+1)$ is based on Claim 4.

We know from Claim 4 that being a distinguished position is definable in $\mathrm{FO}^2(<,+1)$. Let ψ be the formula constructed from φ by restricting all quantifications to quantifications over distinguished positions and replacing all tests $P_b(x)$ by $\alpha_b(x)$.

The correctness of the construction is immediate from Claim 4.

3.2 Proof of Lemma 6

We use an induction on the size $|B|$ and two preorders on elements of S^1 that we define now.

Reachability. Set $s, t \in S^1$. We say that s is $\mathcal{R}$-reachable from t iff there is $r \in S^1$ such that $s = t \cdot r$. Symmetrically we say that s is $\mathcal{L}$-reachable from t iff there exists $r \in S^1$ such that $s = r \cdot t$. For $r \in S^1$, we write $\mathrm{dp}_{\mathcal{R}}(r)$ (resp. $\mathrm{dp}_{\mathcal{L}}(r)$) the number of elements $s \in S^1$ such that s is $\mathcal{R}$-reachable (resp. $\mathcal{L}$-reachable) from r. We can now state our induction lemma.

Lemma 7. *Set $C \subseteq B$, $v_1, v_2 \in C^*$, $s \in S^1$ and $L = \{u \in C^* \mid \beta(v_1uv_2) = s\}$. There exists a formula $\varphi \in \mathrm{FO}^2(<)$ such that for all $u \in C^*$ such that v_1uv_2 is well-formed we have*

$$u \models \varphi \text{ iff } u \in L$$

Observe that by setting $C = B$ and $v_1 = v_2 = \varepsilon$, we get Lemma 6. It now remains to prove Lemma 7. The proof is by induction on three parameters that we list below by order of importance.

1. $|C|$
2. $\mathrm{dp}_{\mathcal{R}}(\beta(v_1))$
3. $\mathrm{dp}_{\mathcal{L}}(\beta(v_2))$

When $|C| = 0$, $C^* = \{\varepsilon\}$. Hence it suffices to take $\varphi = \top$ if $s = 1_S$ or $\varphi = \perp$ if $s \neq 1_S$. Otherwise, we distinguish two cases. We say that $u \in C^*$ is *bad* for v_1 iff v_1u is well-formed and $\beta(v_1)$ is not $\mathcal{R}$-reachable from $\beta(v_1u)$. Similarly, we say that u is *bad* for v_2 iff uv_2 is well-formed and $\beta(v_2)$ is not $\mathcal{L}$-reachable from $\beta(uv_2)$.

Case 1: There exists $u \in C^+$ that is bad for v_1 or v_2. We assume that we are in the first case, i.e. u is bad for v_1 (the other case is symmetrical). We use induction on the first and second parameter (note that induction on the third parameter is used in the symmetrical case). Consider a word u of minimal size that is bad for v_1, i.e. $u = u'c$ with $u' \in C^*$, $c \in C$ and $\beta(v_1)$ is $\mathcal{R}$-reachable from $\beta(v_1u')$. Set $T = \{\beta(v_1vc) \mid v \in (C \setminus \{c\})^* \text{and } v_1vc \text{ is well-formed}\}$. The construction relies on the following key claim which is a consequence of Equation (1) and our choice of c.

Claim 8. *For all $t \in T$, $dp_{\mathcal{R}}(t) < dp_{\mathcal{R}}(\beta(v_1))$.*

Proof. By definition, $t = \beta(v_1vc)$ with $v_1vc \in C^+$ and is well-formed. Set $c = (e, s, f)$, first observe that we may assume that $e \in S$ (i.e. $e \neq 1_S$). Indeed if $e = 1_S$, since v_1vc is well-formed, $v_1vc = c$ and $v_1 = \varepsilon$. It is then immediate by definition of 1_S that $\mathrm{dp}_{\mathcal{R}}(1_S) < \mathrm{dp}_{\mathcal{R}}(t) \in S$. We now assume that $e \in S$.

It suffices to prove that $\beta(v_1)$ is not $\mathcal{R}$-reachable from $\beta(v_1vc)$. We proceed by contradiction and assume that there exists $r \in S^1$ such that $\beta(v_1vc)r = \beta(v_1)$. By definition of c, we have $u' \in C^*$ such that $v_1u'c$ is well-formed, $\beta(v_1)$ is $\mathcal{R}$-reachable from $\beta(v_1u')$ and $\beta(v_1)$ is not $\mathcal{R}$-reachable from $\beta(v_1u'c)$. Let $r' \in S^1$ such that $\beta(v_1u')r' = \beta(v_1)$. Since $v_1u'c$ is well-formed we have the following equalities:

$$e\beta(c) = \beta(c) \qquad \text{and} \qquad \beta(v_1u')e = \beta(v_1u') \tag{3}$$

A little algebra then yields

$$\begin{array}{rcll} \beta(v_1u') & = & \beta(v_1u')r'\beta(vc)r\beta(u') & \\ & = & \beta(v_1u')er'\beta(v)e\beta(c)r\beta(u')e & \text{Using (3)} \\ & = & \beta(v_1u')(er'\beta(v)e\beta(c)r\beta(u')e)^{\omega} & \\ & = & \beta(v_1u')\beta(c)r\beta(u')(er'\beta(v)e\beta(c)r\beta(u')e)^{\omega} & \text{Using (1)} \end{array}$$

Therefore $\beta(v_1u')$ (and hence $\beta(v_1)$) is $\mathcal{R}$-reachable from $\beta(v_1u'c)$, which contradicts our hypothesis on c. ☐

We now construct the formula φ by combining simpler formulas that we obtain by induction. Set $L' = L \cap (C \setminus \{c\})^*$ and for all $t \in T$, $L_t = \{v \in (C \setminus \{c\})^* \mid \beta(v_1vc) = t\}$ and $K_t = \{u \in C^* \mid t\beta(uv_2) = s\}$. Observe that $L = L' \cup \bigcup_{t \in T} L_tcK_t$.

It is immediate by induction on $|C|$ that there exists a formula φ' that agrees with L' over words $u \in (C \setminus \{c\})^*$ such that v_1uv_2 is well-formed. We prove that this is also true for the languages L_tcK_t for all t. The formula φ can then be obtained by making the union of all these formulas. For all $t \in T$, we use induction to construct two $\mathrm{FO}^2(<)$ formulas Γ_t and Ψ_t such that:

1. for any well-formed $v \in (C \setminus \{c\})^*$, $v \models \Gamma_t$ iff $v \in L_t$.
2. for any $u \in C^*$ such that cuv_2 is well-formed, $u \models \Psi_t$ iff $u \in K_t$.

The desired formula φ_t is then defined as the formula

$$\varphi_t = \exists x\ c(x) \wedge \Gamma_t^{\leq} \wedge \Psi_t^{\geq},$$

where $\Gamma_t^{\leq}$ is constructed from Γ_t by replacing all quantifications $\exists y$ by $\exists y(\forall x \leq y \neg c(x))$ while $\Psi_t^{\geq}$ is constructed from Ψ_t by replacing all quantifications $\exists y$ by $\exists y(\exists x < y c(x))$. From the definitions it is immediate that for any $u \in C^*$ such that $v_1 uv_2$ is well-formed,

$$u \models \varphi_t \text{ iff } u \in L_t c K_t$$

It now remains to construct Γ_t and Ψ_t. We begin with Γ_t which is obtained by induction on $|C|$. By induction hypothesis on $|C|$, for all $r \in S^1$, there exists an $\mathrm{FO}^2(<)$ formula ϕ_r such that for any well-formed word $v \in (C \setminus \{c\})^*$, $v \models \phi_r$ iff $\beta(v) = r$. The formula Γ_t is obtained by making the disjunction of all formulas ϕ_r for r such that $\beta(v_1)r\beta(c) = t$.

We finish with the construction of Ψ_t. Set v such that v_1vc is well-formed and $\beta(v_1vc) = t$ (v exists by definition of T). By definition $K_t = \{u \mid \beta(v_1vc)\beta(u)\beta(v_2) = s\}$. By Claim 8, we can apply our induction on $\mathrm{dp}_{\mathcal{R}}(\beta(v_1))$ and we get a formula Ψ_t such that for any $u' \in C^*$ such that $v_1vcu'v_2$ is well-formed $u' \models \Psi_t$ iff $u' \in K_t$. Since, v_1vc is well-formed by definition, $v_1vcu'v_2$ is well-formed iff $cu'v_2$ is well-formed. Hence Ψ_t is the desired formula which terminates the proof.

Case 2: there exists no $u \in C^*$ that is bad for v_1 or v_2. We prove that for any $u, u' \in C^*$ such that both v_1uv_2 and $v_1u'v_2$ are well-formed, $\beta(v_1uv_2) = \beta(v_1u'v_2)$. Therefore it suffices to take $\varphi = \top$ if $s = \beta(v_1uv_2)$ or $\varphi = \bot$ is $s \neq \beta(v_1uv_2)$.

Set $u, u' \in C^*$ such that v_1uv_2 and $v_1u'v_2$ are well-formed, by hypothesis we know that uv_2 is not bad for v_1 and that v_1u' is not bad for v_2. Therefore, we have $r_1, r_2 \in S^1$ such that $\beta(v_1uv_2)r_2 = \beta(v_1u'v_2)$ and $r_1\beta(v_1u'v_2) = \beta(v_1uv_2)$. A little algebra then yields

$$\begin{array}{rcll}
\beta(v_1uv_2) & = & r_1\beta(v_1uv_2)r_2 & \\
& = & r_1^{\omega}\beta(v_1uv_2)r_2^{\omega} & \\
& = & r_1^{\omega}\beta(v_1uv_2)r_2^{\omega+1} & \text{as (1) implies } r_2^{\omega+1} = r_2^{\omega} \\
& = & \beta(v_1uv_2)r_2 & \\
& = & \beta(v_1u'v_2) & \text{by definition of } r_2
\end{array}$$

4 Proof of necessity of (1)

The proof of Proposition 2 is a simple classical Ehrenfeucht-Fraïssé argument. We include a sketch below for completeness. We begin with the definition of the Ehrenfeucht-Fraïssé game associated to $\mathrm{FO}^2(<, +1)$.

There are two players, Duplicator and Spoiler and the board consists in two words and a number k of rounds that is fixed in advance. At any time during the game there is one pebble placed on a position of one word and one pebble placed on a position of the other word and both positions have the same label. If the initial position is not specified, the game starts with the two pebbles placed on the first position of each word. Each round starts with Spoiler moving one of the pebbles inside its word from its original position x to a new position y. Duplicator must answer by moving the pebble in the other word from its original position x' to a new position y'. Moreover, the positions x' and y' must satisfy the same atomic formulas as x and y, i.e. the same predicates among $<$, $+1$ and the label predicates.

If at some point Duplicator cannot answer Spoiler's move, then Spoiler wins the game. If Duplicator is able to respond to all k moves of Spoiler then she wins the game. Winning strategies are defined as usual. If Duplicator has a winning strategy for the k-round game played on the words w, w' then we say that w and w' are k-equivalent and denote this by $w \simeq_k^+ w'$. The following result is classical and simple to prove.

Lemma 9 (Folklore). *If L is definable in* $\mathrm{FO}^2(<,+1)$ *then there is a* k *such that* $w \simeq_k^+ w'$ *implies* $w \in L$ *iff* $w' \in L$.

We can now use Lemma 9 to prove Proposition 2.

Proof of Proposition 2. Let L be a language definable in $\mathrm{FO}^2(<,+1)$. Let $\alpha : A^+ \to S$ its syntactic morphism. Let s, t and e be elements of S with e idempotent. Let U, V, E be words such that $s = \alpha(U)$, $t = \alpha(V)$, $e = \alpha(E)$. For all $k \in \mathbb{N}$, let w_k be the word $(E^kUE^kVE^k)^{k\omega}$ and let w'_k be the word $(E^kUE^kVE^k)^{k\omega}V(E^kUE^kVE^k)^{k\omega}$. Note that for all k, $\alpha(w_k)$ is $(esete)^\omega$ while $\alpha(w'_k)$ is $(esete)^\omega t(esete)^\omega$.

In view of Lemma 9, it is enough for each number k and each words u_ℓ, u_r, to give a winning strategy for Duplicator in the k-move Ehrenfeucht-Fraïssé game played on $u_\ell w_k u_r$ and $u_\ell w'_k u_r$.

This is done by induction on the number i of remaining moves. At each step of the game one pebble is at position x of $u_\ell w_k u_r$ and another one is at position x' of $u_\ell w'_k u_r$. The inductive hypothesis $H(i)$ that Duplicator maintains is:

1. x and x' have the same label.
2. If x is in a copy of E (resp. u_ℓ, u_r, U, V) then x' is in a copy of E (resp. u_ℓ, u_r, U, V) at the same relative position as x.
3. If x has less than i blocks $(E^kUE^kVE^k)$ to its left (resp. to its right) then x' is at the same distance as x from the beginning of the word (resp. from the end of the word).

It is immediate to check that $H(i)$ holds at the beginning of the game. It is also simple to verify that this inductive hypothesis can be maintained during k moves of the game. □

5 Conclusion

We have shown that languages definable in $\mathrm{FO}^2(<,+1)$ are exactly those whose syntactic semigroup satisfies $(esete)^\omega t(esete)^\omega = (esete)^\omega$. In other words and with abuse of notations we have shown that $\mathrm{FO}^2(<,+1) = \mathbf{LDA}$.

Recall from [7] that languages definable in $\mathrm{FO}^2(<)$ are exactly those whose syntactic semigroup is in the variety **DA**. From this and a "wreath product argument", essentially Lemma 5, it follows that languages definable in $\mathrm{FO}^2(<,+1)$ are exactly those whose syntactic semigroup is in $\mathbf{DA} * \mathbf{D}$.

Therefore it follows from our result that $\mathbf{DA} * \mathbf{D} = \mathbf{LDA}$. This in turns is equivalent to the locality of **DA** (see for example [8]).

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