

# Abstraction

# State-space explosion

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In practice, model-checking encounters the problem of **state-space explosion**:

due to **data**: var with  $n$  bits  $\rightarrow 2^n$  states

due to **concurrency**:  $n$  parallel components with  $n!$  different orderings

Countermeasures:

**Compression**: efficient representations (e.g. BDDs)

**Abstraction**: ignore “unimportant” information

**Approximation**: over- or underapproximate

**Reduction**: find a simpler, equivalent problem

# Example 1 (loop)

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Consider the following program with three numeric variables  $x$ ,  $y$ ,  $z$ .

$l_1$ : `y = x+1;`

$l_2$ : `z = 0;`

$l_3$ : `while (z < 100) z = z+1;`

$l_4$ : `if (y < x) error;`

**Question:** Is the error location reachable?

## Example 2 (Sorting)

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Another program with three numeric variables  $x, y, z$ .

$l_1$ :   if  $x > y$  then   swap  $x, y$    else   skip;

$l_2$ :   if  $y > z$  then   swap  $y, z$    else   skip;

$l_3$ :   if  $x > y$  then   swap  $x, y$    else   skip;

$l_4$ :   skip

**Assumption:** initially,  $x, y, z$  are all different

**Question:** Are  $x, y, z$  sorted in ascending order when reaching  $l_4$ ?

## Example 3 (Device driver)

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C code for Windows device driver

Operations on a semaphore: lock, release

Lock and release must be used alternately

# Abstraction

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**Idea:** throw away (abstract from) “unimportant” information

Handling *infinite* state spaces

Reduce (large) finite problems to smaller ones

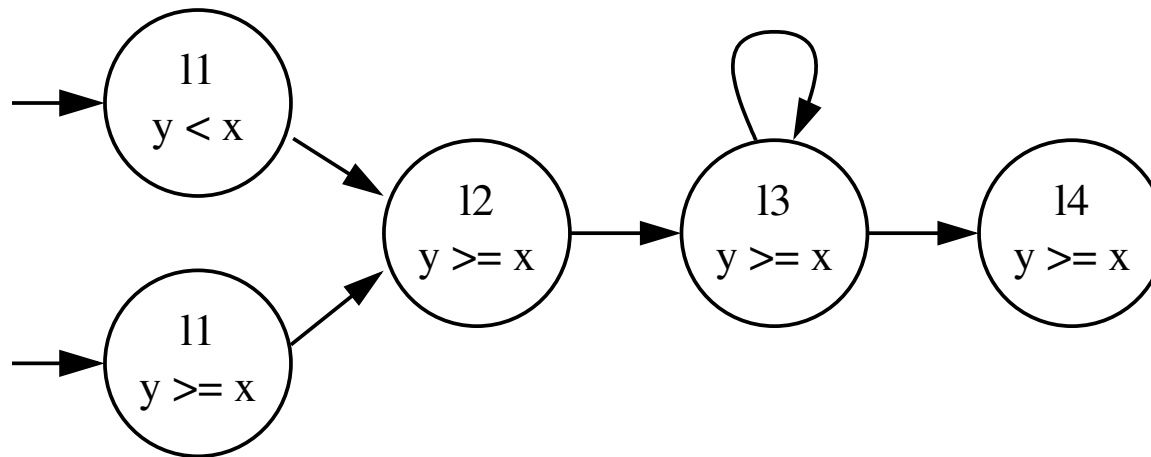
Alternative point of view: merge “equivalent” states

# Example 1

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Omit concrete values of  $x, y, z$ ; retain only the following information: program counter, predicate  $y < x$

Resulting (abstract) Kripke structure:

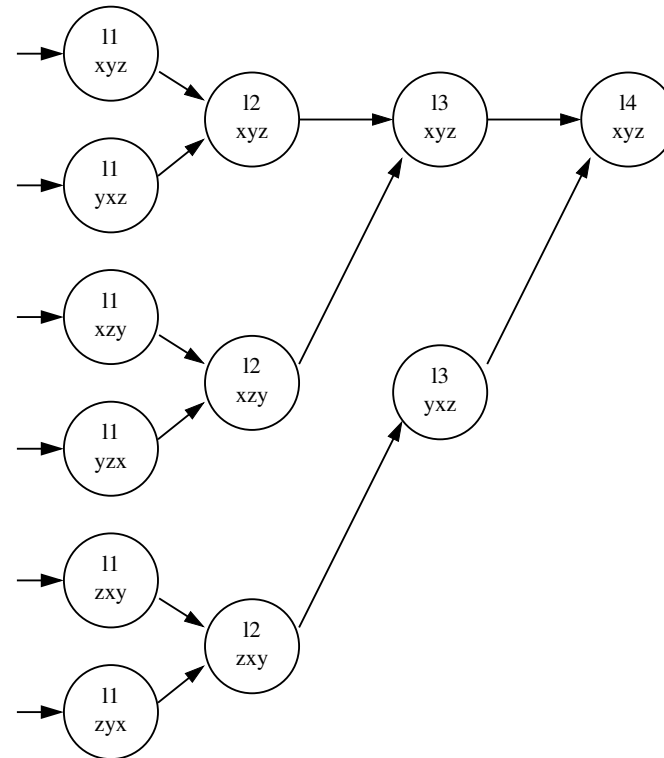


Result:  $l_4$  is reachable only with  $y \geq x$ ; the error will not happen.

## Example 2

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Omit concrete values of  $x, y, z$ ; retain only program counter and permutation of  $x, y, z$



Result:  $l_4$  is reachable only with  $xyz$ ; no error.

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**Questions:** What is the logical relation between the original programs and their abstract versions? What do the abstract versions really say about the original programs?

In Example 1, the error is unreachable in both the original and the abstract version.

However, in Example 1, the original structure terminates but the abstract version does not.

*Which conditions* must hold for the abstract structure in order to draw meaningful conclusions about the original structure?

# Simulation

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Let  $\mathcal{K}_1 = (S, \rightarrow_1, s_0, AP, \nu)$  and  $\mathcal{K}_2 = (T, \rightarrow_2, t_0, AP, \mu)$  be two Kripke structures ( $S, T$  are possibly *infinite*), and let  $H \subseteq S \times T$  be a relation.

$H$  is called a **simulation from  $\mathcal{K}_1$  to  $\mathcal{K}_2$**  iff

(i)  $(s_0, t_0) \in H$ ;

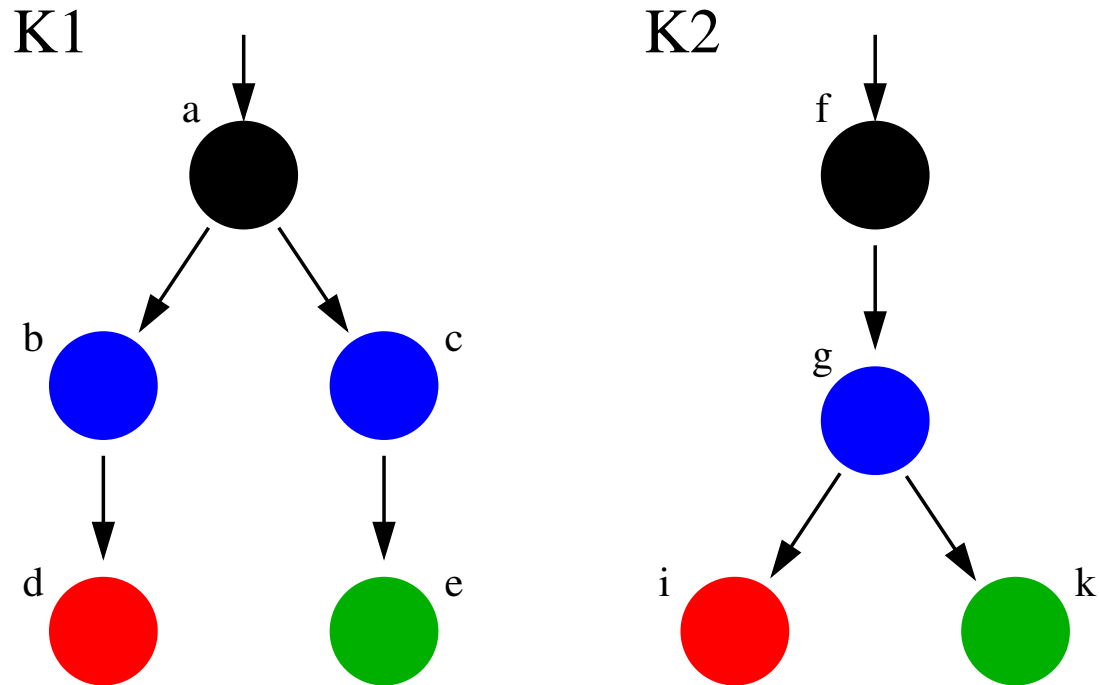
(ii) for all  $(s, t) \in H$  we have:  $\nu(s) = \mu(t)$ ;

(iii) if  $(s, t) \in H$  and  $s \rightarrow_1 s'$ , then there exists  $t'$  such that  $t \rightarrow_2 t'$  and  $(s', t') \in H$ .

We say:  $\mathcal{K}_2$  **simulates**  $\mathcal{K}_1$  (written  $\mathcal{K}_1 \leq \mathcal{K}_2$ ) if such a simulation  $H$  exists.

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Intuition:  $\mathcal{K}_2$  can do anything that is possible in  $\mathcal{K}_1$ .



$\mathcal{K}_2$  simulates  $\mathcal{K}_1$  (with  $H = \{(a, f), (b, g), (c, g), (d, i), (e, k)\}$ ).

However,  $\mathcal{K}_1$  does *not* simulate  $\mathcal{K}_2$ !

# Bisimulation

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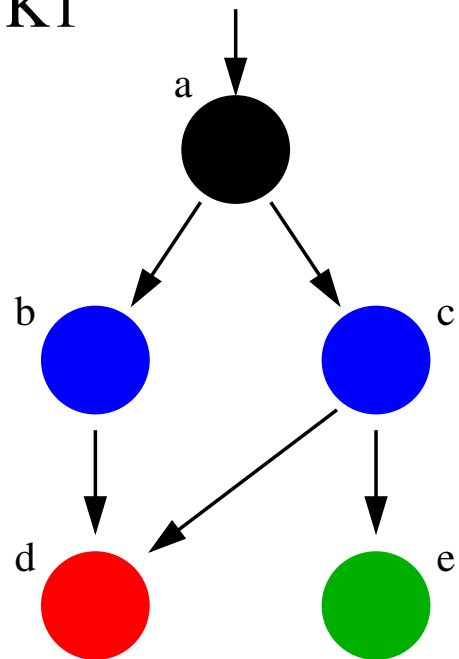
A relation  $H$  is called a **bisimulation between  $\mathcal{K}_1$  and  $\mathcal{K}_2$**  iff  $H$  is a simulation from  $\mathcal{K}_1$  to  $\mathcal{K}_2$  and  $\{(t, s) \mid (s, t) \in H\}$  is a simulation from  $\mathcal{K}_2$  to  $\mathcal{K}_1$ .

We say:  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are **bisimilar** (written  $\mathcal{K}_1 \equiv \mathcal{K}_2$ ) iff such a relation  $H$  exists.

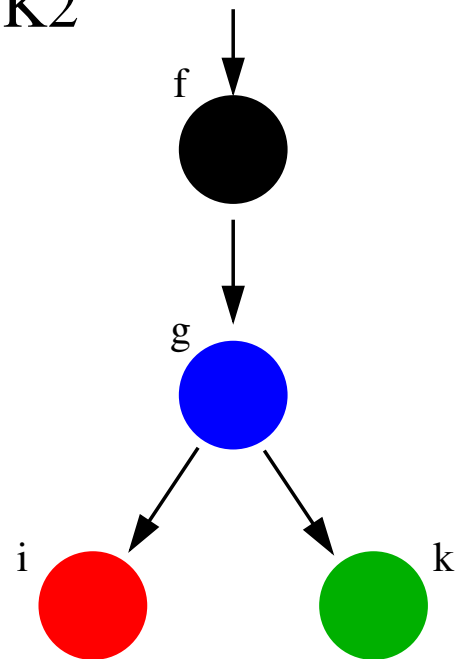
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Careful: In general,  $\mathcal{K}_1 \leq \mathcal{K}_2$  and  $\mathcal{K}_2 \leq \mathcal{K}_1$  do *not* imply  $\mathcal{K}_1 \equiv \mathcal{K}_2$ !

K1



K2



# Notes

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In the literature, one can find other, slightly varying notions of (bi-)simulation:

**Action-based simulation:** If the transitions of  $\mathcal{K}_1, \mathcal{K}_2$  are labelled by actions, one replaces condition (iii) by:

if  $(s, t) \in H$  and  $s \xrightarrow{a}_1 s'$ , then there exists  $t'$  such that  $t \xrightarrow{a}_2 t'$  and  $(s', t') \in H$ .

Simulation relation defined between states of the **same Kripke structure**:

imagine two structures being juxtaposed

**Bisimulation** defined directly:

repeat condition (iii) for the other direction

# (Bi-)Simulation and model checking

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Let  $\mathcal{K}_1 \leq \mathcal{K}_2$  and  $\phi$  an LTL formula.

Then we have:  $\mathcal{K}_2 \models \phi$  implies  $\mathcal{K}_1 \models \phi$ .

Let  $\mathcal{K}_1 \equiv \mathcal{K}_2$  and  $\phi$  a CTL or LTL formula.

Then we have:  $\mathcal{K}_1 \models \phi$  iff  $\mathcal{K}_2 \models \phi$ .

# Proofs

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**Lemma:** Bisimulation on paths

Assume  $\mathcal{K}_1 \leq \mathcal{K}_2$  with relation  $H$  and  $(s_0, t_0) \in H$ . Let  $s_0s_1 \dots$  be a path in  $\mathcal{K}_1$ . Then there exists a path  $t_0t_1 \dots$  in  $\mathcal{K}_2$  such that  $(s_i, t_i) \in H$  for all  $i \geq 0$ .

**Proof:** Immediate for  $(s_0, t_0)$ , proceed by induction on  $i$ .

**Note:** The claim about LTL follows directly from the lemma. For the claim on CTL, one rewrites  $\phi$  to an equivalent form using no temporal operators other than **EX**, **EG**, and **EU**, then proceeds by structural induction over the formula, using the lemma above.

# Notes

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Some remarks (without proof):

If two structures are not equivalent, then one can find a CTL formula that distinguishes them.

The previous claims about CTL can be lifted to CTL\*.

# Existential abstraction

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Let  $\mathcal{K} = (S, \rightarrow, r, AP, \nu)$  be a Kripke structure (**concrete structure**).

Let  $\approx$  be an equivalence relation on  $S$  such that for all  $s \approx t$  we have  $\nu(s) = \nu(t)$  (we say:  $\approx$  **respects**  $\nu$ ).

Let  $[s] := \{t \mid s \approx t\}$  denote the equivalence class of  $s$ ;  
 $[S]$  denotes the set of all equivalence classes.

The **abstraction of  $S$  w.r.t.  $\approx$**  denotes the structure  $\mathcal{K}' = ([S], \rightarrow', [r], AP, \nu')$ ,  
where

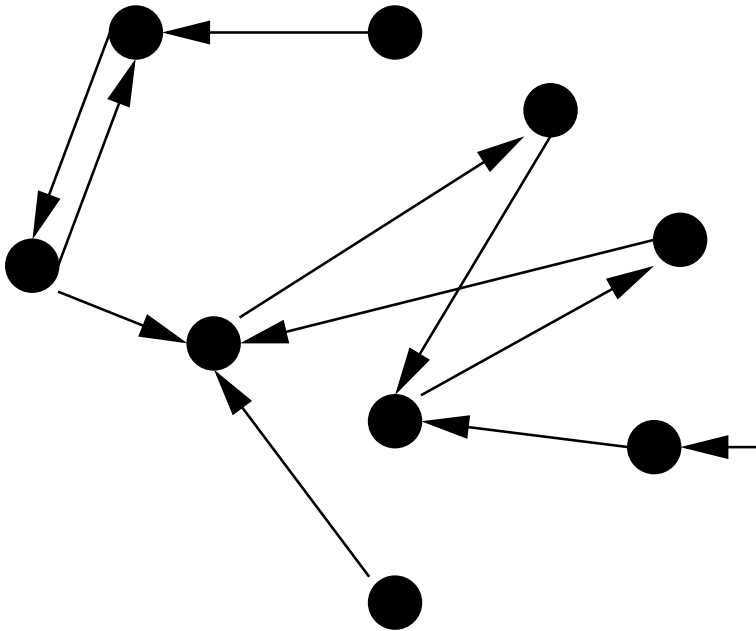
$[s] \rightarrow' [t]$  for all  $s \rightarrow t$ ;

$\nu'([s]) = \nu(s)$  (this is well-defined!).

# Example

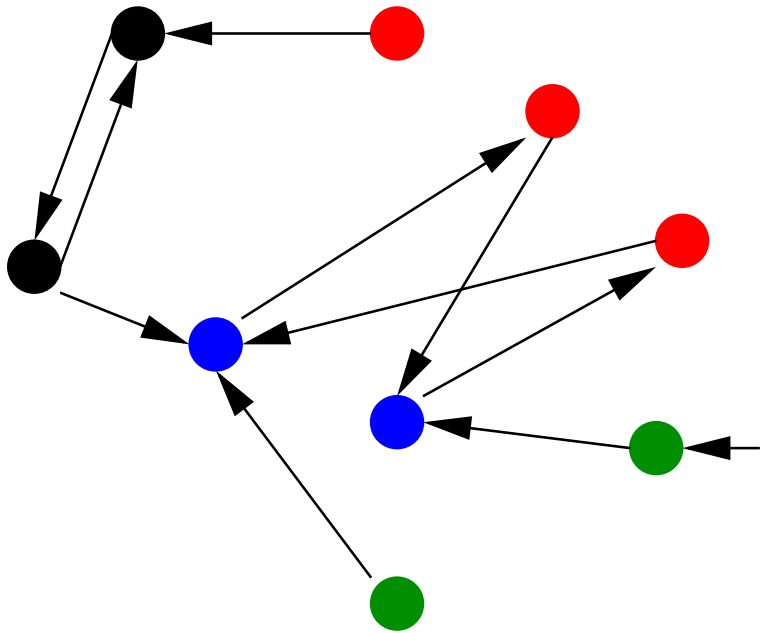
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Consider the Kripke structure below:



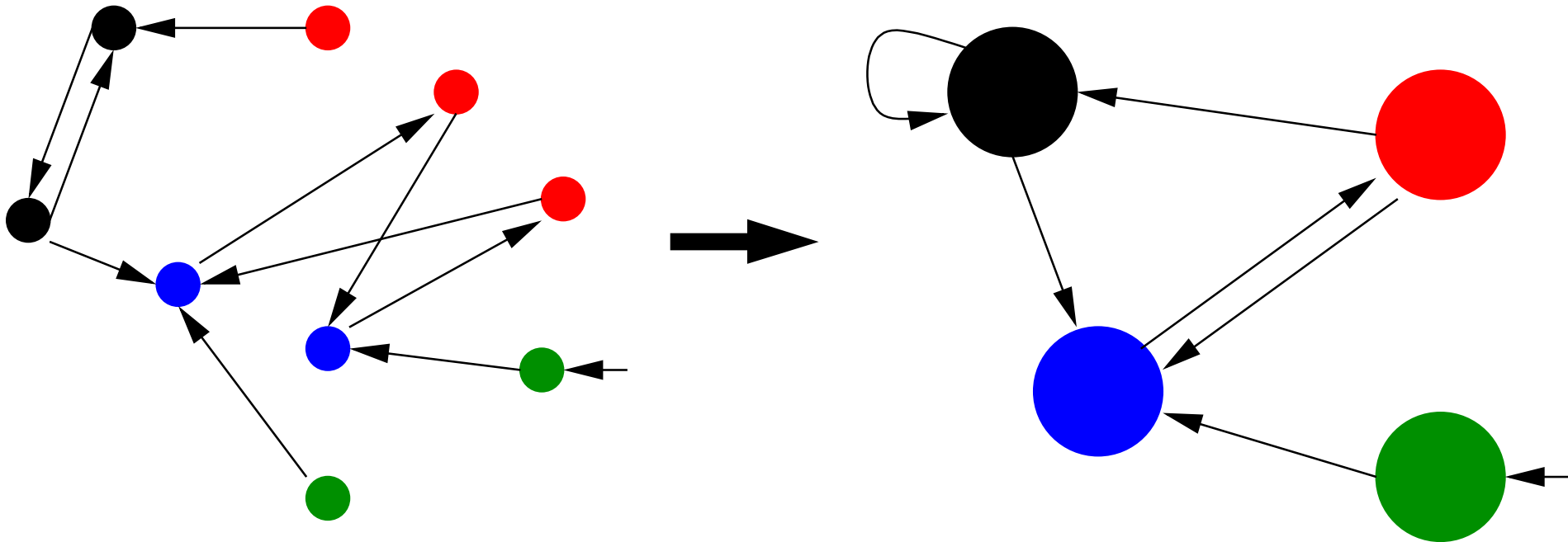
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States partitioned into equivalence classes:



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Abstract structure obtained by quotienting:



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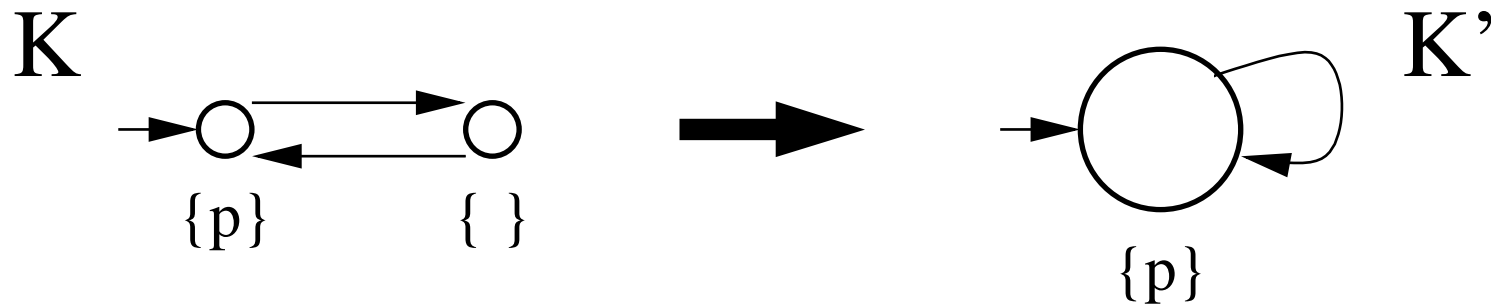
Let  $\mathcal{K}'$  be a structure obtained by abstraction from  $\mathcal{K}$ .

Then  $\mathcal{K} \leq \mathcal{K}'$  holds.

Thus, if  $\mathcal{K}'$  satisfies some LTL formula, so does  $\mathcal{K}$ .

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What happens if  $\approx$  does *not* respect  $\nu$ ?



Then  $\mathcal{K} \not\approx \mathcal{K}'$  does not hold.

Example: The abstraction satisfies  $\mathbf{G}p$ , the concrete system does not.

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Let  $\mathcal{K}'$  be a structure obtained by abstracting  $\mathcal{K}$ .

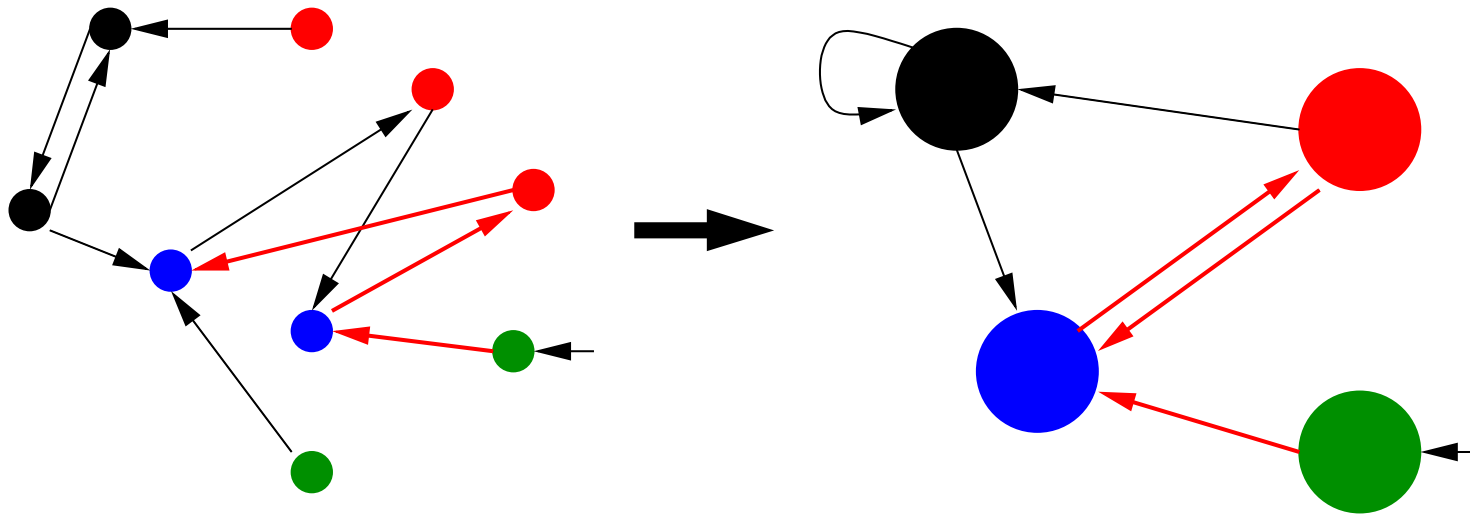
Then  $\mathcal{K} \leq \mathcal{K}'$  holds; thus, if  $\mathcal{K}'$  satisfies some LTL formula, then so does  $\mathcal{K}$ .

However, if  $\mathcal{K}' \not\models \phi$ , then  $\mathcal{K} \models \phi$  may or may not hold!

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Abstraction gives rise to additional paths in the system:

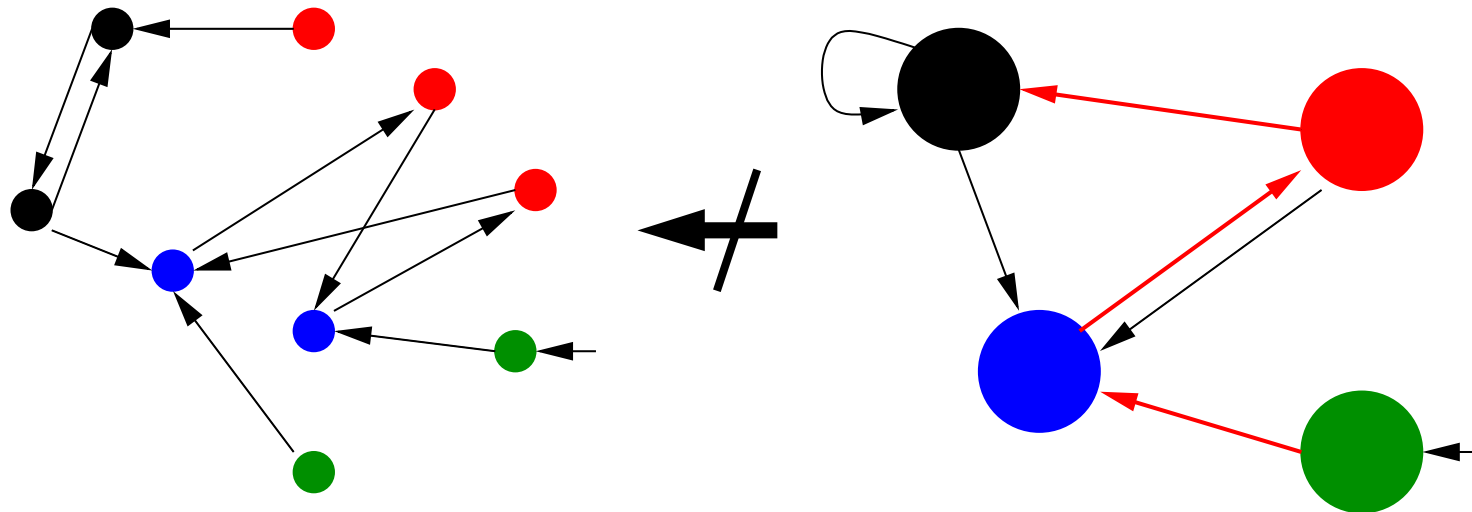
Every concrete run has got a corresponding run in the abstraction . . .



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Abstraction gives rise to additional paths in the system:

...but not every abstract run has got a corresponding run in the concrete system.



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Suppose that  $\mathcal{K}' \not\models \phi$ , where  $\rho$  is a counterexample.

Check whether there is a run in  $\mathcal{K}$  that “corresponds” to  $\rho$ .

If yes, then  $\mathcal{K} \not\models \phi$ .

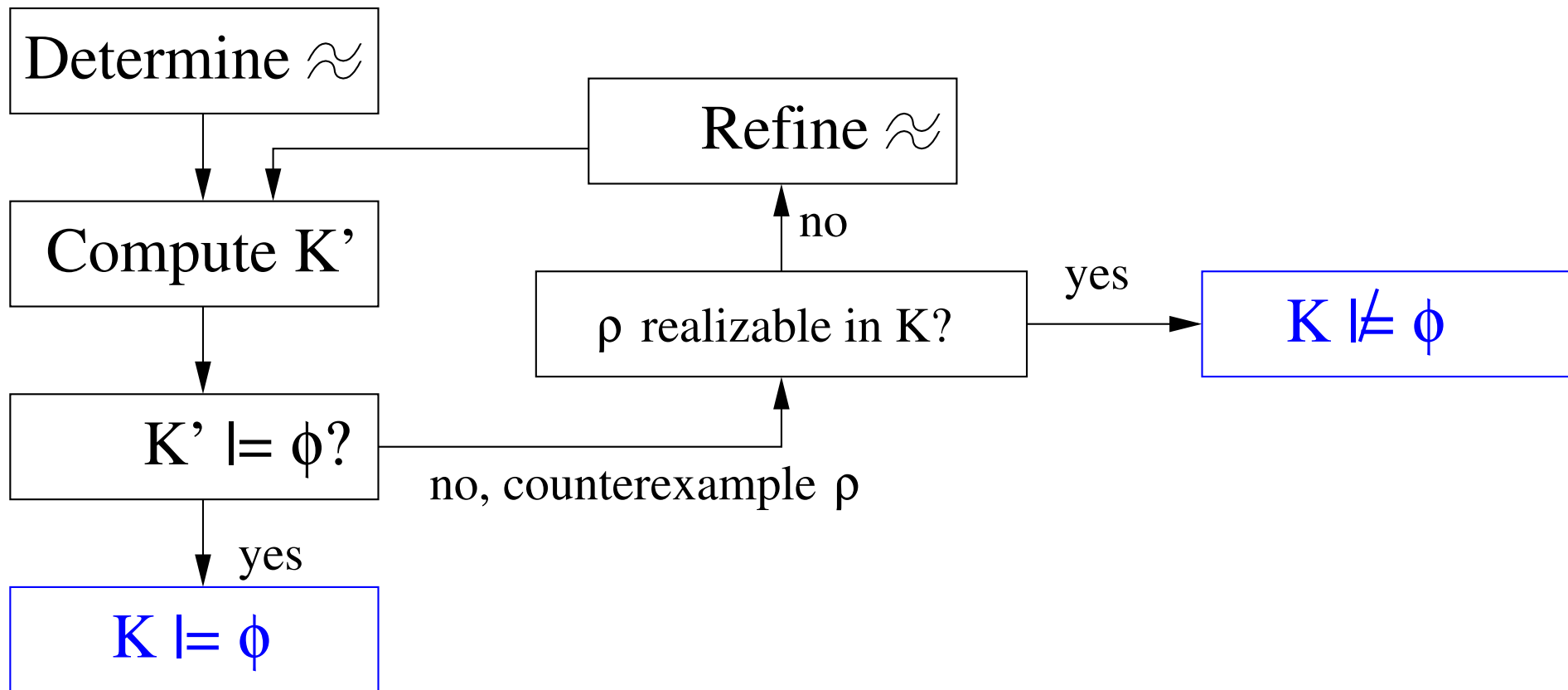
If no, then we can use  $\rho$  to **refine** the abstraction; i.e. we remove some equivalences from the relation  $H$ , introducing additional distinct states in  $\mathcal{K}'$  so that  $\rho$  disappears.

The refinement can be repeated until a definite answer for  $\mathcal{K} \models \phi$  (positive or negative) can be determined. This technique is called **counterexample-guided abstraction refinement** (CEGAR) [Clarke et al, 1994].

# The abstraction-refinement cycle

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Input:  $\mathcal{K}, \phi$



# Simulation of $\rho$

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**Problem:** Given a counterexample  $\rho$ , is there a run corresponding to  $\rho$  in  $\mathcal{K}$ ?

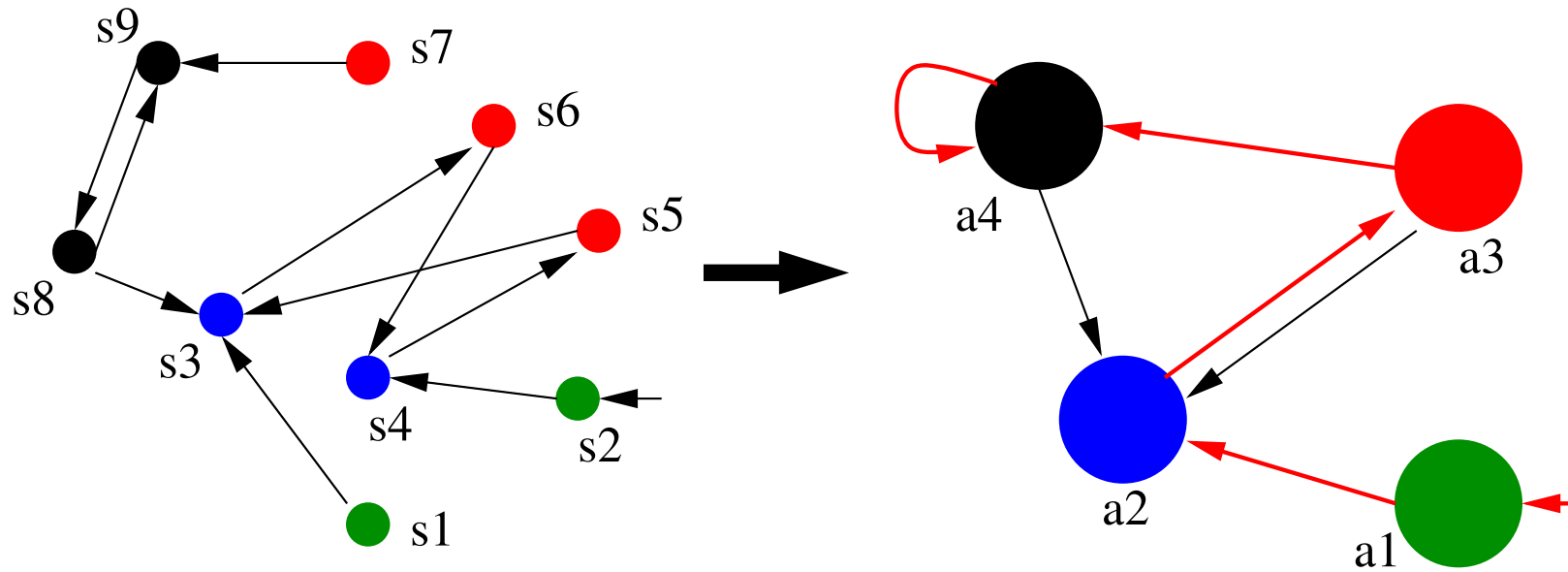
**Solution:** “Simulate”  $\rho$  on  $\mathcal{K}$ .

**Remark:** Any counterexample  $\rho$  can be partitioned into a finite **stem** and a finite **loop**, i.e.  $\rho = w_S w_L^\omega$  for suitable  $w_S, w_L$ .

**Case distinction:** The simulation may fail in the stem or in the loop.

# Example 1: $G \not\models \text{black}$

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Abstraction yields a counterexample with stem  $a_1 a_2 a_3 a_4$  and loop  $a_4$ .

# Simulating the stem

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Let  $w_S = b_0 \cdots b_k$ .

Start with  $S_0 = \{r\}$ . (We have  $b_0 = [r]$ .)

For  $i = 1, \dots, k$ , compute  $S_i = \{t \mid t \in b_i \wedge \exists s \in S_{i-1} : s \rightarrow t\}$ .

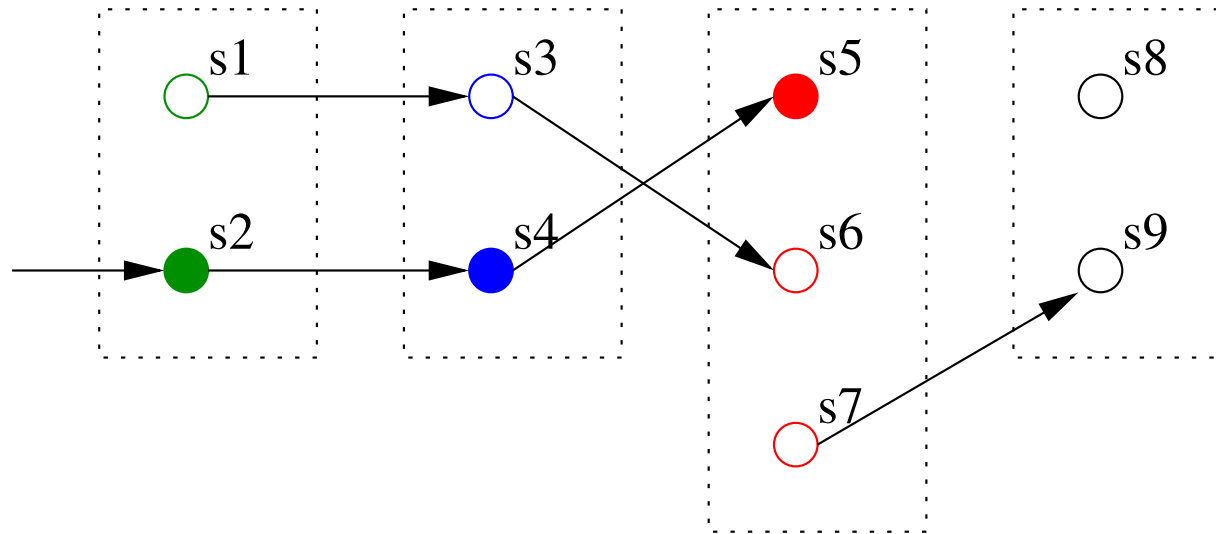
If  $S_k \neq \emptyset$ , then there is a concrete correspondence for  $w_S$ .

If  $S_k = \emptyset$ : Find the smallest index  $\ell$  with  $S_\ell = \emptyset$ : The refinement should distinguish the states in  $S_{\ell-1}$  and those  $b_{\ell-1}$ -states that have immediate successors in  $b_\ell$ .

# Example: $w_S = a_1 a_2 a_3 a_4$

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$$S_0 = \{s_2\}, \quad S_1 = \{s_4\}, \quad S_2 = \{s_5\}, \quad S_3 = \emptyset.$$

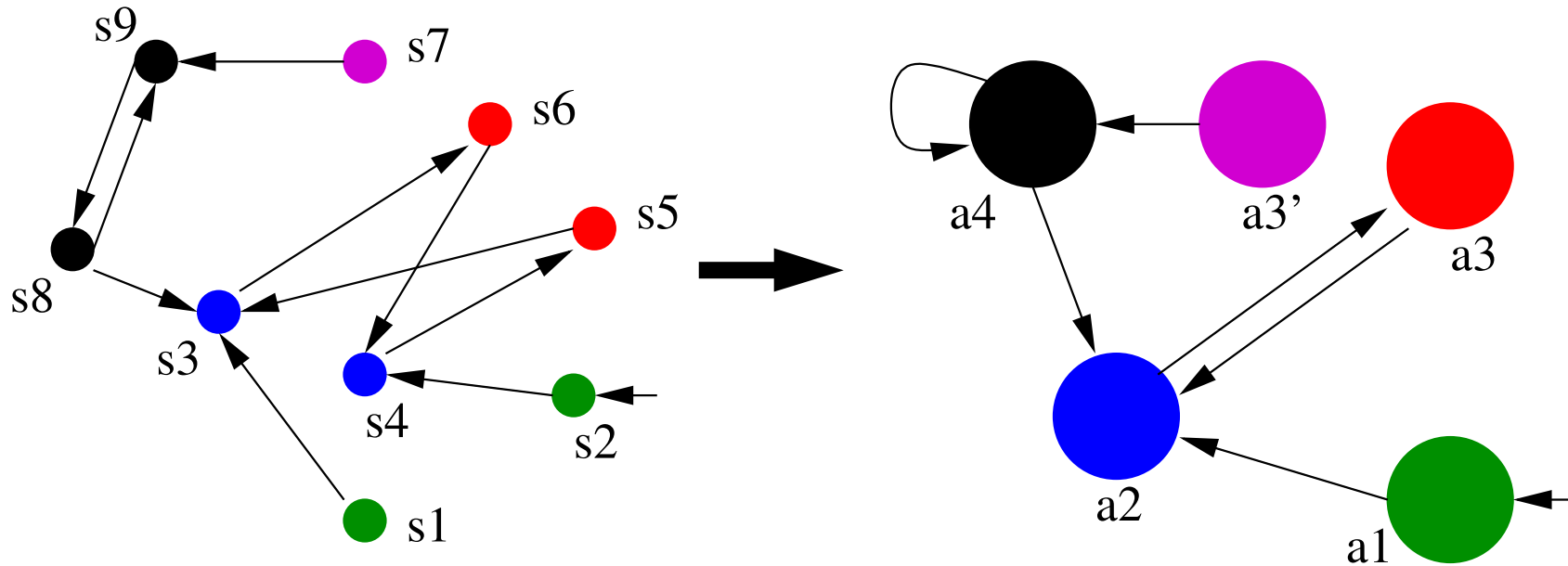


In the next refinement,  $s_5$  and  $s_7$  must be distinguished.

Possible new equivalence classes:  $\{s_5, s_6\}$ ,  $\{s_7\}$  or  $\{s_5\}$ ,  $\{s_6, s_7\}$ .

## Next try: $G \neg black$ with refinement

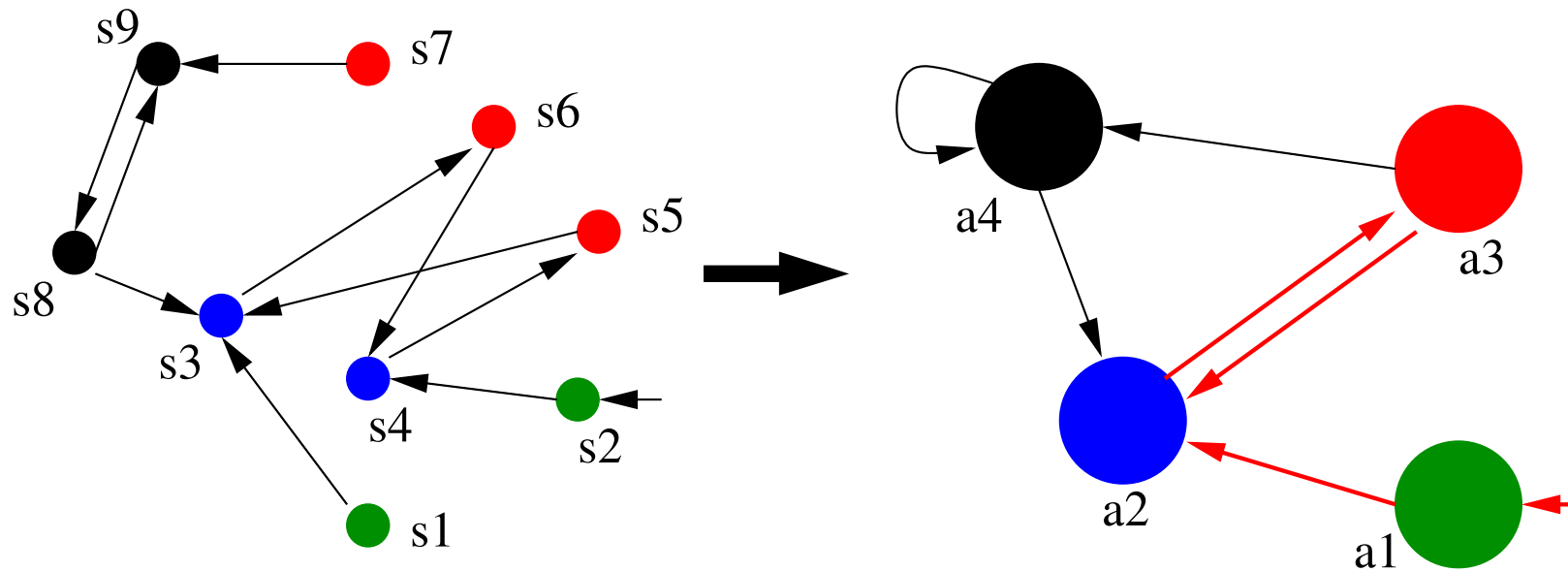
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The new abstraction does not yield any counterexample; therefore,  $G \neg black$  also holds in the concrete system.

## Example 2: $\text{FG red}$

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The abstraction yields a counterexample with stem  $a_1 a_2$  and loop  $a_3 a_2$ .

# Simulating a loop

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Assume  $w_S = b_0 \cdots b_k$ ,  $w_L = c_1 \cdots c_\ell$

$w_S$  is simulated as before, however  $w_L$  may have to be simulated multiple times.

Let  $m$  be the size of the smallest equivalence class in  $w_L$ :

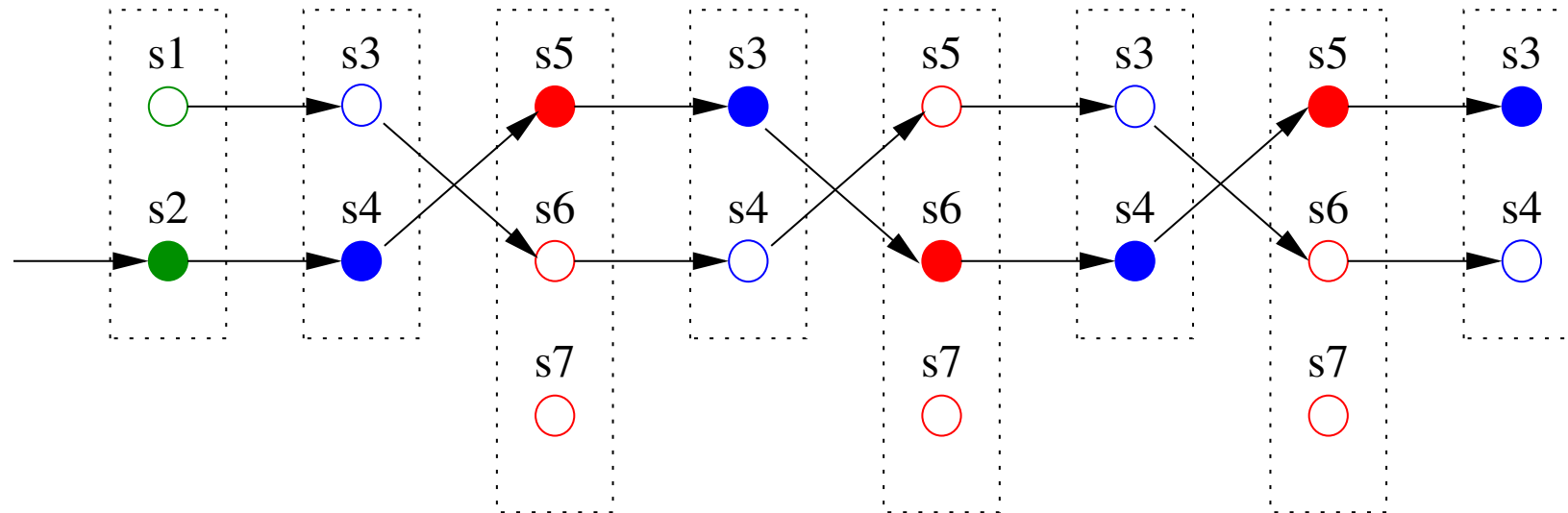
$$m = \min_{i=1, \dots, \ell} |c_i|$$

Then we simulate the path  $w_S w_L^{m+1}$ ; doing so, either the simulation will fail, or we will discover a real counterexample.

Refinement: same as before.

Example:  $w_S = a_1 a_2$ ,  $w_L = a_3 a_2$ ,  $m = 2$

---



The simulation succeeds because there is a loop around  $s_4$ .  
 Thus, there is a real counterexample, so  $\mathcal{K} \neq \phi$ .

# Constructing the abstraction

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**Problem:** Abstract system should be generated without generating the concrete system! (too big or even infinite)

**Here:** Example for the case where the concrete system is given using BDDs.

$\mathcal{K} = (\mathcal{S}, \rightarrow, r, AP, \nu)$ : concrete transition relation  $\rightarrow \subseteq \mathcal{S} \times \mathcal{S}$  given as BDD  $R$  with variables  $\vec{x}, \vec{y}$

Equivalence classes given as BDDs  $V_1, \dots, V_m$  (over  $\vec{x}$ )

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Introduce two sets of  $m$  new BDD variables  $\vec{v}, \vec{w}$ .

Compute  $R' := \exists \vec{x}, \vec{y}: R \wedge \bigwedge_{i=1}^m (v_i \leftrightarrow V_i) \wedge \bigwedge_{i=1}^m (w_i \leftrightarrow V_i[\vec{x}/\vec{y}])$

Then  $R'$  is the abstract transition relation.

**Checking a counterexample:** use adequate operations to compute  $S_i = \{ t \mid t \in b_i \wedge \exists s \in S_{i-1}: s \rightarrow t \}$ .

**Refinement:** If  $S_{\ell+1} = \emptyset$ , choose new class  $S_\ell$ .

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CEGAR: first discussed in 1994

Since about 2000: software model checkers using CEGAR

SLAM / SDV (Microsoft Research)

Blast (Stanford / EPFL)

Magic (CMU)

“canonical application:” device drivers (many lines of code but few data dependencies)

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Typical properties of existing CEGAR systems:

Input: C program

Property: Reachability or safety property

Abstraction: predicates, boolean programs

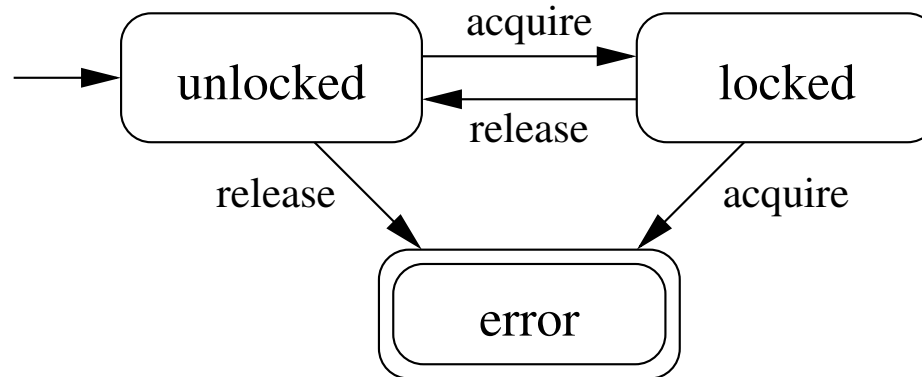
# Overview of SLAM / SDV

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Developed by **Ball, Rajamani et al**, since 2000

Input: C program (Windows device driver)

Safety property given by a finite automaton,  
e.g. “acquire/release” operations must be in proper order:



More complicated property: up to 30 states

Preprocessing: embed the automaton into the C program using assertions

# Abstraction in SLAM

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Set of predicates  $P$  (boolean expressions, e.g.  $x==y$ )

equivalence classes distinguish lines of code and fulfilled predicates

**Implementation:** Translate the C program into a “boolean program”, where all variables are of type Boolean.

BDD-based model checker, tests for reachability of failed assertions and returns an execution (of the boolean program) that reaches the failed assertion.

# Generating the predicates

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Initial predicates: State of the property automaton.

Every instruction of the C program translated into an assignment of the predicate variables.

Compute the weakest precondition for validity or invalidity of each predicate after the instruction.

Find out which predicates imply the weakest precondition.

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Example: predicate  $p := (x = y)$ , instruction  $x = x + 1$

weakest precondition:  $x + 1 = y$

$p$  implies  $\neg(x + 1 = y)$ .

Neither  $p$  nor  $\neg p$  imply  $x + 1 = y$ .

$\Rightarrow$  translate the statement into:

```
if p then p:=false else p:=unknown
```

---

Use of **theorem provers** to find the right preconditions (theories for arithmetics, pointers etc required!)

Not always decidable!

(The theorem prover may reply “I don’t know”.)

Many calls to the theorem prover, generating the abstraction becomes the bottleneck.

# Checking a counterexample

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A counterexample is a sequence of instructions, e.g.:

```
x = c;
```

```
d = c+1;
```

```
y = d;
```

```
if (y != x+1) error;
```

Obviously, this sequence cannot lead to the error (it is a “spurious” counterexample).

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Translate the instructions into a formula  $F = F_1 \wedge F_2 \wedge F_3 \wedge F_4$ :

$$F_1 = x_1 = c_0 \wedge y_1 = y_0 \wedge d_1 = d_0 \wedge c_1 = c_0$$

$$F_2 = x_2 = x_1 \wedge y_2 = y_1 \wedge d_2 = c_1 + 1 \wedge c_2 = c_1$$

$$F_3 = x_3 = x_2 \wedge y_3 = d_2 \wedge d_3 = d_2 \wedge c_3 = c_2$$

$$F_4 = y_3 \neq x_3 + 1$$

**Observation:** The example is spurious iff  $F$  is unsatisfiable.

How do we gain new predicates from the formula?

# Interpolants

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Let  $F, G$  be two formulae s.t.  $F \wedge G$  is unsatisfiable.

$J$  is called a (Craig) interpolant between  $F$  and  $G$  iff

- (i)  $F$  implies  $J$ ;
- (ii)  $J \wedge G$  is unsatisfiable;
- (iii)  $J$  contains only atomic propositions that also appear in  $F$  and in  $G$ .

Let  $Z$  be the variables occurring in  $F$  but not in  $G$ .

Then  $\exists Z : F$  is a Craig interpolant between  $F$  and  $G$ .

Let  $Z$  be the variables occurring in  $G$  but not in  $F$ .

Then  $\forall Z : \neg G$  is a Craig interpolant between  $F$  and  $G$ .

# Application of interpolation

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Compute interpolants between  $F_1$  and  $F_2 \wedge F_3 \wedge F_4$ ,  
between  $F_1 \wedge F_2$  and  $F_3 \wedge F_4$  etc.

In the example (using existential quantification):

$$x_1 = c_1$$

$$d_2 = x_2 + 1$$

$$y_3 = x_3 + 1$$

Add those interpolants as predicates (for the respective line of code):

Optimization: Reduce the sequence to those instructions that actually influence the error condition (“slicing”).

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Let  $J$  be an interpolant between (z.B.)  $F_1$  and  $F_2 \wedge F_3 \wedge F_4$ .

$J$  depends only on variables with index 1, i.e. it is a predicate over the state after the first instruction.

$J$  is implied by  $F_1$ , i.e. we know that after executing the first statement  $J$  holds.

⇒ In the new abstraction  $J$  will be true after the first instruction.

$J \wedge G$  is unsatisfiable, i.e. the states represented by  $J$  cannot reach the error state.

⇒ The new abstraction excludes the previous error path.

**Conclusion:** Adding the predicate  $J$  will improve the abstraction.

# Computing interpolants

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(depends on the logic in which  $F$  and  $G$  are expressed!)

In general:

the strongest postcondition of  $F$  w.r.t. *true* is an interpolant.

the weakest precondition  $G$  w.r.t. *false* is an interpolant.

Let us consider the case of propositional logic.

# Example

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Consider the following sequence. The assume statement cannot be executed:

$x := \text{true}; \quad y := x; \quad \text{assume}(\neg y \wedge z);$

Translation into propositional-logic formulae:

$$F_1 = x_1 \wedge (y_1 \leftrightarrow y_0) \wedge (z_1 \leftrightarrow z_0)$$

$$F_2 = (x_2 \leftrightarrow x_1) \wedge (y_2 \leftrightarrow x_1) \wedge (z_2 \leftrightarrow z_1)$$

$$F_3 = (\neg y_2 \wedge z_2) \wedge (x_3 \leftrightarrow x_2) \wedge (y_3 \leftrightarrow y_2) \wedge (z_3 \leftrightarrow z_2)$$

---

Computation of the strongest interpolants:

between  $F_1$  and  $F_2 \wedge F_3$ :  $J_1 := x_1$

between  $F_1 \wedge F_2$  and  $F_3$ :  $J_2 := x_2 \wedge y_2$

Computation of the weakest interpolants:

between  $F_1 \wedge F_2$  and  $F_3$ :  $K_2 := y_2 \vee \neg z_2$

between  $F_1$  and  $F_2 \wedge F_3$ :  $K_1 := x_1 \vee \neg z_1$

Both  $J_1, J_2$  and  $K_1, K_2$  are Hoare annotations proving spuriousness of the sequence:

$\{\text{true}\} \quad x:=\text{true}; \quad \{x\} \quad y:=x; \quad \{x \wedge y\} \quad \text{assume}(\neg y \wedge z); \quad \{\text{false}\}$

$\{\text{true}\} \quad x:=\text{true}; \quad \{x \vee \neg z\} \quad y:=x; \quad \{y \vee \neg z\} \quad \text{assume}(\neg y \wedge z); \quad \{\text{false}\}$