Logic homework – possible solutions

Question 1

Proof 1 (π_1)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi}\ \text{Axiom}}{\phi, \neg\phi \Rightarrow \bot}\ \neg\ \text{left}}{\phi \Rightarrow \neg\neg\phi}\ \neg\ \text{right}}{\Rightarrow \phi \to \neg\neg\phi}\ \to\ \text{right}$$

Proof 2 (π_2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi}\ \text{Axiom}}{\phi \Rightarrow \phi \vee \neg\phi}\ \vee\ \text{right}}{\phi, \neg(\phi \vee \neg\phi) \Rightarrow \bot}\ \neg\ \text{left}}{\neg(\phi \vee \neg\phi) \Rightarrow \neg\phi}\ \neg\ \text{right}}{\neg(\phi \vee \neg\phi) \Rightarrow \phi \vee \neg\phi}\ \vee\ \text{right}}{\neg(\phi \vee \neg\phi), \neg(\phi \vee \neg\phi) \Rightarrow \bot}\ \neg\ \text{left}}{\neg(\phi \vee \neg\phi) \Rightarrow \bot}\ \text{CL}}{\Rightarrow \neg\neg(\phi \vee \neg\phi)}\ \neg\ \text{right}$$

Proof 3 (π_3)

$$\dfrac{\dfrac{}{\phi \Rightarrow \phi}\ \text{Axiom} \qquad \dfrac{\dfrac{\dfrac{}{\neg\phi \Rightarrow \neg\phi}\ \text{Axiom}}{\neg\phi, \neg\neg\phi \Rightarrow \bot}\ \neg\ \text{left}}{\neg\phi, \neg\neg\phi \Rightarrow \phi}\ \text{WR}}{\phi \vee \neg\phi, \neg\neg\phi \Rightarrow \phi}\ \vee\ \text{left}$$

Proof 4 (π_4)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi}\ \text{Axiom}}{\phi \Rightarrow \phi \lor \psi}\ \lor\ \text{right}}{\phi, \neg(\phi \lor \psi) \Rightarrow \bot}\ \neg\ \text{left}}{\neg(\phi \lor \psi) \Rightarrow \neg\phi}\ \neg\ \text{right}}{\neg(\phi \lor \psi) \Rightarrow \neg\phi \lor \neg\psi}\ \lor\ \text{right}$$

Question 2

Prove RA using DN:

$$\dfrac{\dfrac{\neg\phi, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \neg\neg\phi}\ \neg\ \text{right} \qquad \dfrac{}{\neg\neg\phi \Rightarrow \phi}\ \text{DN}}{\Gamma \Rightarrow \phi}\ \text{Cut}$$

Prove DN using TE: We re-use Proof 3 of Question 1.

$$\dfrac{\dfrac{}{\Rightarrow \phi \lor \neg\phi}\ \text{TE} \qquad \dfrac{\pi_3}{\phi \lor \neg\phi, \neg\neg\phi \Rightarrow \phi}}{\neg\neg\phi \Rightarrow \phi}\ \text{Cut}$$

Prove TE using RA: We have already shown $\neg(\phi \lor \neg\phi) \Rightarrow \bot$ in π_2.

$$\dfrac{\dfrac{\pi_2}{\neg(\phi \lor \neg\phi) \Rightarrow \bot}}{\Rightarrow \phi \lor \neg\phi}\ \text{RA}$$

This shows that all the extensions of LJ_0 are equivalent. It remains to show that they are equivalent to LK_0. If Δ is a multiset, let us denote by $\overline{\Delta}$ the multiset containing an occurrence of $\neg\phi$ for every occurrence of ϕ in Δ.

First, one shows the following for all the rules in LJ_0 plus extensions: if the premises are valid, then the conclusion is valid. This is trivial, in fact most rules are special cases of the rules in LK_0, for which correctness is known, and the others are very simple. (However, note that the reverse implication does not hold!) Thus, any sequent provable in LJ_0 plus extensions is valid, therefore provable in LK_0.

For the reverse direction, assume that $\Gamma_0 \Rightarrow \phi_0$ is provable in LK_0 by a proof π. By the completeness theorem, we can w.l.o.g. assume that π uses only axioms and the eight rules for introducing operators on the left and right. Suppose that one can do the following things:

1. for every axiom $\Gamma \Rightarrow \Delta$ used in π, one constructs a proof of $\Gamma, \overline{\Delta} \Rightarrow \bot$;

2. for every step in π with premises $\Gamma_i \Rightarrow \Delta_i$ and conclusion $\Gamma \Rightarrow \Delta$, one constructs a proof with premises $\Gamma_i, \overline{\Delta}_i \Rightarrow \bot$ and conclusion $\Gamma, \overline{\Delta} \Rightarrow \bot$;

3. one constructs a proof with premise $\Gamma_0, \neg\phi_0 \Rightarrow \bot$ and conclusion $\Gamma_0 \Rightarrow \phi_0$.

Then, putting those constructions together, one obtains another proof of $\Gamma_0 \Rightarrow \phi_0$. We will show that these constructions are possible using only rules from LJ_0 plus extensions.

For step 1, every axiom $\Gamma, A \Rightarrow \Delta, A$ in LK_0 is replaced by an axiom $\Gamma, \overline{\Delta}, A \Rightarrow A$ in LJ_0, followed by an application of $\neg$ left.

Step 3 is achieved by an application of RA.

For step 2, the transformation depends on the rule of LK_0 that was used:

- For $\neg$ left, nothing needs to be done.

- For $\wedge$ right, we do the following:

$$\dfrac{\dfrac{\dfrac{\Gamma, \overline{\Delta}, \neg\phi \Rightarrow \bot}{\Gamma, \overline{\Delta} \Rightarrow \phi}\ \text{RA} \qquad \dfrac{\Gamma, \overline{\Delta}, \neg\psi \Rightarrow \bot}{\Gamma, \overline{\Delta} \Rightarrow \psi}\ \text{RA}}{\Gamma, \overline{\Delta} \Rightarrow \phi \wedge \psi}\ \neg\ \text{right}}{\Gamma, \overline{\Delta}, \neg(\phi \wedge \psi) \Rightarrow \bot}\ \neg\ \text{left}$$

- The cases $\wedge$ left, $\vee$ left, $\rightarrow$ left, $\rightarrow$ right, and $\neg$ right are analogous. (In fact, all are simpler, because one needs just one premise, or no RA step, or no $\neg$ left step, or combinations thereof.)

- The remaining case is $\vee$ right. We replace it by

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow \phi \vee \neg\phi}\ \text{TE} \qquad \dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi}\ \text{Axiom}}{\phi \Rightarrow \phi \vee \psi}\ \vee\ \text{right} \qquad \dfrac{\dfrac{\Gamma, \overline{\Delta}, \neg\phi, \neg\psi \Rightarrow \bot}{\Gamma, \overline{\Delta}, \neg\phi \Rightarrow \psi}\ \neg\ \text{right}}{\Gamma, \overline{\Delta}, \neg\phi \Rightarrow \phi \vee \psi}\ \vee\ \text{right}}{\Gamma, \overline{\Delta}, \phi \vee \neg\phi \Rightarrow \phi \vee \psi}\ \vee\ \text{left}}{\Gamma, \overline{\Delta} \Rightarrow \phi \vee \psi}\ \text{Cut}}{\Gamma, \overline{\Delta}, \neg(\phi \vee \psi) \Rightarrow \bot}\ \neg\text{left}$$

Question 3

Part 1

One can apply the same rewriting techniques as in LK_0, often a bit simplified because of the simpler structure of the rules. The only significant change is in the case where a disjunction is generated on both sides just before the cut:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \phi_1}{\Gamma_2 \Rightarrow \phi_1 \vee \phi_2}\;\vee\text{ right} \qquad \dfrac{\Gamma_2, \phi_1 \Rightarrow \psi \quad \Gamma_3, \phi_2 \Rightarrow \psi}{\Gamma_2, \Gamma_3, \phi_1 \vee \phi_2 \Rightarrow \psi}\;\vee\text{ left}}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \psi}\;\text{Cut}$$

This can be rewritten as follows:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \phi_1 \quad \Gamma_2, \phi_1 \Rightarrow \psi}{\Gamma_1, \Gamma_2 \Rightarrow \psi}\;\text{Cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \psi}\;\text{WL}$$

Part 2

We know from the answer to Question 2 that only valid sequents can be obtained in LJ_0. Moreover, according to Part 1, we can w.l.o.g. assume a proof does not contain the cut rule. For each sequent ϕ in question, we list the possible last rules in a proof. It is then easy to verify that the sequents from which ϕ can be derived are invalid, therefore a proof cannot exist. We use ϕ^+ to denote one or more occurrence of ϕ, and ϕ^* for zero or more occurrences of ϕ and CL^* to denote arbitrarily many applications of CL.

1. $\Rightarrow A \vee \neg A$ can be derived through application of
   - $\vee$ right from $\Rightarrow A$ or $\Rightarrow \neg A$;
   - WR from $\Rightarrow \bot$.
2. $A \rightarrow B \Rightarrow \neg A \vee B$ can be derived through application of
   - $\vee$ right and CL^* from $(A \rightarrow B)^+ \Rightarrow \neg A$ or $(A \rightarrow B)^+ \Rightarrow B$;
   - WR and CL^* from $(A \rightarrow B)^+ \Rightarrow \bot$;
   - WL from $\Rightarrow \neg A \vee B$;
   - $\rightarrow$ left and CL^* from $(A \rightarrow B)^* \Rightarrow A$ and $(A \rightarrow B)^*, B \Rightarrow \neg A \vee B$.
3. $\neg(A \wedge B) \Rightarrow \neg A \vee \neg B$ can be derived through application of
   - $\vee$ right and CL^* from $\neg(A \wedge B)^+ \Rightarrow \neg A$ or $\neg(A \wedge B)^+ \Rightarrow \neg B$.
   - WR and CL^* from $\neg(A \wedge B)^* \Rightarrow \bot$;
   - WL from $\Rightarrow \neg A \vee \neg B$;

Part 3

One can easily check that, apart from Cut, every rule in LJ_0 preserves all the subformulae from the premises in the consequence. Thus, once a subformula occurs in a proof, it is guaranteed to occur in the final sequent.

Question 4

It suffices to check whether a cut-free proof exists. Moreover, because of WL and CL it suffices to consider the case $\Gamma \Rightarrow \phi$ where Γ is a set (rather than the general multiset case). More precisely, whenever we write $\Gamma \Rightarrow \phi$ we really mean the equivalence class of sequents that can be obtained by inflating or deflating non-zero multiplicities of formulae using WL and CL. Below, we propose two algorithms based on "backward" and "forward" search.

The "backward" solution consists of a procedure that takes as argument a sequent $\Sigma = \Gamma \Rightarrow \phi$. First, it checks whether Σ is valid (e.g., by taking $\neg\phi \wedge \bigwedge_{\psi \in \Gamma} \psi$, converting it to clausal form, and checking whether the result is unsatisfiable using resolution). If Σ is invalid, the procedure returns false. If Σ is an axiom, it returns true. Otherwise, it recursively checks for all the (pairs of) possible predecessor sequents, of which there are finitely many, whether they are provable in LJ_0. This latter step requires a little care:

- For instance, if $\Sigma = \Gamma \cup \{\phi \vee \psi\} \Rightarrow \theta$, then – because of the use of WL and CL mentioned above – the predecessor sequents include all pairs $\Gamma_1, \phi \Rightarrow \theta$ and $\Gamma_2 \psi \Rightarrow \theta$, where $\Gamma \subseteq \Gamma_1 \cup \Gamma_2 \subseteq \Gamma \cup \{\phi \vee \psi\}$. In other words, every formula of Γ must be contained in at least one of Γ_1 and Γ_2, maybe in both, and $\phi \vee \psi$ may or may not be contained in one or both.

- Because of CL it is difficult to make an argument that the sequents appearing during the recursion are of decreasing size. However, all sequents that are constructed throughout the procedure are subformulae of formulae from Σ, and there are only finitely many of them. Thus, termination can be ensured by checking, before each recursive call, whether the current call stack already contains the sequent in question, and aborting if that is the case. (Granted, this is not very elegant.)

For the "forward" solution, let $\mathcal{F}$ be the set of subformulae occurring in $\Gamma \Rightarrow \phi$, plus $\bot$. We take $\mathcal{S}$ to be the set of sequents $\Gamma' \Rightarrow \phi'$, where $\Gamma' \subseteq \mathcal{F}$ and $\phi' \in \mathcal{F}$. Moreover, initialize $\mathcal{S}_0$ to be the set of all axioms in $\mathcal{S}$. We then apply a fixed-point computation: In each iteration, we extend $\mathcal{S}_0$ by those sequents in $\mathcal{S} \setminus \mathcal{S}_0$ that are derivable with one rule (other than Cut) from premises in $\mathcal{S}_0$. Since $\mathcal{S}$ is finite, the computation terminates, and $\mathcal{S}_0$ contains all the derivable sequents within $\mathcal{S}$. We then check whether $\Gamma \Rightarrow \phi$ is contained in $\mathcal{S}_0$.