TD 8: Unfoldings, Vector Addition Systems

1 Unfoldings

Exercise 1 (Adequate Partial Orders)**.** A partial order $\prec$ between events is *adequate* if the three following conditions are verified:

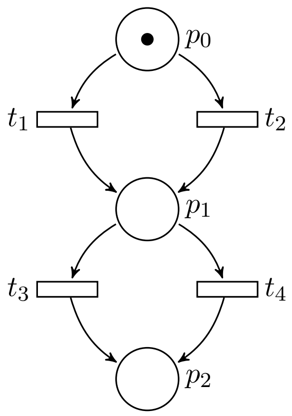
(a) $\prec$ is well-founded,

(b) $\lfloor t \rfloor \subsetneq \lfloor t' \rfloor$ implies $t \prec t'$, and

(c) $\prec$ is preserved by finite extensions: as in the lecture notes, if $t \prec t'$ and $B(t) = B(t')$, and E and E' are two isomorphic extensions of $\lfloor t \rfloor$ and $\lfloor t' \rfloor$ with $\lfloor u \rfloor = \lfloor t \rfloor \oplus E$ and $\lfloor u' \rfloor = \lfloor t' \rfloor \oplus E'$, then $u \prec u'$.

As you can guess, adequate partial orders result in complete unfoldings.

1. Show that $\prec_s$ defined by $t \prec_s t'$ iff $|\lfloor t \rfloor| < |\lfloor t' \rfloor|$ is adequate.
2. Construct the finite unfolding of the following Petri net using $\prec_s$; how does the size of this unfolding relate to the number of reachable markings?



3. Suppose we define an arbitrary total order $\ll$ on the transitions T of the Petri net, i.e. they are $t_1 \ll \cdots \ll t_n$. Given a set S of events and conditions of $\mathcal{Q}$, $\varphi(S)$ is the sequence $t_1^{i_1} \cdots t_n^{i_n}$ in T^* where i_j is the number of events labeled by t_j in S. We also note $\ll$ for the lexicographic order on T^*.

   Show that $\prec_e$ defined by $t \prec_e t'$ iff $|\lfloor t \rfloor| < |\lfloor t' \rfloor|$ or $|\lfloor t \rfloor| = |\lfloor t' \rfloor|$ and $\varphi(\lfloor t \rfloor) \ll \varphi(\lfloor t' \rfloor)$ is adequate. Construct the finite unfolding for the previous Petri net using $\prec_e$.
4. There might still be examples where $\prec_e$ performs poorly. One solution would be to use a *total* adequate order; why? Give a 1-safe Petri net that shows that $\prec_e$ is not total.

Exercise 2 (Coverability in Unfoldings)**.** We consider the coverability problem in a finite complete prefix $\mathcal{Q}$ of the unfolding of a 1-safe Petri net $\mathcal{N}$: given a target marking m in $\mathbb{N}^P$, is there an event e of $\mathcal{Q}$ with $m_e \geq m$?

Show that the coverability problem for complete unfoldings is NP-hard.

2 Vector Addition Systems

Exercise 3 (VASS)**.** An n-dimensional *vector addition system with states* (VASS) is a tuple $\mathcal{V} = \langle Q, \delta, q_0 \rangle$ where Q is a finite set of states, $q_0 \in Q$ the initial state, and $\delta \subseteq Q \times \mathbb{Z}^n \times Q$ the transition relation. A configuration of $\mathcal{V}$ is a pair (q, v) in $Q \times \mathbb{N}^n$. An execution of $\mathcal{V}$ is a sequence of configurations $(q_0, v_0)(q_1, v_1) \cdots (q_m, v_m)$ such that $v_0 = \bar{0}$, and for $0 < i \leq m$, $(q_{i-1}, v_i - v_{i-1}, q_i)$ is in δ.

1. Show that any VASS can be simulated by a Petri net—we can give a formal meaning to 'simulation', but you haven't seen it in class yet, so do it at an intuitive level...
2. Show that, conversely, any Petri net can be simulated by a VASS.

Exercise 4 (VAS)**.** An n-dimensional *vector addition system* (VAS) is a pair (v_0, W) where $v_0 \in \mathbb{N}^n$ is the initial vector and $W \subseteq \mathbb{Z}^n$ is the set of transition vectors. An execution of (v_0, W) is a sequence $v_0 v_1 \cdots v_m$ where $v_i \in \mathbb{N}$ for all $0 \leq i \leq m$ and $v_i - v_{i-1} \in W$ for all $0 < i \leq m$.

We want to show that any n-dimensional VASS $\mathcal{V}$ can be simulated by an $(n+3)$-dimensional VAS (v_0, W).

Hint: Let $k = |Q|$, and define the two functions $a(i) = i + 1$ and $b(i) = (k+1)(k-i)$. Encode a configuration (q_i, v) of $\mathcal{V}$ as the vector $(v(1), \ldots, v(n), a(i), b(i), 0)$. For every state q_i, $0 \leq i < k$, we add two transition vectors to W:

$$t_i = (0, \ldots, 0, -a(i), a(k-i) - b(i), b(k-i))$$
$$t'_i = (0, \ldots, 0, b(i), -a(k-i), a(i) - b(k-i))$$

For every transition $d = (q_i, w, q_j)$ of $\mathcal{V}$, we add one transition vector to W:

$$t_d = (w(1), \ldots, w(n), a(j) - b(i), b(j), -a(i))$$

1. Show that any execution of $\mathcal{V}$ can be simulated by (v_0, W) for a suitable v_0.
2. Conversely, show that this VAS (v_0, W) simulates $\mathcal{V}$ faithfully.