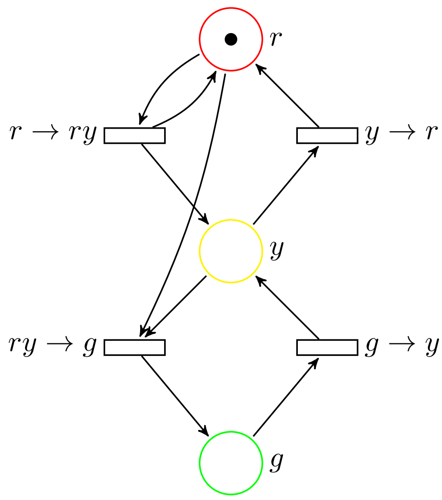
TD 7: Petri Nets

1 Modeling Using Petri Nets

Exercise 1 (Traffic Lights)**.** Consider again the traffic lights example from the lecture notes:



1. How can you correct this Petri net to avert unwanted behaviours (like $r \to ry \to rr$) in a 1-safe manner?

2. Extend your Petri net to model two traffic lights handling a street intersection.

Exercise 2 (Producer/Consumer)**.** A producer/consumer system gathers two types of processes:

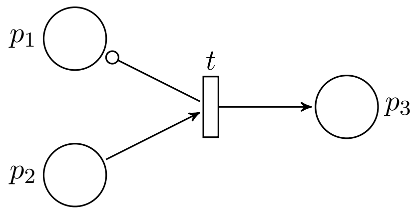
producers who can make the actions *produce* (p) or *deliver* (d), and

consumers with the actions *receive* (r) and *consume* (c).

All the producers and consumers communicate through a single unordered channel.

1. Model a producer/consumer system with two producers and three consumers. How can you modify this system to enforce a maximal capacity of ten simultaneous items in the channel?

2. An *inhibitor arc* between a place p and a transition t makes t firable only if the current marking at p is zero. In the following example, there is such an inhibitor arc between p_1 and t. A marking $(0, 2, 1)$ allows to fire t to reach $(0, 1, 2)$, but $(1, 1, 1)$ does not allow to fire t.



Using inhibitor arcs, enforce a priority for the first producer and the first consumer on the channel: the other processes can use the channel only if it is not currently used by the first producer and the first consumer.

2 Model Checking Petri Nets

Exercise 3 (Upper Bounds)**.** Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider as usual propositional LTL, with a set of atomic propositions AP equal to P the set of places of the Petri net. We define proposition p to hold in a marking m in $\mathbb{N}^P$ if $m(p) > 0$.

The models of our LTL formulæ are *computations* $m_0 m_1 \cdots$ in $(\mathbb{N}^P)^\omega$ such that, for all $i \in \mathbb{N}$, $m_i \rightarrow_{\mathcal{N}} m_{i+1}$ is a transition step of the Petri net $\mathcal{N}$.

1. We want to prove that state-based LTL model checking can be performed in polynomial space for 1-safe Petri nets. For this, prove that one can construct an exponential-sized Büchi automaton $\mathcal{B}_{\mathcal{N}}$ from a 1-safe Petri net that recognizes all the infinite computations of $\mathcal{N}$ starting in m_0.

2. In the general case, state-based LTL model checking is undecidable. Prove it for Petri nets with at least two unbounded places, by a reduction from the halting problem for 2-counter Minsky machines.

3. We consider now a different set of atomic propositions, such that $\Sigma = 2^{\mathrm{AP}}$, and a labeled Petri net, with a labeling homomorphism $\lambda : T \rightarrow \Sigma$. The models of our LTL formulæ are infinite words $a_0 a_1 \cdots$ in Σ^ω such that $m_0 \xrightarrow{t_0}_{\mathcal{N}} m_1 \xrightarrow{t_1}_{\mathcal{N}} m_2 \cdots$ is an execution of $\mathcal{N}$ and $\lambda(t_i) = a_i$ for all i.

   Prove that *action-based* LTL model checking can be performed in polynomial space for labeled 1-safe Petri nets.

3 Coverability Graphs

Exercise 4 (Dickson's Lemma)**.** A *quasi-order* $(A, \leq)$ is a set A endowed with a reflexive and transitive ordering relation $\leq$. A *well quasi order* (wqo) is a quasi order $(A, \leq)$ s.t., for any infinite sequence $a_0 a_1 \cdots$ in A^ω, there exist indices $i < j$ with $a_i \leq a_j$.

1. Let $(A, \leq)$ be a wqo and $B \subseteq A$. Show that $(B, \leq)$ is a wqo.

2. Show that $(\mathbb{N} \uplus \{\omega\}, \leq)$ is a wqo.

3. Let $(A, \leq)$ be a wqo. Show that any infinite sequence $a_0 a_1 \cdots$ in A^ω embeds an infinite increasing subsequence $a_{i_0} \leq a_{i_1} \leq a_{i_2} \leq \cdots$ with $i_0 < i_1 < i_2 < \cdots$.

4. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two wqo's. Show that the cartesian product $(A \times B, \leq_\times)$, where the product ordering is defined by $(a, b) \leq_\times (a', b')$ iff $a \leq_A a'$ and $b \leq_B b'$, is a wqo.

Exercise 5 (Coverability Graph)**.** The *coverability problem* for Petri nets is the following decision problem:

Instance: A Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$ and a marking m_1 in $\mathbb{N}^P$.

Question: Does there exist m_2 in $\mathsf{reach}_\mathcal{N}(m_0)$ such that $m_1 \leq m_2$?

For 1-safe Petri nets, coverability coincides with reachability, and is thus PSPACE-complete.

One way to decide the general coverability problem is to use Karp and Miller's coverability graph (see the lecture notes). Indeed, we have the equivalence between the two statements:

i. there exists m_2 in $\mathsf{reach}_\mathcal{N}(m_0)$ such that $m_1 \leq m_2$, and

ii. there exists m_3 in $\mathsf{CoverabilityGraph}_\mathcal{N}(m_0)$ such that $m_1 \leq m_3$.

1. In order to prove that (*i*) implies (*ii*), we will prove a stronger statement: for a marking m in $(\mathbb{N} \uplus \{\omega\})^P$, write $\Omega(m) = \{p \in P \mid m(p) = \omega\}$ be the set of ω-places of m.

   Show that, if $m_0 \xrightarrow{u}_\mathcal{N} m_2$ in the Petri net $\mathcal{N}$ for some u in T^*, then there exists m_3 in $(\mathbb{N} \uplus \{\omega\})^P$ such that $m_2(p) = m_3(p)$ for all p in $P \setminus \Omega(m_3)$ and $m_0 \xrightarrow{u}_G m_3$ in the coverability graph.

2. Let us prove that (*ii*) implies (*i*). The idea is that we can find reachable markings that agree with m_3 on its finite places, and that can be made arbitrarily high on its ω-places. For this, we need to identify the graph nodes where new ω values were introduced, which we call *ω-nodes*.

   (a) The *threshold* $\Theta(u)$ of a transition sequence u in T^* is the minimal marking m in $\mathbb{N}^P$ s.t. u is enabled from m. Show how to compute $\Theta(u)$. Show that $\Theta(u \cdot v) \leq \Theta(u) + \Theta(v)$ for all u, v in T^*.

   (b) Recall that an ω value is introduced in the coverability graph thanks to Algorithm 1.

   Let $\{v_1, \ldots, v_\ell\}$ be the set of "v" sequences found on line 1 of the algorithm that resulted in adding at least one ω value to m' on line 1 during a single call

repeat
  $saved \leftarrow m'$
  foreach $m'' \in V \ s.t. \ \exists v \in T^+, m'' \xrightarrow{v}_G m$ **do**
    if $m'' < m'$ **then**
      $m' \leftarrow m' + ((m' - m'') \cdot \omega)$
until $saved = m'$
return m'

Algorithm 1: AddOmegas(m, m', V)

to AddOmegas(m, m', V) on line 8 of the CoverabilityGraph algorithm from the course notes. Let $w = v_1 \cdots v_\ell$. Show that, for any k in $\mathbb{N}$, the marking ν_k defined by

$$\nu_k(p) = \begin{cases} m'(p) & \text{if } p \in P \setminus \Omega(m) \\ \Theta(w^k)(p) & \text{if } p \in \Omega(m) \end{cases}$$

allows to fire w^k. How does the marking ν'_k with $\nu_k \xrightarrow{w^k}_{\mathcal{N}} \nu'_k$ compare to ν_k?

(c) Prove that, if $m_0 \xrightarrow{u}_G m_3$ for some u in T^* in the coverability graph and m' in $\mathbb{N}^{\Omega(m_3)}$ is a partial marking on the places of $\Omega(m_3)$, then there are

- n in $\mathbb{N}$,
- a decomposition $u = u_1 u_2 \cdots u_{n+1}$ with each u_i in T^* (where the markings μ_i reached by $m \xrightarrow{u_1 \cdots u_i}_G \mu_i$ for $i \leq n$ have new ω values),
- sequences $w_1, \ldots, w_n$ in T^+,
- numbers $k_1, \ldots, k_n$ in $\mathbb{N}$,

such that $m_0 \xrightarrow{u_1 w_1^{k_1} u_2 \cdots u_n w_n^{k_n} u_{n+1}}_{\mathcal{N}} m_2$ with $m_2(p) = m_3(p)$ for all p in $P \setminus \Omega(m_3)$ and $m_2(p) \geq m'(p)$ for all p in $\Omega(m_3)$.

Exercise 6 (Decidability of Model-checking Action-based LTL)**.**

1. Let $\mathcal{N}$ be Petri net, G its coverability graph, and m some marking in $\mathbb{N}^P$. An infinite *computation* is a sequence $m_0 m_1 \cdots$ in $(\mathbb{N}^P)^\omega$ where for all $i \in \mathbb{N}$, $m_i \rightarrow_{\mathcal{N}} m_{i+1}$ is a transition step. The *effect* $\Delta(u)$ of a transition sequence u in T^* is defined by $\Delta(\varepsilon) = 0^P$ and $\Delta(ut) = \Delta(u) - W(P, t) + W(t, P)$.

   Show that there exists an infinite computation s.t. $m \leq m_i$ for infinitely many indices i iff there exists an accessible loop $m' \xrightarrow{v}_G m'$ in G s.t. $m \leq m'$ and $\Delta(v) \geq 0^P$.

2. Show that action-based LTL model-checking is decidable for labeled Petri nets.