TD 5: EF Games, Separation

1 Separation

Exercise 1 (Expressiveness and Separation)**.** Consider the FO(AP, <) formula

$$\psi(x) = P_a(x) \wedge \forall y.P_a(y) \to \exists z.(y < x \to P_b(z) \wedge y < z < x)$$
$$\wedge(y > x \to P_c(z) \wedge z > y)\ .$$

1. Separate $\psi(x)$, i.e. provide pure formulæ $\psi_i(x)$ such that $\psi(x)$ is equivalent to a boolean combination of the $\psi_i(x)$, and each $\psi_i(x)$ only contains separated subformulæ.
2. Provide equivalent $\mathrm{TL}(\mathrm{AP}, \mathsf{SS}, \mathsf{SU})$ formulæ φ_i for the $\psi_i(x)$.

Exercise 2 (Deciding Semantic Purity)**.** Let us consider time flows in $(\mathbb{N}, <)$. Show that the problem whether a $\mathrm{TL}(\mathrm{AP}, \mathsf{SS}, \mathsf{SU})$ formula φ is *semantically* pure future is in PSpace.

2 EF Games

Exercise 3 (Non-Strict Until)**.**

1. Show that SU is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ over $(\mathbb{R}, <)$.
2. Show that SU is not expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{S}, \mathsf{U})$ over $(\mathbb{N}, <)$.

Exercise 4 (Periodic Properties)**.**

1. Show that the fact that a finite temporal time flow is of "even length" cannot be expressed in $\mathrm{TL}(\mathrm{AP}, \mathsf{SS}, \mathsf{SU})$.
2. Recall Exercise 3 of TD 2: Show that the set $(\{p\}\Sigma)^\omega$ cannot be expressed in $\mathrm{TL}(\{p\}, \mathsf{SS}, \mathsf{SU})$ over $(\mathbb{N}, <)$.

3 LTL with Past

Exercise 5 (Succinctness of Past Formulæ)**.** Consider the time flow $(\mathbb{N}, <)$. Let $\mathrm{AP}_{n+1} = \{p_0, \ldots, p_n\} = \mathrm{AP}_n \cup \{p_n\}$ be a set of atomic propositions, defining the alphabet $\Sigma_{n+1} = 2^{\mathrm{AP}_{n+1}}$. We want to show the existence of an $O(n)$-sized LTL formula with past such that any equivalent pure future LTL formula is of size $\Omega(2^n)$.

First consider the following LTL formula of exponential size:

$$\bigwedge_{S\subseteq \mathrm{AP}_n} \Big((\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j \wedge p_n) \Rightarrow \mathsf{G}((\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j) \Rightarrow p_n)$$
$$\wedge (\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j \wedge \neg p_n) \Rightarrow \mathsf{G}((\bigwedge_{p_i\in S} p_i \wedge \bigwedge_{p_j\notin S} \neg p_j) \Rightarrow \neg p_n)\Big) \qquad (\varphi_n)$$

1. Describe 'intuitively' which words of Σ_{n+1}^{ω} are the models of φ_n.
2. Can an LTL formula with past modalities check whether it is at the initial position of a word?
3. Provide an LTL formula with past ψ_n of size $O(n)$ *initially* equivalent to φ_n.
4. Consider the language $L_n = \{\sigma \in \Sigma_{n+1}^{\omega} \mid \sigma \models \mathsf{G}\,\varphi_n\}$. We want to prove that any generalized Büchi automaton that recognizes L_n requires at least 2^{2^n} states.

   For this we fix a permutation $a_0 \cdots a_{2^n-1}$ of the symbols in Σ_n and we consider all the different subsets $K \subseteq \{0, \ldots, 2^n - 1\}$. For each K we consider the word

   $$w_K = b_0 \cdots b_{2^n-1}$$

   in $\Sigma_{n+1}^{2^n}$, defined for each i in $\{0, \ldots, 2^n - 1\}$ by

   $$b_i = a_i \qquad \text{if } i \in K$$
   $$b_i = a_i \cup \{p_n\} \qquad \text{otherwise.}$$

   Thus K is the set of positions of w_K where p_n does not hold.

   Using the w_K for different values of K, prove that any generalized Büchi automaton for $\mathsf{G}\,\varphi_n$ requires at least 2^{2^n} states.
5. Conclude using the fact that any pure future LTL formula φ can be given a generalized Büchi automaton with at most $2^{|\varphi|}$ states.

4 Stavi Connectives

Exercise 6 (Linear Orders with Gaps)**.** In this exercise we assume $(\mathbb{T}, <)$ to be a linear time flow.

1. Let us define a new unary "gap" modality gap:

$$\begin{aligned} w, i \models \mathsf{gap}\varphi \text{ iff } & \forall k.k > i \rightarrow (\exists \ell.k < \ell \wedge \forall j.i < j < \ell \rightarrow w, j \models \varphi) \\ & \qquad \vee (\exists j.i < j < k \wedge w, j \models \neg\varphi) \\ & \wedge \exists k_1.k_1 > i \wedge \forall j.i < j \leq k_1 \rightarrow w, j \models \varphi \\ & \wedge \exists k_2.k_2 > i \wedge w, k_2 \models \neg\varphi\ . \end{aligned}$$

The intuition behind gap is that φ should hold for some time until a gap occurs in the time flow, after which $\neg\varphi$ holds at points arbitrarily close to the gap.

(a) Express $\mathsf{gap}\varphi$ using the standard SU modality.

(b) Show that, if $(\mathbb{T}, <)$ is Dedekind-complete, then $\mathsf{gap}p$ for $p \in \mathrm{AP}$ cannot be satisfied.

2. Consider the temporal flow $(\{0\} \times \mathbb{Z}_{<0} \times \mathbb{Z} \cup \{1\} \times \mathbb{Z} \times \mathbb{Z}, <)$ where $<$ is the lexicographic ordering and $\mathrm{AP} = \{p\}$. Let n be an even integer in $\mathbb{Z}$, and define

$$h_0(p) = \{(0,i,j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1,i,j) \in \mathbb{T} \mid i \text{ is odd}\}$$
$$h_1(p) = \{(0,i,j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1,i,j) \in \mathbb{T} \mid i > n \text{ is odd}\} .$$

(a) Show that $w_0, (x, i, j) \models \mathsf{gap}p$ for any $x \in \{0, 1\}$, odd i, and j.

(b) Show that no $\mathrm{TL}(\{p\}, \mathsf{SS}, \mathsf{SU})$ formula can distinguish between $(w_0, (0, -1, 0))$ and $(w_1, (0, -1, 0))$.

(c) Here is the definition of the Stavi "until" modality:

$$\begin{aligned} w, i \models \varphi \,\overline{\mathsf{U}}\, \psi \text{ iff } \exists \ell. i < \ell \\ \wedge \forall k. i < k < \ell \to [\exists j_1. k < j_1 \wedge \forall j. i < j < j_1 \to w, j \models \varphi] \\ \vee [(\forall j_2. k < j_2 < \ell \to w, j_2 \models \psi) \\ \wedge (\exists j_3. i < j_3 < k \wedge w, j_3 \models \neg\varphi)] \\ \wedge \exists k_1. i < k_1 < \ell \wedge w, k_1 \models \neg\varphi \\ \wedge \exists k_2. i < k_2 < \ell \wedge \forall j. i < j < k_2 \to w, j \models \varphi \end{aligned}$$

This modality is quite similar to $\mathsf{gap}\varphi$, but further requires ψ to hold for some time after the gap (the "j_2" condition above).

Show that $w_1, (0, -1, 0) \models p \,\overline{\mathsf{U}}\, \neg\, \mathsf{gap}\, p$ but $w_0, (0, -1, 0) \not\models p \,\overline{\mathsf{U}}\, \neg\, \mathsf{gap}\, p$.