TD 3, Exercise 7.3

Let $\Sigma = \{a, b\}$. Construct a prophetic automaton for the language $(a\Sigma)^\omega$.

Thanks to Exercise 7.1, we know that we need to partition Σ^ω in a suitable way. The main insight is that factors of the form $ba^{2n}b$ with $n \geq 0$ must be excluded in order to be in $(a\Sigma)^\omega$. We are going to distinguish three disjoint cases depending on the ultimate behaviour of the words in Σ^ω:

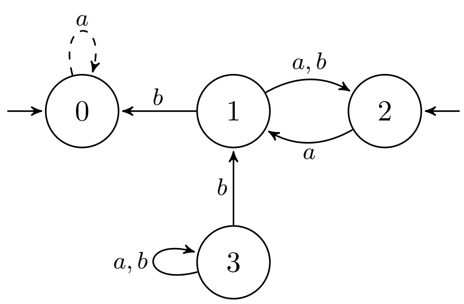
1. the word contains finitely many b's, i.e. is in
$$\Sigma^* a^\omega \; ; \tag{1}$$
2. the word contains infinitely many b's, but only finitely many factors of the form $ba^{2n}b$ for some $n \geq 0$, i.e. is in
$$\Sigma^*(b(aa)^*a)^\omega \; ; \tag{2}$$
3. the word contains infinitely many factors of the form $ba^{2n}b$ for some $n \geq 0$, i.e. is in
$$\Sigma^* \left(b\big((aa)^*ab\big)^*(aa)^*\right)^\omega . \tag{3}$$

Each case will give rise to a subautomaton. To ease the construction, we will construct a Büchi automaton with an accepting set of *transitions* rather than an accepting set of states; we show those transitions with dashed lines in the figures.

Case 1. Let us start with the case where, ultimately, the words are in a^ω, i.e. words in $\Sigma^* a^\omega$. This gives rise to four languages:

$$L_0 = a^\omega \; , \qquad L_1 = (\Sigma a)^* b a^\omega \; , \qquad L_2 = (a\Sigma)^* aba^\omega \; , \qquad L_3 = \Sigma^* b(aa)^* b(a\Sigma)^* a^\omega \; .$$

The rationale for this particular split is to distinguish between words that belong to $(a\Sigma)^\omega$ and the others, since $\Sigma^* a^\omega \cap (a\Sigma)^\omega = L_0 \cup L_2$. Hence the first sub-automaton, which is a co-deterministic automaton for $\Sigma^* a^\omega$:

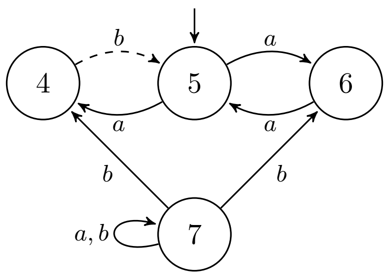


State 3 recognizes the words that contain an occurrence of a $ba^{2n}b$ factor: the transition from state 3 to state 1 has to be taken on the last occurrence of such a factor, on the first of the two b's. Then, in states 0 to 2, we cannot have any such factor and we can only have finitely many b's, using the transition from state 1 to state 0 on the last occurrence of a b; state 1 recognizes those words where this last b is in an even position, and state 2 those words where it is in an odd position.

Case 2. The next case to handle is that of words in $\Sigma^*(b(aa)^*a)^\omega$. The case is interesting because, among the words in $(a\Sigma)^\omega$, those with infinitely many bs are necessarily in this set, and therefore we need to distinguish those words from those of Case 3.

$$L_4 = (b(aa)^*a)^\omega\ ,\quad L_5 = ((aa)^*ab)^\omega\ ,\quad L_6 = a((aa)^*ab)^\omega\ ,\quad L_7 = \Sigma^* b\Sigma((aa)^*ab)^\omega\ .$$

The corresponding subautomaton is show below:

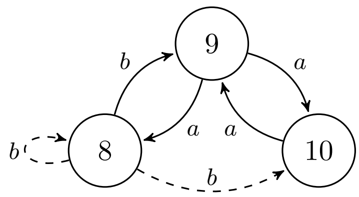


Let w be a word in $\Sigma^*(b(aa)^*a)^\omega$, and consider the last occurrence of a factor $ba^{2n}b$ with $n \geq 0$ in w. If there is such an occurrence, then w has a single final run from L_7 that uses one of the two b transitions to state 4 or 6 (corresponding the first b in this last occurrence of a $ba^{2n}b$ factor). Otherwise, if w does not contain a single occurrence of a factor $ba^{2n}b$ with $n \geq 0$, then it has to be recognized starting from state 4, 5, or 6. If w contains a b in an even position, then it has a unique final run from state 4 if it starts with a b or from state 6 if it starts with an a. The remaining case is for w to belong to $(a\Sigma)^\omega$, and then it has a unique final run from state 5: $L_5 = (a\Sigma)^\omega \cap \Sigma^*(b(aa)^*a)^\omega$.

Case 3. The last case considers the words recognized in the first two cases, i.e. the words in $\Sigma^*\left(b\big((aa)^*ab\big)^*(aa)^*\right)^\omega$. Note that no such word can be in $(a\Sigma)^\omega$, hence our only concern is to construct a co-deterministic co-complete automaton for this language. We split it in the following way:

$$L_8 = \left(b\big((aa)^*ab\big)^*(aa)^*\right)^\omega\ ,\qquad L_9 = (aa)^*aL_8\ ,\qquad L_{10} = (aa)^*aaL_8\ .$$

The three languages are indeed disjoint since the first b occurrence in L_8 is at the first position in L_8, after an odd number of a's in L_9, and after an non-zero even number of a's in L_{10}.



Consider now any word w in $\Sigma^* \left(b\big((aa)^*ab\big)^*(aa)^*\right)^\omega$. It can be split in a unique way as an infinite sequence of factors

$$w = w_0\, a^{2n_0} b\, w_1\, a^{2n_1} b\, w_2 \cdots$$

where each w_i is a finite word ending with a b, but none of the bw_i contains a factor $ba^{2n}b$ for any n. (The initial prefix w_0 can be seen as a word bw_0 since there is a transition from state 8 on b towards any other state.) Each bw_i can in turn be split as

$$bw_i = ba^{2m_{i,1}+1}ba^{2m_{i,2}+1}b \cdots ba^{2m_{i,n_i}+1}b$$

where each b except the last uses the non-accepting transition to state 9. Then, the last b uses either the transition to state 8 if $n_i = 0$, or the transition to state 9 if $n_i > 0$.

Merging Some States. Observe that we can "merge" states 1 with 4 and 6, 2 with 5, and 3 with 7: