TD 8: BDDs

Exercise 1 (Some BDDs)**.** Draw the reduced BDDs for the following functions, using the order of your choice on the variables $\{x_1, x_2, x_3\}$:

1. $(x_1 \Leftrightarrow x_2) \vee (x_1 \Leftrightarrow x_3)$,
2. the majority function $m(x_1, x_2, x_3)$: its value is 1 iff the majority of the input bits are 1's,
3. the constant sum function $s_c(x_1, x_2, x_3)$ for $c = 1$: its value is 1 iff $c = \sum_{i=1}^{3} x_i$,
4. the hidden weighted bit function $h(x_1, x_2, x_3)$: its value is that of variable x_s, where $s = \sum_{i=1}^{3} x_i$ and x_0 is defined as 0.

Exercise 2 (Symmetric Functions)**.** A *symmetric function* of n variables has the same value for all permutations of the same n tuple of arguments. Clearly, all variable orderings lead to the same reduced BDD size for symmetric functions.

Show that a reduced BDD for a symmetric function has at most $\binom{n+1}{2} + 1$ nodes (when using a single sink for $\top$).

Exercise 3 (Counting Solutions)**.** Write a linear time algorithm for counting the number of solutions of a boolean function f represented by a reduced BDD, i.e. of the number of valuations ν s.t. $\nu \models f$.

Exercise 4 (Shared BDDs)**.** When dealing with several boolean functions at once, with a fixed order on the variables, one can share the reduced BDDs for identical subfunctions. A *shared BDD* between m functions is a reduced BDD with m *root pointers* assigning a root node to each of the functions.

Let $x_1, \ldots, x_{2n}$ be the ordered set of variables. We want to compute the $n+1$ bits $f_{n+1} f_n \cdots f_1$ of the sum of two n bits numbers $x_1 x_3 \cdots x_{2n-1}$ and $x_2 x_4 \cdots x_{2n}$. Represent the shared BDD for the functions $f_3, \ldots, f_1$, i.e. for $n = 2$.

Exercise 5 (An Upper Bound on the Size of BDDs)**.** The size $B(f)$ of a reduced BDD for a function f is defined as the number of its nodes. Consider an arbitrary boolean function f on the ordered set $x_1 \cdots x_n$, and consider a variable x_k.

1. Show that we can bound the number of nodes labeled by $\{x_1, \ldots, x_k\}$ by $2^k - 1$.
2. How many different subfunctions on the ordered set of variables $x_{k+1} \cdots x_n$ exist? Deduce another bound for the number of nodes labeled by $\{x_{k+1}, \ldots, x_n\}$.

3. What global bound do you obtain for $k = n - \log_2(n - \log_2 n)$?

Exercise 6 (Finding the Optimal Order)**.** There are in general $n!$ different orders for the variables $\{x_1, \ldots, x_n\}$, and building the ORBDD for each of these is computationally expensive. One can nevertheless design an exponential time algorithm for finding the optimal order. Indeed, an optimal ordering on a subset X of variables does not depend on the order in which $X' = \{x_1, \ldots, x_n\} \backslash X$ has been accessed.

1. Fix a boolean function f from $\{x_1, \ldots, x_n\}$ to $\mathbb{B}$. We assume that f is provided as an ORBDD B for the ordering $x_1, x_2, \ldots, x_n$.

   Given a subset X of $\{x_1, \ldots, x_n\}$ and a variable x in X, how many nodes labeled by x does any BDD B' for f has if it first treats $X' = \{x_1, \ldots, x_n\} \backslash X$, then x, and last $X \backslash \{x\}$? How can you compute this number on the provided BDD B for f?

2. Reduce the optimal order problem to the search of a path of minimal weight in a weighted graph with subsets of $\{x_1, \ldots, x_n\}$ as vertices.

Exercise 7 (Quasi Reduced BDDs)**.** An ordered BDD for a boolean function f on $\{x_1, \ldots, x_n\}$ is *complete* if all paths from the root to a sink are of length n. A BDD is *quasi reduced* if it is complete and no two nodes define the same subfunction.

1. Give the quasi reduced BDD for the majority function of Exercise 1.2.

2. Show that a quasi reduced BDD is unique up to isomorphism for an ordered set of variables $x_1 \cdots x_n$.

3. Let $Q(f)$ be the size of the quasi reduced BDD for the boolean function f on the ordered set of variables $x_1 \cdots x_n$. Show that $Q(f) \leq (n+1)B(f)$.

Exercise 8 (Minimal DFAs)**.** A deterministic finite automaton $\mathcal{A}$ recognizes a boolean function f on the ordered set of variables $x_1 \cdots x_n$ if $L(\mathcal{A}) = \{\nu \in \{0,1\}^n \mid \nu \models f\}$, i.e. $\mathcal{A}$ recognizes exactly the solutions of f.

What are the relations between the reduced BDD, the quasi reduced BDD, and the minimal DFA recognizing the same boolean function f on the ordered set of variables $x_1 \cdots x_n$?