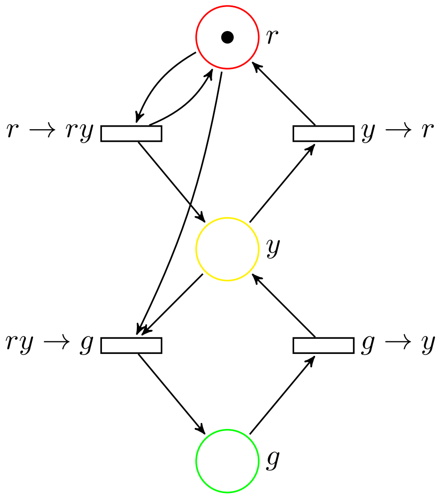
# TD 6: Petri Nets

## 1 Modeling Using Petri Nets

**Exercise 1** (Traffic Lights)**.** Consider again the traffic lights example from the lecture notes:




1. How can you correct this Petri net to avert unwanted behaviours (like $r \to ry \to rr$) in a 1-safe manner?
2. Extend your Petri net to model two traffic lights handling a street intersection.

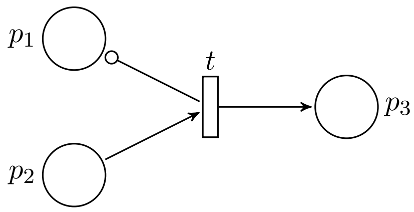
**Exercise 2** (Producer/Consumer)**.** A producer/consumer system gathers two types of processes:

**producers** who can make the actions *produce* ($p$) or *deliver* ($d$), and

**consumers** with the actions *receive* ($r$) and *consume* ($c$).

All the producers and consumers communicate through a single unordered channel.

1. Model a producer/consumer system with two producers and three consumers. How can you modify this system to enforce a maximal capacity of ten simultaneous items in the channel?
2. An *inhibitor arc* between a place $p$ and a transition $t$ makes $t$ firable only if the current marking at $p$ is zero. In the following example, there is such an inhibitor arc between $p_1$ and $t$. A marking $(0, 2, 1)$ allows to fire $t$ to reach $(0, 1, 2)$, but $(1, 1, 1)$ does not allow to fire $t$.




Using inhibitor arcs, enforce a priority for the first producer and the first consumer on the channel: the other processes can use the channel only if it is not currently used by the first producer and the first consumer.

## 2 Model Checking Petri Nets

**Exercise 3** (Upper Bounds)**.** Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider as usual propositional LTL, with a set of atomic propositions AP equal to $P$ the set of places of the Petri net. We define proposition $p$ to hold in a marking $m$ in $\mathbb{N}^P$ if $m(p) > 0$.

The models of our LTL formulæ are *computations* $m_0 m_1 \cdots$ in $(\mathbb{N}^P)^\omega$ such that, for all $i \in \mathbb{N}$, $m_i \to_{\mathcal{N}} m_{i+1}$ is a transition step of the Petri net $\mathcal{N}$.

1. We want to prove that state-based LTL model checking can be performed in polynomial space for 1-safe Petri nets. For this, prove that one can construct an exponential-sized Büchi automaton $\mathcal{B}_{\mathcal{N}}$ from a 1-safe Petri net that recognizes all the infinite computations of $\mathcal{N}$ starting in $m_0$.

2. In the general case, state-based LTL model checking is undecidable. Prove it for Petri nets with at least two unbounded places, by a reduction from the halting problem for 2-counter Minsky machines.

3. We consider now a different set of atomic propositions, such that $\Sigma = 2^{\mathrm{AP}}$, and a labeled Petri net, with a labeling homomorphism $\lambda : T \to \Sigma$. The models of our LTL formulæ are infinite words $a_0 a_1 \cdots$ in $\Sigma^\omega$ such that $m_0 \xrightarrow{t_0}_{\mathcal{N}} m_1 \xrightarrow{t_1}_{\mathcal{N}} m_2 \cdots$ is an execution of $\mathcal{N}$ and $\lambda(t_i) = a_i$ for all $i$.

   Prove that *action-based* LTL model checking can be performed in polynomial space for labeled 1-safe Petri nets.

## 3 Unfoldings

**Exercise 4** (Adequate Partial Orders)**.** A partial order $\prec$ between events is *adequate* if the three following conditions are verified:

(a) $\prec$ is well-founded,

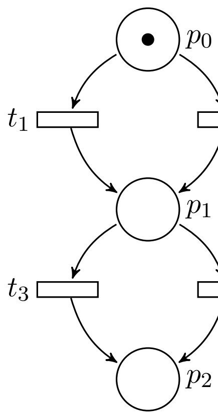
(b) $\lfloor t \rfloor \subsetneq \lfloor t' \rfloor$ implies $t \prec t'$, and

(c) $\prec$ is preserved by finite extensions: as in the lecture notes, if $t \prec t'$ and $B(t) = B(t')$, and $E$ and $E'$ are two isomorphic extensions of $\lfloor t \rfloor$ and $\lfloor t' \rfloor$ with $\lfloor u \rfloor = \lfloor t \rfloor \oplus E$ and $\lfloor u' \rfloor = \lfloor t' \rfloor \oplus E'$, then $u \prec u'$.

As you can guess, adequate partial orders result in complete unfoldings.

1. Show that $\prec_s$ defined by $t \prec_s t'$ iff $|\lfloor t \rfloor| < |\lfloor t' \rfloor|$ is adequate.

2. Construct the finite unfolding of the following Petri net using $\prec_s$; how does the size of this unfolding relate to the number of reachable markings?




3. Suppose we define an arbitrary total order $\ll$ on the transitions $T$ of the Petri net, i.e. they are $t_1 \ll \cdots \ll t_n$. Given a set $S$ of events and conditions of $\mathcal{Q}$, $\varphi(S)$ is the sequence $t_1^{i_1} \cdots t_n^{i_n}$ in $T^*$ where $i_j$ is the number of events labeled by $t_j$ in $S$. We also note $\ll$ for the lexicographic order on $T^*$.

   Show that $\prec_e$ defined by $t \prec_e t'$ iff $|\lfloor t \rfloor| < |\lfloor t' \rfloor|$ or $|\lfloor t \rfloor| = |\lfloor t' \rfloor|$ and $\varphi(\lfloor t \rfloor) \ll \varphi(\lfloor t' \rfloor)$ is adequate. Construct the finite unfolding for the previous Petri net using $\prec_e$.

4. There might still be examples where $\prec_e$ performs poorly. One solution would be to use a *total* adequate order; why? Give a 1-safe Petri net that shows that $\prec_e$ is not total.

## 4 Vector Addition Systems

**Exercise 5** (VASS)**.** An $n$-dimensional *vector addition system with states* (VASS) is a tuple $\mathcal{V} = \langle Q, \delta, q_0 \rangle$ where $Q$ is a finite set of states, $q_0 \in Q$ the initial state, and $\delta \subseteq Q \times \mathbb{Z}^n \times Q$ the transition relation. A configuration of $\mathcal{V}$ is a pair $(q, v)$ in $Q \times \mathbb{N}^n$. An execution of $\mathcal{V}$ is a sequence of configurations $(q_0, v_0)(q_1, v_1) \cdots (q_m, v_m)$ such that $v_0 = \bar{0}$, and for $0 < i \leq m$, $(q_{i-1}, v_i - v_{i-1}, q_i)$ is in $\delta$.

1. Show that any VASS can be simulated by a Petri net—we can give a formal meaning to 'simulation', but you haven't seen it in class yet, so do it at an intuitive level...

2. Show that, conversely, any Petri net can be simulated by a VASS.

**Exercise 6** (VAS)**.** An $n$-dimensional *vector addition system* (VAS) is a pair $(v_0, W)$ where $v_0 \in \mathbb{N}^n$ is the initial vector and $W \subseteq \mathbb{Z}^n$ is the set of transition vectors. An execution of $(v_0, W)$ is a sequence $v_0 v_1 \cdots v_m$ where $v_i \in \mathbb{N}$ for all $0 \leq i \leq m$ and $v_i - v_{i-1} \in W$ for all $0 < i \leq m$.

We want to show that any $n$-dimensional VASS $\mathcal{V}$ can be simulated by an $(n+3)$-dimensional VAS $(v_0, W)$.

Hint: Let $k = |Q|$, and define the two functions $a(i) = i + 1$ and $b(i) = (k+1)(k-i)$. Encode a configuration $(q_i, v)$ of $\mathcal{V}$ as the vector $(v(1), \ldots, v(n), a(i), b(i), 0)$. For every state $q_i$, $0 \leq i < k$, we add two transition vectors to $W$:

$$t_i = (0, \ldots, 0, -a(i), a(k-i) - b(i), b(k-i))$$
$$t'_i = (0, \ldots, 0, b(i), -a(k-i), a(i) - b(k-i))$$

For every transition $d = (q_i, w, q_j)$ of $\mathcal{V}$, we add one transition vector to $W$:

$$t_d = (w(1), \ldots, w(n), a(j) - b(i), b(j), -a(i))$$

1. Show that any execution of $\mathcal{V}$ can be simulated by $(v_0, W)$ for a suitable $v_0$.
2. Conversely, show that this VAS $(v_0, W)$ simulates $\mathcal{V}$ faithfully.