Home Assignment 2: Fairness and Petri Nets

To hand in before or on February 17, 2013.

	28	29	30	31			
February					1	2	3
	4	5	6	7	8	9	10
	11	12	13	14	15	16	17

Electronic versions (PDF only) can be sent by email to ⟨schmitz@lsv.ens-cachan.fr⟩, paper versions should be handed in on the 17th or put in my mailbox at LSV, ENS Cachan.

The following formula will be applied to late assignments: $g - 3d + 1$, where $0 \leq g \leq 20$ is the original grade and $d > 0$ is the number of days of delay, rounded up onward from the 13th at midnight.

This assignment is concerned with *fairness* in Petri nets. Fairness properties are employed to rule out behaviours where a process might wait indefinitely before being activated.

The numbers in the margins next to exercises are indications of time and difficulty.

1 The Fairness Fragment

We restrict ourselves to a fragment $\mathrm{TL}(\mathrm{AP}, \mathsf{GF})$ where the only temporal modality is GF (sometimes also written $\overset{\infty}{\mathsf{F}}$), i.e. with syntax

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{GF}\,\varphi$$

where p ranges over AP.

Let us consider the case of state-based LTL model-checking for Petri nets. In this framework, we are checking infinite sequences of markings of a Petri net $\mathcal{N} = \langle P, T, W, m_0 \rangle$, i.e. infinite sequences $m_0 m_1 \cdots m_i \cdots$ in $(\mathbb{N}^P)^\omega$ such that, for all i in $\mathbb{N}$, $m_i \to_{\mathcal{N}} m_{i+1}$ is a transition step of $\mathcal{N}$ according to some t in T, thus verifying for all p in P that $m_i(p) \geq W(p,t)$ and $m_{i+1}(p) = m_i(p) - W(p,t) + W(t,p)$. More generally, the *effect* $\Delta(u)$ of a transition sequence u in T^* is defined by $\Delta(\varepsilon) = 0^P$ and $\Delta(ut) = \Delta(u) - W(P,t) + W(t,P)$.

The atomic propositions in $\mathrm{AP} \subseteq P$ represent places, and are interpreted over such a sequence by $m_i \models p$ iff $m_i(p) > 0$. Put differently, a Petri net $\mathcal{N}$ gives rise to an infinite Kripke structure $\mathfrak{M}_{\mathcal{N}} = \langle \mathbb{N}^P, \to_{\mathcal{N}}, \{m_0\}, \mathrm{AP}, \ell \rangle$ where $\ell(m) = \{p \in \mathrm{AP} \mid m(p) > 0\}$.

Recall (from TD 6, Exercise 3) that the model-checking problem for state-based TL(AP, X, U) is undecidable in general and PSPACE-complete for safe nets. In the following we restrict ourselves to the model-checking problem for *ordinary* Petri nets, which verify $W(t, p) \leq 1$ and $W(p, t) \leq 1$ for all p in P and t in T (note that this does *not* imply that the net is safe).

[4] **Exercise 1** (Fairness in Petri Nets)**.** Let $\mathcal{N}$ be an ordinary Petri net, m a marking in $\{0, 1\}^P$, and t be a transition in T. Reduce the following problems to $\text{MC}^\exists(\mathsf{AP}, \mathsf{GF})$ model-checking instances on an ordinary Petri net $\mathcal{N}'$:

repeated coverability ($\text{RC}^\exists$): there exists an infinite execution where m is covered infinitely often, i.e. such that $m_i \geq m$ for an infinite number of indices i,

weak fairness ($\text{WF}^\forall$): every infinite execution either fires t infinitely often, or t is infinitely often not firable,

strong fairness ($\text{SF}^\forall$): in every infinite execution, if t is firable infinitely often, then it is actually fired infinitely often.

One can also consider the existential questions $\text{WF}^\exists$ or $\text{SF}^\exists$, which ask whether there *exist* some fair infinite execution. We will see in sections 2 and 3 that $\text{RC}^\exists$ and $\text{WF}^\exists$ are decidable.

It turns out that $\text{SF}^\exists$ is not decidable. Because the proof of this result is a bit involved, we rather look at a decidable case:

Exercise 2 (Strong Fairness in Safe Nets)**.** We consider the $\text{SF}^\exists$ problem when the Petri net $\mathcal{N}$ is known to be *safe*, i.e. verifying $m(p) \leq 1$ for all reachable m and all places p in P.

[3] Show that $\text{SF}^\exists$ is PSPACE-complete when $\mathcal{N}$ is safe. *Hint: For hardness, reduce from reachability in safe nets.*

2 Repeated Coverability

We show in this section that $\text{RC}^\exists$ is decidable, relying for this on the properties of the coverability graph seen during TD 7.

Exercise 3 (Decidability)**.** We prove in this exercise that repeated coverability is decidable. Let $\mathcal{N}$ be Petri net, G be its coverability graph and m some marking in $\mathbb{N}^P$.

[4] Show that there exists an infinite computation s.t. $m \leq m_i$ for infinitely many indices i iff there exists an accessible loop $m' \xrightarrow{v}_G m'$ in G s.t. $m \leq m'$ and $\Delta(v) \geq 0^P$. Conclude. *Hint: Use Exercise 2 from TD 7.*

Exercise 4 (Action-Based LTL)**.** Recall from Exercise 3 of TD 6 that *action-based* LTL considers *labeled* Petri nets $\langle \mathcal{N}, \lambda \rangle$ where λ is a labelling from T to 2^{AP}. The model-checking problem then considers the infinite sequences $\lambda(t_1)\lambda(t_2)\cdots$ of transition labels along an execution $m_0 \xrightarrow{t_1}_{\mathcal{N}} m_1 \xrightarrow{t_2}_{\mathcal{N}} m_2 \cdots$; alternatively, we can consider the Kripke structure associated with $\mathcal{N}$ to be $\mathfrak{M}_{\mathcal{N}}^{\lambda} = \langle \mathbb{N}^P \times T, T', I, \ell \rangle$ where $T' = \{((m,t),(m',t')) \mid m \xrightarrow{t}_{\mathcal{N}} m' \wedge m' \geq W(P,t')\}$, $I = \{m_0\} \times \{t \in T \mid m_0 \geq W(P,t)\}$, and $\ell(m,t) = \lambda(t)$.

[3] Show that action-based LTL model-checking is decidable for labeled Petri nets.

3 Weak Fairness

We examine in this section the relationship between the existential weak fairness problem $\mathrm{WF}^{\exists}$ and the *reachability problem* (RP) in Petri nets: given a Petri net $\mathcal{N}$ and a marking m in $\mathbb{N}^P$, does there exist a finite run $m_0 \to_{\mathcal{N}}^* m$? This problem is known to be decidable [3, 1, 2], although its exact complexity is still open.

Exercise 5 (Lower Bounds)**.** We want to exhibit a reduction from RP to $\mathrm{WF}^{\exists}$.

[2] 1. Show that RP can be reduced to the following problem: given a Petri net $\mathcal{N}$ and a place p, does there exist a reachable marking m (i.e. verifying $m_0 \to_{\mathcal{N}}^* m$) such that $m(p) = 0$?

[3] 2. Show that RP can be reduced to $\mathrm{WF}^{\exists}$.

Exercise 6 (Decidability)**.** We want to show that $\mathrm{WF}^{\exists}$ is decidable. Let us first concentrate on the subcase where there exists an infinite execution where t is infinitely often *not* firable.

[1] 1. Reduce this case to the question whether there exists a place p in P and an execution $m_0 \to_{\mathcal{N}}^* m \to_{\mathcal{N}}^+ m'$ with $m \leq m'$ and $m(p) = m'(p) = 0$.

[2] 2. Reduce the previous question to an instance of the reachability problem.

[2] 3. Deduce that $\mathrm{WF}^{\exists}$ is decidable.

References

[1] S. R. Kosaraju. Decidability of reachability in vector addition systems. In *STOC'82*, pages 267–281. ACM Press, 1982. doi: 10.1145/800070.802201.

[2] J. Leroux. Vector addition system reachability problem: a short self-contained proof. In *POPL 2011*, pages 307–316. ACM Press, 2011. doi: 10.1145/1926385.1926421.

[3] E. W. Mayr. An algorithm for the general Petri net reachability problem. In *STOC'81*, pages 238–246. ACM Press, 1981. doi: 10.1145/800076.802477.