TD 2: LTL

1 Specification

Exercise 1. We would like to verify the properties of a boolean circuit with input x, output y, and two registers r_1 and r_2. We define accordingly $\mathrm{AP} = \{x, y, r_1, r_2\}$ as our set of atomic propositions and consider the time flow $(\mathbb{N}, <)$ where the runs of the circuit can be seen as temporal structures.

Translate the following properties (a) in TL(AP) and (b) in FO(<):

1. "it is impossible to get two consecutive 1 as output"
2. "each time the input is 1, at most two ticks later the output will be 1"
3. "each time the input is 1, the register contents remains the same over the next tick"
4. "register r_1 is infinitely often 1"

Note that there might be several, non-equivalent formal specifications matching these informal descriptions—that's the whole point of writing specifications!—but your (a) and (b) should be equivalent.

2 LTL

Exercise 2. We fix a set AP of atomic propositions including $\{p, q, r\}$ and some discrete linear time flow $(\mathbb{T}, <)$.

1. Consider the formulæ $\varphi_1 = \mathsf{G}(p \to \mathsf{X}q)$ and $\varphi_2 = \mathsf{G}(p \to ((\neg q) \mathrel{\mathsf{R}} q))$.
   (a) Does φ_2 imply φ_1?
   (b) Does φ_1 imply φ_2?
2. Simplify the following formula:

$$\mathsf{F}(((\mathsf{G}r) \mathrel{\mathsf{U}'} p) \wedge (\neg q \mathrel{\mathsf{U}'} p)) \vee \mathsf{F}(\neg p \vee \mathsf{F}'q)\ .$$

Exercise 3 (Expressiveness)**.** We fix the set $\mathrm{AP} = \{p\}$ of atomic propositions, with an associated alphabet $\Sigma = \{\{p\}, \emptyset\}$, and consider the $(\mathbb{N}, <)$ flow of time, where temporal structures can be seen as infinite words over Σ, i.e. words in Σ^ω.

1. Show that the following subsets of Σ^ω are expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{U}', \mathsf{X})$:

   (a) $\{p\}^* \cdot \emptyset^\omega$, and
   (b) $\{p\}^n \cdot \emptyset^\omega$ for each fixed $n \geq 0$.

2. Is the language $(\{p\} \cdot \emptyset)^\omega$ expressible in $\mathrm{TL}(\mathrm{AP}, \mathsf{U}', \mathsf{X})$?

3. Consider the infinite sequence $\sigma_i = \{p\}^i \cdot \emptyset \cdot \{p\}^\omega$ for $i \geq 0$. Show by induction on $\mathrm{TL}(\mathrm{AP}, \mathsf{U}', \mathsf{X})$ formulæ φ that, for all $n \geq 0$, if φ has less than n X modalities, then for all $i, i' > n$, $\sigma_i \models \varphi$ iff $\sigma_{i'} \models \varphi$. *(Hint: For the case of U', show that $\sigma_i \models \varphi$ iff $\sigma_{n+1} \models \varphi$.)*

4. Using the previous question, show that the set $(\{p\} \cdot \Sigma)^\omega$ is not expressible in $\mathrm{TL}(\mathrm{AP})$ over $(\mathbb{N}, <)$.

3 LTL with Past

Exercise 4 (Specifying with Past)**.** Provide TL formulæ over $\mathrm{AP} = \{\mathrm{ok}, \mathrm{crash}, \mathrm{alarm}, \mathrm{reset}\}$ with and without past modalities for the following properties:

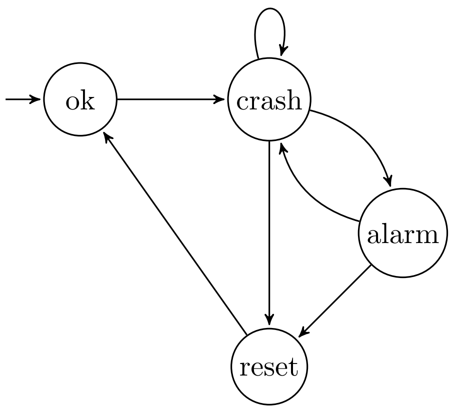
1. "Whenever the alarm rings, there has been a crash immediately before."

2. "Whenever the alarm rings, there has been a crash some time before, and no reset in the meantime."

Exercise 5 (History Variables)**.** Consider the time flow $(\mathbb{N}, <)$. One way of getting rid of *pure* past modalities is to tweak both the model and the formula, by adding *history variables* to the model and by replacing pure past subformulæ by atomic propositions on these variables, i.e. from a pair $\langle M, \varphi \rangle$ where M is a Kripke model and φ a LTL formula with past modalities, construct $\langle M', \varphi' \rangle$ where M' is a modified version of M with extra atomic propositions, and φ' is a pure future LTL formula, such that $M \models \varphi$ iff $M' \models \varphi'$.

For instance, a subformula $\mathsf{Y}\psi$ will be replaced by a boolean variable $h_{\mathsf{Y}\psi}$ in the specification, and the model will update this variable according to whether or not ψ holds in the previous state. Two new atomic propositions are introduced, corresponding to $h_{\mathsf{Y}\psi} = \mathrm{true}$ and $h_{\mathsf{Y}\psi} = \mathrm{false}$.

1. Apply this technique to the specification of the previous exercise and the following alarm system:



2. What is the cost of the model transformation?

Exercise 6 (Succinctness of Past Formulæ)**.** Consider the time flow $(\mathbb{N}, <)$. Let $\mathrm{AP}_{n+1} = \{p_0, \ldots, p_n\} = \mathrm{AP}_n \cup \{p_n\}$ be a set of atomic propositions, defining the alphabet $\Sigma_{n+1} = 2^{\mathrm{AP}_{n+1}}$. We want to show the existence of an $O(n)$-sized LTL formula with past such that any equivalent pure future LTL formula is of size $\Omega(2^n)$.

First consider the following LTL formula of exponential size:

$$\bigwedge_{S \subseteq \mathrm{AP}_n} \Bigg((\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \wedge p_n) \Rightarrow \mathsf{G}((\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j) \Rightarrow p_n)$$
$$\wedge (\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \wedge \neg p_n) \Rightarrow \mathsf{G}((\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j) \Rightarrow \neg p_n) \Bigg) \qquad (\varphi_n)$$

1. Describe which words of Σ_{n+1}^{ω} are the models of φ_n.
2. Can an LTL formula with past modalities check whether it is at the initial position of a word?
3. Provide an LTL formula with past ψ_n of size $O(n)$ initially equivalent to φ_n.
4. Consider the language $L_n = \{\sigma \in \Sigma_{n+1}^{\omega} \mid \sigma \models \mathsf{G}'\varphi_n\}$. We want to prove that any generalized Büchi automaton that recognizes L_n requires at least 2^{2^n} states.

   For this we fix a permutation $a_0 \cdots a_{2^n-1}$ of the symbols in Σ_n and we consider all the different subsets $K \subseteq \{0, \ldots, 2^n - 1\}$. For each K we consider the word

   $$w_K = b_0 \cdots b_{2^n-1}$$

   in $\Sigma_{n+1}^{2^n}$, defined for each i in $\{0, \ldots, 2^n - 1\}$ by

   $$b_i = a_i \qquad \text{if } i \in K$$
   $$b_i = a_i \cup \{p_n\} \qquad \text{otherwise.}$$

Thus K is the set of positions of w_K where p_n does not hold.

Using the w_K for different values of K, prove that any generalized Büchi automaton for $\mathsf{G}'\varphi_n$ requires at least 2^{2^n} states.

5. Conclude using the fact that any pure future LTL formula φ can be given a generalized Büchi automaton with at most $2^{|\varphi|}$ states.