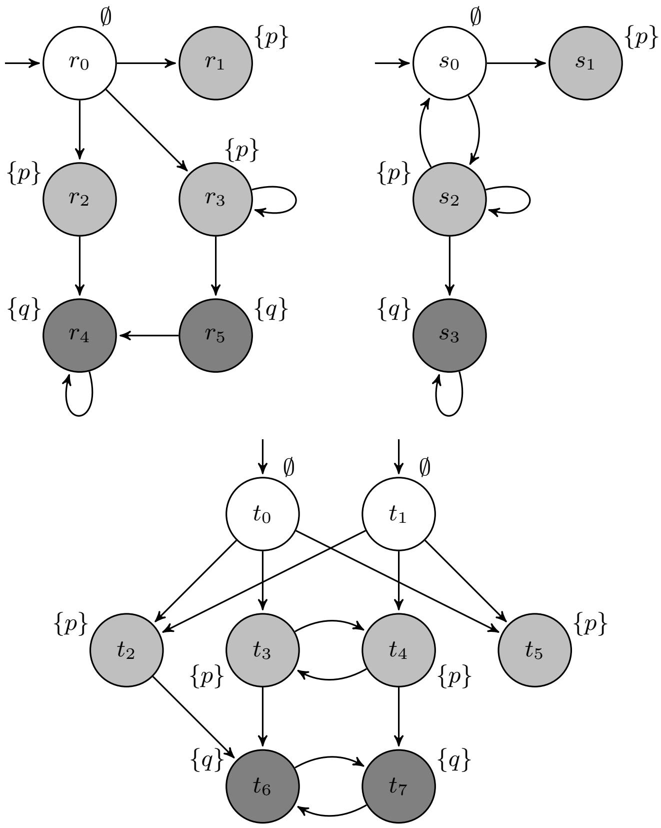
TD 7: Simulation & Bisimulation

Exercise 1 (Bisimulations)**.** Consider the following Kripke structures:



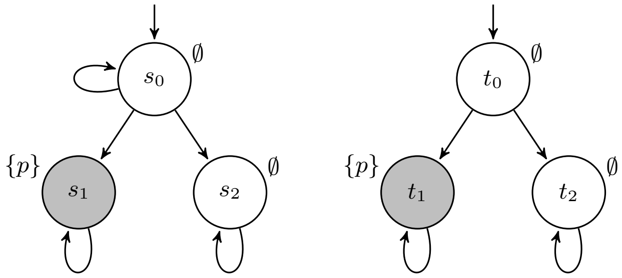
For each couple of structures, exhibit a bisimulation relation if they are bisimilar, or a CTL* formula allowing to distinguish between them if they are not bisimilar.

Exercise 2 (Computing the Coarsest Bisimulation)**.** Computing $\equiv$ on a single Kripke structure is very similar to the computation of a minimal DFA.

1. Design a partition refinement algorithm for computing $\equiv$, i.e. an algorithm that computes an initial relation $\equiv_0$ and refines it successively until $\equiv_k = \equiv$ for some k. Prove that your algorithm terminates and computes $\equiv$.

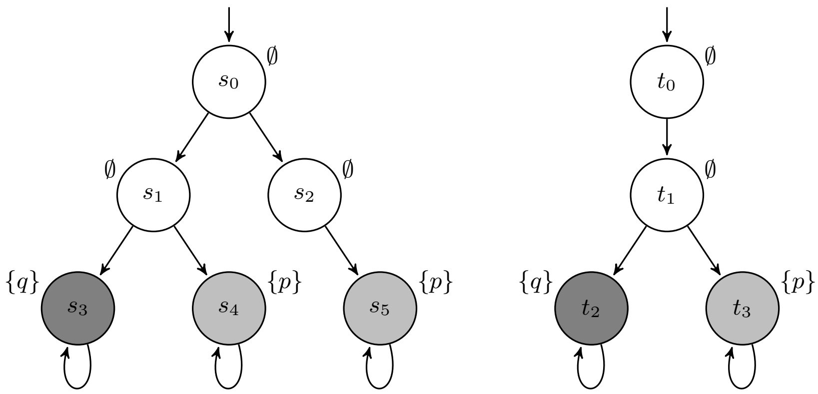
2. Apply your algorithm to the union of two bisimilar systems from the previous exercise and draw the quotiented system.

Exercise 3 (Simulations)**.** Consider the following two systems:



1. Exhibit a simulation to prove $t_0 \preceq s_0$.
2. Show that $s_0 \not\preceq t_0$.
3. Let $M = \langle S, T, I, \mathrm{AP}, l\rangle$ be a single Kripke structure. Show that $\preceq$ is reflexive and transitive on S. Is it symmetric?
4. Propose an algorithm for computing $\preceq$ on a single structure M.

Exercise 4 (Simulation Quotienting)**.** Two Kripke structures M_1 and M_2 are *simulation equivalent*, noted $M_1 \simeq M_2$ if $M_1 \preceq M_2$ and $M_2 \preceq M_1$. The lecture notes provide an example of two simulation equivalent but not bisimilar structures. Consider now the two following structures M_s and M_t:



1. Which of the following relations hold: $M_s \preceq M_t$, $M_t \preceq M_s$, $M_s \simeq M_t$?
2. Construct the quotient of $(M_s \cup M_t)$ by $\simeq$. Is the resulting system bisimilar to $(M_s \cup M_t)$?
3. Prove that if $M/\!\simeq$ is the quotient of M by $\simeq$, then $M/\!\simeq\, \preceq M$ and $M \preceq M/\!\simeq$.
4. Call a Kripke structure $M = \langle S, T, I, \mathrm{AP}, l\rangle$ AP-*deterministic* if

(a) for all the subsets a of AP, $I \cap \{s \in S \mid l(s) = a\}| \leq 1$, i.e. there is at most one initial state labeled with each valuation in 2^{AP}, and

(b) for each state s, if there exist two transitions (s, s_1) and (s, s_2) in T with $l(s_1) = l(s_2)$, then $s_1 = s_2$.

Show that, if two Kripke structures M_1 and M_2 are AP-deterministic, then they are bisimilar iff they are simulation equivalent.

Exercise 5 (Logical Characterization)**.** Let us define *existential CTL** as the fragment of CTL* defined by the following abstract syntax, where p ranges over the set of atomic propositions AP:

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{E}\psi \qquad \text{(state formulæ)}$$
$$\psi ::= \varphi \mid \mathsf{X}\psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \psi \,\mathsf{U}\, \psi \mid \psi \,\mathsf{R}\, \psi \;. \qquad \text{(path formulæ)}$$

Existential CTL* includes both LTL and *existential CTL* (hereafter noted ECTL), which is defined by the following abstract syntax:

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathsf{EX}\varphi \mid \mathsf{E}(\varphi \,\mathsf{U}\, \varphi) \mid \mathsf{E}(\varphi \,\mathsf{R}\, \varphi) \;. \qquad \text{(state formulæ)}$$

Let us consider two (not necessarily different) Kripke structures $M_1 = \langle S_1, T_1, I_1, \text{AP}, l_1 \rangle$ and $M_2 = \langle S_2, T_2, I_2, \text{AP}, l_2 \rangle$. We assume these structures to be *total*, where for any state s there exists some state s' such that (s, s') is a transition.

1. Prove the following two statements, for any two states s_1 and s_2, and any two infinite paths π_1 and π_2 in M_1 and M_2, resp.:

   (a) if $s_1 \preceq s_2$, then for any existential CTL* state formula φ, $s_1 \models \varphi$ implies $s_2 \models \varphi$,

   (b) if $\pi_1 = s_{0,1}s_{1,1}\cdots$ and $\pi_2 = s_{0,2}s_{1,2}\cdots$ with $s_{i,1} \preceq s_{i,2}$ for all i in $\mathbb{N}$, then for any existential CTL* path formula ψ, $\pi_1 \models \psi$ implies $\pi_2 \models \psi$.

2. Let us consider the following relation on $S_1 \times S_2$:

   $$\mathcal{F} = \{(s_1, s_2) \in S_1 \times S_2 \mid \forall \varphi \in \text{ECTL}, s_1 \models \varphi \Rightarrow s_2 \models \varphi\} \;.$$

   Assuming that for all initial states s in I_1, $\mathcal{F}(s) \cap I_2$ is not empty, show that $\mathcal{F}$ is a simulation between M_1 and M_2.

3. Conclude by proving the following theorem:

Theorem 1 (Logical Characterization of Simulation)**.** *Let $M_1 = \langle S_1, T_1, I_1, \text{AP}, l_1 \rangle$ and $M_2 = \langle S_2, T_2, I_2, \text{AP}, l_2 \rangle$ be two total Kripke structures and s_1 and s_2 be two states of S_1 and S_2 resp. The following three statements are equivalent:*

1. $s_1 \preceq s_2$,

2. for all existential CTL formulæ φ: $s_1 \models \varphi$ implies $s_2 \models \varphi$,*

3. for all existential CTL formulæ φ: $s_1 \models \varphi$ implies $s_2 \models \varphi$.