Home Assignment 1a: Multi-focus Games for LTL

To hand in before or on November 2, 2010.
The penalty for delays is 2 points per day.

October					1	2	3
	4	5	6	7	8	9	10
	11	12	13	14	15	16	17
	18	(19)	20	21	22	23	24
	25	26	27	28	29	30	31
November	1	(2)	3	4	5	6	7
	8	9	10	11	12	13	14

Electronic versions (PDF only) can be sent by email to ⟨schmitz@lsv.ens-cachan.fr⟩, paper versions should be handed in on the 2nd or put in my mailbox at LSV, ENS Cachan.

1 Multi-focus Games

1.1 LTL Formulæ

We are interested in a game-theoretic approach to proving satisfiability of LTL formulæ. We consider for this LTL formulæ in *negative normal form*, i.e. using the abstract syntax

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathsf{X}\varphi \mid \varphi \mathbin{\mathsf{U}} \varphi \mid \varphi \mathbin{\mathsf{R}} \varphi\ ,$$

where p ranges over a non-empty finite set of atomic propositions AP. Let $\top \equiv p \vee \neg p$ and $\bot \equiv p \wedge \neg p$ for some atomic proposition p. The F and G modalities are defined as usual by $\mathsf{F}\varphi \equiv \top \mathbin{\mathsf{U}} \varphi$ and $\mathsf{G}\varphi \equiv \bot \mathbin{\mathsf{R}} \varphi$.

Given a LTL formula φ, the *closure* $\mathrm{cl}(\varphi)$ of φ is the smallest set of LTL formulæ such that

- $\varphi \in \mathrm{cl}(\varphi)$,
- if $\neg\psi \in \mathrm{cl}(\varphi)$ or $\mathsf{X}\psi \in \mathrm{cl}(\varphi)$, then $\psi \in \mathrm{cl}(\varphi)$,
- if $\psi_1 \vee \psi_2 \in \mathrm{cl}(\varphi)$, $\psi_1 \wedge \psi_2 \in \mathrm{cl}(\varphi)$, $\psi_1 \mathbin{\mathsf{U}} \psi_2 \in \mathrm{cl}(\varphi)$, or $\psi_1 \mathbin{\mathsf{R}} \psi_2 \in \mathrm{cl}(\varphi)$, then $\psi_1 \in \mathrm{cl}(\varphi)$ and $\psi_2 \in \mathrm{cl}(\varphi)$,
- if $\psi_1 \mathbin{\mathsf{U}} \psi_2 \in \mathrm{cl}(\varphi)$, then $\mathsf{X}(\psi_1 \mathbin{\mathsf{U}} \psi_2) \in \mathrm{cl}(\varphi)$, and
- if $\psi_1 \mathbin{\mathsf{R}} \psi_2 \in \mathrm{cl}(\varphi)$, then $\mathsf{X}(\psi_1 \mathbin{\mathsf{R}} \psi_2) \in \mathrm{cl}(\varphi)$.

1.2 Definition of Multi-focus Games

The *multi-focus game* for φ is a finite graph (the *arena*, where vertices are called *positions*) $G(\varphi) = \langle V, \longrightarrow\rangle$ along with both a *winning condition* $W \subseteq V^*$ and a *losing condition* $L \subseteq V^*$ on sequences of positions in this arena.

A *play* in the game is a sequence $\rho = P_0 P_1 \cdots P_n$ in $W \cup L$ such that P_0 is the *initial position*, and for each $0 \leq i < n$, $P_i \longrightarrow P_{i+1}$ and the prefix $P_0 \cdots P_i$ is not in $W \cup L$. Thus a play stops as soon as it is in W or L. The play is *winning* for the unique player if $\rho \in W$. The game is *winning* if there exists a winning play. A position P in V is *useful* if there exists $P_1, \ldots, P_i, P_i + 1, \ldots, P_n$ in V such that $P_0 P_1 \cdots P_i P P_{i+1} \cdots P_n$ is a play.

Positions In our case $V = \{r, c\} \times \mathrm{cl}(\varphi) \times \mathrm{cl}(\varphi)$ is the set of positions $P = m, [\Gamma], \Delta$ where

- m is a *mode* in $\{r, c\}$, standing respectively for *reset* and *check*,
- $\Gamma \subseteq \mathrm{cl}(\varphi)$ is a set of *focused* formulæ, and
- $\Delta \subseteq \mathrm{cl}(\varphi)$ is a set of formulæ.

A position $P = m, [\Gamma], \Delta$ is *reduced* if all the formulæ in both Γ and Δ are either literals p or $\neg p$ or under the scope of a next modality, i.e. of form $\mathsf{X}\psi$. Given a position $P = m, [\Gamma], \Delta$,

$$\bigwedge P = \bigwedge_{\psi \in \Gamma} \psi \wedge \bigwedge_{\psi \in \Delta} \psi$$

denotes the conjunction of all the formulæ in P, and

$$\Sigma_P = \bigwedge_{p \in \mathrm{AP} \cap \Delta} p \wedge \bigwedge_{\neg p \in \Delta, p \in \mathrm{AP}} \neg p$$

denotes its atomic satisfaction formula. We write "Γ, ψ" for the set $\Gamma \uplus \{\psi\} \subseteq \mathrm{cl}(\varphi)$, as in e.g. "$P = m, [\Gamma, \mathsf{X}(\psi_1 \mathbin{\mathsf{U}} \psi_2)], \Delta, \psi_1$" denoting that a formula $\mathsf{X}(\psi_1 \mathbin{\mathsf{U}} \psi_2)$ is in focus in P and a formula ψ_1 is in its non-focused set.

Initial Position The initial position of the game is $P_0 = r, [\emptyset], \varphi$.

Transitions The set of transitions is defined by

$$m, [\Gamma], \Delta, \psi_1 \vee \psi_2 \longrightarrow m, [\Gamma], \Delta, \psi_1 \qquad (\vee_1)$$
$$m, [\Gamma], \Delta, \psi_1 \vee \psi_2 \longrightarrow m, [\Gamma], \Delta, \psi_2 \qquad (\vee_2)$$
$$m, [\Gamma], \Delta, \psi_1 \wedge \psi_2 \longrightarrow m, [\Gamma], \Delta, \psi_1, \psi_2 \qquad (\wedge)$$
$$m, [\Gamma], \Delta, \psi_1 \mathbin{\mathsf{U}} \psi_2 \longrightarrow m, [\Gamma], \Delta, \psi_2 \qquad (\mathsf{U}_2)$$
$$r, [\Gamma], \Delta, \psi_1 \mathbin{\mathsf{U}} \psi_2 \longrightarrow r, [\Gamma, \mathsf{X}(\psi_1 \mathbin{\mathsf{U}} \psi_2)], \Delta, \psi_1 \qquad (\mathsf{U}_1^r)$$
$$c, [\Gamma], \Delta, \psi_1 \mathbin{\mathsf{U}} \psi_2 \longrightarrow c, [\Gamma], \Delta, \psi_1, \mathsf{X}(\psi_1 \mathbin{\mathsf{U}} \psi_2) \qquad (\mathsf{U}_1^c)$$
$$c, [\Gamma, \psi_1 \mathbin{\mathsf{U}} \psi_2], \Delta \longrightarrow c, [\Gamma], \Delta, \psi_2 \qquad (\mathsf{U}_2^f)$$
$$c, [\Gamma, \psi_1 \mathbin{\mathsf{U}} \psi_2], \Delta \longrightarrow c, [\Gamma, \mathsf{X}(\psi_1 \mathbin{\mathsf{U}} \psi_2)], \Delta, \psi_1 \qquad (\mathsf{U}_1^f)$$

$$m, [\Gamma], \Delta, \psi_1 \mathsf{R}\, \psi_2 \longrightarrow m, [\Gamma], \Delta, \psi_1, \psi_2 \qquad (\mathsf{R}_1)$$

$$m, [\Gamma], \Delta, \psi_1 \mathsf{R}\, \psi_2 \longrightarrow m, [\Gamma], \Delta, \psi_2, \mathsf{X}(\psi_1 \mathsf{R}\, \psi_2) \qquad (\mathsf{R}_2)$$

$$m, [\emptyset], \mathsf{X}\psi_1, \ldots, \mathsf{X}\psi_r, l_1, \ldots, l_s \longrightarrow r, [\emptyset], \psi_1, \ldots, \psi_r \qquad (\mathsf{X})$$

$$m, [\mathsf{X}\psi_1, \ldots, \mathsf{X}\psi_r], \mathsf{X}\psi_{r+1}, \ldots, \mathsf{X}\psi_t, l_1, \ldots, l_s \longrightarrow c, [\psi_1, \ldots, \psi_r], \psi_{r+1}, \ldots, \psi_t \qquad (\mathsf{X}^f)$$

for all $r \geq 1$, $s, t \geq 0$, $m \in \{r, c\}$, $\Gamma, \Delta \subseteq \mathrm{cl}(\varphi)$, $\psi_1, \psi_2, \ldots \in \mathrm{cl}(\varphi)$, and $l_1, l_2, \ldots$ literals of form p or $\neg p$ for some $p \in \mathrm{AP} \cap \mathrm{cl}(\varphi)$.

Observe that an until subformula $\psi_1 \mathsf{U}\, \psi_2$ can enter the focus by rule (U_1^r) if it is not immediately fulfilled (i.e. ψ_2 does not hold), and conversely leave the focus by rule (U_2^f) when fulfilled. Only formulæ of form $\psi_1 \mathsf{U}\, \psi_2$ or of form $\mathsf{X}(\psi_1 \mathsf{U}\, \psi_2)$ can ever be in focus.

Also note that transitions of type (X) and (X^f) are possible if and only if the position is reduced. It is generally useful to distinguish between applications of rules $(\vee_1)$ to (R_2) and applications of rules (X) and (X^f); we write $P \longrightarrow_\varepsilon P'$ in the former case and $P \longrightarrow_{\mathsf{X}} P'$ in the latter case.

Winning and Losing Conditions Consider a play $\rho = P_0 P_1 \cdots P_n$ with $P_i = m_i, [\Gamma_i], \Delta_i$ for all $0 \leq i \leq n$.

The play is *winning* if either

$$\Gamma_n = \emptyset \wedge \Delta_n = l_1, \ldots, l_s \wedge \Sigma_{P_n} \text{ is satisfiable, or} \qquad (W_1)$$

$$P_n \text{ is reduced} \wedge \exists 0 \leq i < n. (\Gamma_i = \Gamma_n \wedge \Delta_i = \Delta_n \wedge \exists i \leq j \leq n. (\Gamma_j = \emptyset)) \qquad (W_2)$$

for some $l_1, l_2, \ldots$ literals of form p or $\neg p$ with $p \in \mathrm{AP} \cap \mathrm{cl}(\varphi)$. Note that condition (W_1) also implies that P_n is reduced.

The play is *losing* if either

$$\Sigma_{P_n} \text{ is unsatisfiable, or} \qquad (L_1)$$

$$P_n \text{ is reduced} \wedge \Gamma_n \neq \emptyset \wedge \exists 0 \leq i < n. (\Gamma_i = \Gamma_n \wedge \Delta_i = \Delta_n \wedge \forall i \leq j \leq n. (\Gamma_j \neq \emptyset))\ . \qquad (L_2)$$

2 Exercises

Indications of difficulty are given in the margin. The questions are not always detailed, and some intermediate lemmata can be required in order to provide a clean proof—you are expected to come up with these lemmata on your own.

Exercise 1 (Game Example)**.** Let $\varphi = p \mathsf{R}\, (\neg q \mathsf{U}\, q)$. We are interested in plays $P_0 \xrightarrow{(a)} P_1 \xrightarrow{(b)} \cdots P_n$ in the arena $G(\varphi)$ with appropriate transition labels $a, b, \ldots$ in $\{\vee_1, \vee_2, \wedge, \mathsf{U}_2, \ldots, \mathsf{X}^f\}$:

[1] 1. exhibit a play that loses according to (L_1),

[1] 2. exhibit a play that loses according to (L_2),

[1] 3. exhibit a play that wins according to (W_1) and give a model for φ corresponding to this particular winning play, and

[1] 4. exhibit a play that wins according to (W_2) and give a model for φ corresponding to this particular winning play.

Exercise 2 (Determinacy)**.**

[1] 1. Show that a play cannot be both winning and losing: $W \cap L = \emptyset$.

[2] 2. Show that if a game is *not* winning, then there exists a losing play: $W = \emptyset$ implies $L \neq \emptyset$.

[2] **Exercise 3** (Direct Transitions)**.** Which transitions rules are needed in the game arena in order to treat the case of modalities F and G directly?

Exercise 4 (A Winning Strategy)**.** The unique player of the multi-focus game has an optimal strategy that will always build a winning game from a satisfiable formula φ. The point of the strategy is to avoid both losing conditions (L_1) and (L_2).

The strategy maintains a *context formula* γ along with the current position $P = m, [\psi_1 \mathsf{U}\, \psi'_1, \ldots, \psi_r \mathsf{U}\, \psi'_r, \mathsf{X}(\psi_{r+1} \mathsf{U}\, \psi'_{r+1}), \ldots, \mathsf{X}(\psi_t \mathsf{U}\, \psi'_t)], \Delta$, and makes sure that

$$f(\gamma, P) = \gamma \mathsf{U}\, (\gamma \wedge \bigvee_{i=1}^{t} \psi'_i) \wedge \bigwedge P$$

remains satisfiable throughout the play—initially this context formula is $\top$, thus $f(\top, r, [\emptyset], \varphi)$ is initially satisfiable.

The idea is to reset the context to $\top$ whenever rule (U_2^f) is used. Conversely, the context is augmented to $\gamma \wedge \neg \bigwedge P = \gamma \wedge (\bigvee_{i=1}^{t} \mathsf{X}\neg\psi_i \vee \bigvee_{j=1}^{s} \neg l_j)$ upon firing rule (X^f). In all the other cases, the context remains the same.

[1] 1. There are a number of choices of moves in the game, i.e. $(\vee_1)$ vs. $(\vee_2)$, (U_2) vs. (U_1^r) and (U_1^c), (U_2^f) vs. (U_1^f), and (R_1) vs. (R_2). The strategy consists in all these cases to choose the first alternative if it yields a satisfiable $f(\gamma, P)$, and the second alternative otherwise. Show that this strategy also preserves the satisfiability of $f(\gamma, P)$ when it is forced to take the second alternative or one of $(\wedge)$ or (X).

2. We want to show that the strategy also preserves the satisfiability of $f(\gamma, P)$ upon firing (X^f).

[2] (a) Show that, if $\gamma \wedge (\varphi \mathsf{U}\, \psi)$ is satisfiable, then $\gamma \wedge (\psi \vee (\varphi \wedge \mathsf{X}((\varphi \wedge \neg\gamma) \mathsf{U}\, (\psi \wedge \neg\gamma))))$ is also satisfiable, for any LTL formulæ γ, φ, and ψ.

[1] (b) Show that in a reduced position $P = m, [\mathsf{X}(\psi_1 \mathbin{\mathsf{U}} \psi'_1), \ldots, \mathsf{X}(\psi_t \mathbin{\mathsf{U}} \psi'_t)], \Delta$ reached through the strategy, $\gamma \wedge \bigvee_{i=1}^{t} \psi'_i \wedge \bigwedge P$ is not satisfiable.

[1] (c) Conclude.

[0] 3. Can losing condition (L_1) be fulfilled with this strategy?

[3] 4. Show that losing condition (L_2) cannot be fulfilled either.

Exercise 5 (Extracting Models from Winning Plays)**.** We want to show that the existence of a winning play $\rho = P_0 \cdots P_n$ in $G(\varphi)$ implies that φ is satisfiable. Define $P_i = m_i, [\Gamma_i], \Delta_i$.

To this end, we consider the indices $i_1, \ldots, i_k$ such that P_{i_j} is reduced in the play. If ρ is winning by condition (W_1), then define the infinite word $w = a_1 \cdots a_k \emptyset^\omega$ in $(2^{\mathrm{AP}})^\omega$ with $a_j \models \Sigma_{P_{i_j}}$ for all $1 \leq j \leq k$. If ρ is winning by condition (W_2), then consider the index $i = i_\ell$ of the condition and define instead the infinite word $w = a_1 \cdots a_{\ell-1}(a_\ell \cdots a_k)^\omega$ in $(2^{\mathrm{AP}})^\omega$ with $a_j \models \Sigma_{P_{i_j}}$ for all $1 \leq j \leq k$.

[5] 1. Show by induction on ψ in $\mathrm{cl}(\varphi)$ that, if $\psi \in \Gamma_i \cup \Delta_i$ and $i_j < i \leq i_{j+1}$, then $w, j+1 \models \psi$.

[0] 2. Conclude.

[2] **Exercise 6** (Complexity)**.** Show that this multi-focus game view of LTL satisfiability yields a PSPACE algorithm.