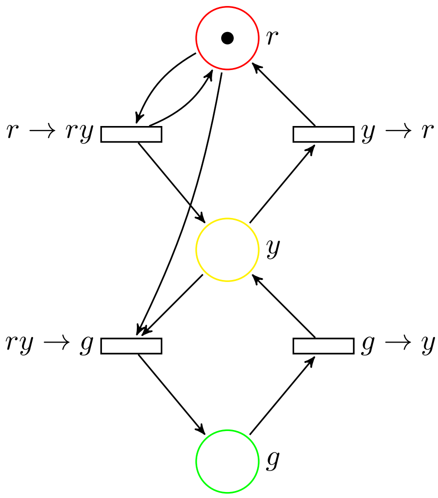
TD 8: Petri Nets

1 Modeling Using Petri Nets

Exercise 1 (Traffic Lights)**.** Consider again the traffic lights example from the lecture notes:



1. How can you modify this Petri net so that it becomes 1-safe?
2. Extend your Petri net to model two traffic lights handling a street intersection.

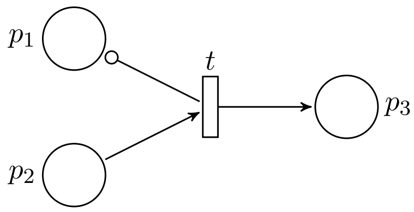
Exercise 2 (Producer/Consumer)**.** A producer/consumer system gathers two types of processes:

producers who can make the actions *produce* (p) or *deliver* (d), and

consumers with the actions *receive* (r) and *consume* (c).

All the producers and consumers communicate through a single unordered channel.

1. Model a producer/consumer system with two producers and three consumers. How can you modify this system to enforce a maximal capacity of ten simultaneous items in the channel?
2. An *inhibitor arc* between a place p and a transition t makes t firable only if the current marking at p is zero. In the following example, there is such an inhibitor arc between p_1 and t. A marking $(0,2,1)$ allows to fire t to reach $(0,1,2)$, but $(1,1,1)$ does not allow to fire t.



Using inhibitor arcs, enforce a priority for the first producer and the first consumer on the channel: the other processes can use the channel only if it is empty it is not currently used by the first producer and the first consumer.

2 Model Checking Petri Nets

Exercise 3 (Upper Bounds)**.** Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider as usual propositional LTL, with a set of atomic propositions AP equal to P the set of places of the Petri net. We define proposition p to hold in a marking m in $\mathbb{N}^P$ if $m(p) > 0$.

The models of our LTL formulæ are *computations* $m_0 m_1 \cdots$ in $(\mathbb{N}^P)^\omega$ such that, for all $i \in \mathbb{N}$, $m_i \rightarrow_{\mathcal{N}} m_{i+1}$ is a transition step of the Petri net $\mathcal{N}$.

1. We want to prove that state-based LTL model checking can be performed in polynomial space for 1-safe Petri nets. For this, prove that one can construct an exponential-sized Büchi automaton $\mathcal{B}_{\mathcal{N}}$ from a 1-safe Petri net that recognizes all the infinite computations of $\mathcal{N}$ starting in m_0.

2. In the general case, state-based LTL model checking is undecidable. Prove it for Petri nets with at least two unbounded places, by a reduction from the halting problem for 2-counter Minsky machines.

3. We consider now a different set of atomic propositions, such that $\Sigma = 2^{\text{AP}}$, and a labeled Petri net, with a labeling homomorphism $\lambda : T \rightarrow \Sigma$. The models of our LTL formulæ are infinite words $a_0 a_1 \cdots$ in Σ^ω such that $m_0 \xrightarrow{t_0}_{\mathcal{N}} m_1 \xrightarrow{t_1}_{\mathcal{N}} m_2 \cdots$ is an execution of $\mathcal{N}$ and $\lambda(t_i) = a_i$ for all i.

   Prove that action-based LTL model checking can be performed in polynomial space for labeled 1-safe Petri nets.

Exercise 4 (Lower Bounds for 1-Safe Petri Nets)**.** A *linear bounded automaton* (LBA) $\mathcal{M} = \langle Q, \Sigma \uplus \{\dashv, \vdash\}, \Gamma, \delta, q_0, \#, F \rangle$ is a Turing machine with a left endmarker $\dashv$ and a right endmarker $\vdash$,

- that cannot move left from $\dashv$ nor right from $\vdash$,
- that cannot print over $\dashv$ or $\vdash$, and

- that starts with input $\dashv x \vdash$ for some x in Σ^*.

A LBA is thus restricted to its initial tape contents. The membership problem for a LBA with input $\dashv x \vdash$ is PSpace-hard.

1. Show how to simulate a LBA with input $\dashv x \vdash$ by a 1-safe Petri net of quadratic size.
2. Show that state-based LTL model checking is PSpace-hard in the size of the Petri net for 1-safe Petri nets.
3. Show that action-based LTL model checking is PSpace-hard in the size of the Petri net for labeled 1-safe Petri nets.

3 Coverability

The *coverability problem* for Petri nets is the following decision problem:

Instance: A Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$ and a marking m_1 in $\mathbb{N}^P$.

Question: Does there exist m_2 in $\mathsf{Reach}_{\mathcal{N}}(m_0)$ such that $m_1 \leq m_2$?

For 1-safe Petri nets, coverability coincides with reachability, and is thus PSpace-complete according to the previous exercises.

Exercise 5 (Inhibitor Arcs)**.** Prove that the coverability problem is undecidable for Petri nets having two inhibitor arcs.
(Hint: start by proving its undecidability for Petri nets with two places that are the sources of inhibitor arcs.)

Exercise 6 (Coverability Graph)**.** One way to decide the coverability problem is to use Karp and Miller's coverability graph (see the lecture notes). Indeed, we have the equivalence between the two statements:

i. there exists m_2 in $\mathsf{Reach}_{\mathcal{N}}(m_0)$ such that $m_1 \leq m_2$, and

ii. there exists m_3 in $\mathsf{CoverabilityGraph}_{\mathcal{N}}(m_0)$ such that $m_1 \leq m_3$.

1. Prove that (i) implies (ii).
(Hint: prove that if $m \xrightarrow{u}_{\mathcal{N}} m_2$ in the Petri net $\mathcal{N}$ for some m in $\mathbb{N}^P$ and u in T^*, then there exists m_3 in $(\mathbb{N} \cup \{\omega\})^P$ such that $m_2 \leq m_3$ and $m \xrightarrow{u}_G m_3$ in the coverability graph.)

2. Let us prove that (ii) implies (i). The idea is that we can find reachable markings that agree with m_3 on its finite places, and that can be made arbitrarily high on its ω-places. For this, we need to identify the graph nodes where new ω values were introduced, which we call *ω-nodes*. Moreover, for a marking m in $(\mathbb{N} \cup \{\omega\})^P$, we define $\Omega(m)$ as the set of places p such that $m(p) = \omega$.

   (a) Recall that an ω value is introduced in the coverability graph thanks to Algorithm 1.

```
1 repeat
2     saved ← m'
3     foreach m'' ∈ V s.t. ∃v ∈ T⁺, m'' →_G^v m do
4         if m'' < m' then
5             m' ← m' + ((m' − m'') · ω)
6         end
7     end
8 until saved = m'
9 return m'
```

Algorithm 1: AddOmegas(m, t, m', V, E)

   Let $\{v_1, \ldots, v_l\}$ be the set of v sequences found on line 3 of the algorithm that resulted in an ω value for m' on line 5 during a call to AddOmegas(m, t, m', V, E). For each i, let n_i in $\mathbb{N}$ be a value such that the sequence v_i can be fired from the marking $(n_i, n_i, \ldots, n_i)$.
   Show that, for any j in $\mathbb{N}$, there exists a marking ν_j such that

$$\nu_j(p) = \begin{cases} m(p) - W(p,t) + W(t,p) & \text{if } p \in P \backslash \Omega(m) \\ j \cdot \sum_{i=1}^{l} n_i & \text{if } p \in \Omega(m) \end{cases}$$

   that allows to fire the sequence $v_1^j \cdots v_l^j$. How does the marking ν'_j with $\nu_j \xrightarrow{v_1^j \cdots v_l^j}_{\mathcal{N}} \nu'_j$ compare to ν_j?

   (b) Prove that, if $m \xrightarrow{u}_G m_3$ for some u in T^* in the coverability graph and m' in $\mathbb{N}^{\Omega(m_3)}$ is a partial marking on the places of $\Omega(m_3)$, then there are

   - a decomposition $u = u_1 u_2 \cdots u_{n+1}$ with each u_i in T^* (where the markings μ_i reached by $m \xrightarrow{u_1 \cdots u_i}_G \mu_i$ are ω-nodes),
   - sequences $w_1, \ldots, w_n$ in T^+,
   - numbers $k_1, \ldots, k_n$ in $\mathbb{N}$,

   such that $m \xrightarrow{u_1 w_1^{k_1} u_2 \cdots u_n w_n^{k_n} u_{n+1}}_{\mathcal{N}} m_2$ with $m_2(p) = m_3(p)$ for all p in $P \backslash \Omega(m_3)$ and $m_2(p) \geq m'(p)$ for all p in $\Omega(m_3)$.

Exercise 7 (Rackoff's Algorithm)**.** A rather severe issue with the coverability graph construction (see Exercise 6) is that it can generate a graph of non primitive recursive size compared to that of the original Petri net. We show here a much more decent ExpSpace upper bound, which is matched by an ExpSpace hardness proof by Lipton.

Let us fix a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$. We consider *generalized markings* in $\mathbb{Z}^P$. A *generalized computation* is a sequence $\mu_1 \cdots \mu_n$ in $(\mathbb{Z}^P)^*$ such that, for all $1 \leq i < n$, there is a transition t in T with $\mu_{i+1}(p) = \mu_i(p) - W(p,t) + W(t,p)$ for all $p \in P$ (i.e. we do not enforce enabling conditions). For a subset I of P, a generalized sequence is I*-admissible* if furthermore $\mu_i(p) \geq W(p,t)$ for all p in I at each step $1 \leq i < n$. For a value B in $\mathbb{N}$, it is I*–*B*-bounded* if furthermore $\mu_i(p) < B$ for all p in I at each step $1 \leq i \leq n$. A generalized sequence is an I*-covering* for m_1 if $\mu_1 = m_0$ and $\mu_n(p) \geq m_1(p)$ for all p in I.

Thus a computation is a P-admissible generalized computation, and a P-admissible P-covering for m_1 answers the coverability problem.

For a Petri net $\mathcal{N} = \langle P, T, F, W, m_0 \rangle$ and a marking m_1 in $\mathbb{N}^P$, let $\ell(\mathcal{N}, m_1)$ be the length of the shortest P-admissible P-covering for m_1 in $\mathcal{N}$ if one exists, and otherwise $\ell(\mathcal{N}, m_1) = 0$. For L, k in $\mathbb{N}$, define

$$M_L(k) = \sup\{\ell(\mathcal{N}, m_1) \mid |P| = k,\\ \max\{W(p,t) \mid p \in P, t \in T\} + \max\{m_1(p) \mid p \in P\} \leq L\}\;.$$

1. Show that $M_L(0) \leq 1$.

2. We want to show that

   $$M_L(k) \leq (L \cdot M_L(k-1))^k + M_L(k-1)$$

   for all $k \geq 1$. To this end, we prove that, for every marking m_1 in $\mathbb{N}^P$ for a Petri net $\mathcal{N}$ with $|P| = k$,

   $$\ell(\mathcal{N}, m_1) \leq (L \cdot M_L(k-1))^k + M_L(k-1)\;. \qquad (*)$$

   Let

   $$B = M_L(k-1) \cdot \max\{W(p,t) \mid p \in P, t \in T\} + \max\{m_1(p) \mid p \in P\}\;.$$

   and suppose that there exists a P-admissible P-covering $w = \mu_1 \cdots \mu_n$ for m_1 in $\mathcal{N}$.

   (a) Show that, if w is P–B-bounded, then $(*)$ holds.

   (b) Assume the contrary: we can split w as $w_1 w_2$ such that w_1 is P–B-bounded and w_2 starts with a marking μ_j with a place p such that $\mu_j(p) \geq B$. Show that $(*)$ also holds.

3. Show that $M_L(|P|) \leq L^{(3\cdot|P|)!}$ for $L = 2 + \max\{W(p,t) \mid p \in P, t \in T\} + \max\{m_1(p) \mid p \in P\}$.

4. Assuming that the size n of the instance $(\mathcal{N}, m_1)$ of the coverability problem is more than
$$\max\{\log L, |P|, \max\{\log W(t,p) \mid t \in T, p \in P\}\} ,$$
deduce that we can guess a P-admissible P-covering for m_1 of length at most $2^{2^{c \cdot n \log n}}$ for some constant c. Conclude.