Home Assignment 2: Stuttering and Bisimulation

To hand in before or on January 11, 2010.
The penalty for delays is 2 points per day.

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Electronic versions (PDF only) can be sent by email to ⟨schmitz@lsv.ens-cachan.fr⟩, paper versions should be handed in on the 11th or put in my mailbox at LSV, ENS Cachan. Any mistake spotted in the subject should be reported so that everyone can benefit from its correction.

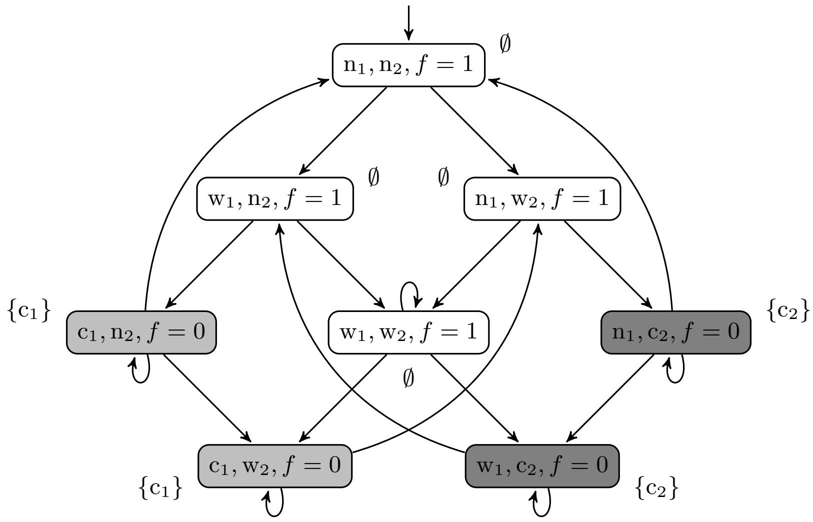
This homework investigates bisimulation relations that oversee stuttering steps in finite Kripke structures. These bisimulations are also known as branching bisimulations in the process calculi literature.

Definition 1 (Stutter Bisimulation)**.** Consider two (not necessarily different) Kripke structures $M_1 = \langle S_1, T_1, I_1, \mathrm{AP}, \ell_1\rangle$ and $M_2 = \langle S_2, T_2, I_2, \mathrm{AP}, \ell_2\rangle$. A *stutter simulation* between M_1 and M_2 is a relation $R \subseteq S_1 \times S_2$ satisfying:

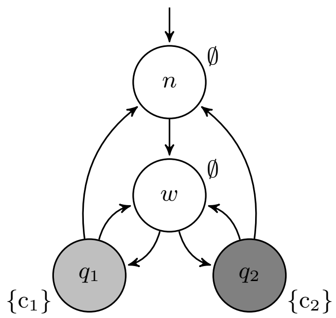
1. for any initial state s_1 in I_1, there exists an initial state s_2 in I_2, such that (s_1, s_2) is in R,
2. for all (s_1, s_2) in R,
   (a) $\ell_1(s_1) = \ell_2(s_2)$,
   (b) if (s_1, s_1') is a transition in T_1 with $(s_1', s_2) \notin R$, then there exist an integer n in $\mathbb{N}$ and $n+2$ states $u_0, \ldots, u_{n+1}$ in S_2 such that $u_0 = s_2$, (s_1', u_{n+1}) is in R, and for each $0 \le i \le n$, (s_1, u_i) is in R and (u_i, u_{i+1}) is a transition in T_2.

A *stutter bisimulation* on $S_1 \times S_2 \cup S_2 \times S_1$ between M_1 and M_2 is a union $R \cup R^{-1}$ where R is a stutter simulation between M_1 and M_2 and its inverse R^{-1} a stutter simulation between M_2 and M_1. Two states s_1 and s_2 (resp. two systems M_1 and M_2) are *stutter bisimilar*, noted $s_1 \approx s_2$ (resp. $M_1 \approx M_2$), if there exists such a stutter bisimulation with (s_1, s_2) in R (resp. such a stutter bisimulation between M_1 and M_2). A stutter bisimulation on a single system M is a stutter bisimulation between M and itself.

Exercise 1 (Mutual Exclusion)**.** The following system is an abstract mutual exclusion protocol for two processes. We are interested in the mutual exclusion property, i.e. whether $\mathrm{G}(\neg c_1 \vee \neg c_2)$ holds for all traces of the system; thus we can restrict ourselves to the set of atomic propositions $\mathrm{AP} = \{c_1, c_2\}$ (state labels are displayed next to the states and match the gray levels). Observe that with this new labeling our system starts displaying quite a bit of stuttering.



A more abstract system for the same functionality could be:



1. Are the two systems stutter bisimilar?
2. Are they bisimilar?

Justify your answers.

Exercise 2 (Coarsest Stutter Bisimulation)**.** Let $M = \langle S, T, I, \mathrm{AP}, \ell \rangle$ be a Kripke structure.

1. Show that $\approx$ is an equivalence relation on S.

2. Show that $\approx$ is a stutter bisimulation for M.

3. Show that $\approx$ is the coarsest stutter bisimulation for M and coincides with the union of all stutter bisimulations for M.

Exercise 3 (Quotients)**.** Since $\approx$ is an equivalence relation, we note $[s]$ for the equivalence class of s under $\approx$. The *quotient* of $M = \langle S, T, I, \mathrm{AP}, \ell\rangle$ by $\approx$ is defined as

$$M/{\approx} = \langle S/{\approx}, T/{\approx}, I/{\approx}, \mathrm{AP}, \ell_{\approx}\rangle$$

where

$$\begin{aligned} S/{\approx} &= \{[s] \mid s \in S\},\\ T/{\approx} &= \{([s],[s']) \mid (s,s') \in T \text{ and } s \not\approx s'\},\\ I/{\approx} &= \{[s] \mid s \in I\}, \text{ and}\\ \ell_{\approx}([s]) &= \ell(s)\ . \end{aligned}$$

1. Construct the quotient of the system of Exercise 1 by its coarsest stutter bisimulation.

2. Prove that M is always stutter bisimilar to $M/{\approx}$.

Exercise 4 (Stutter Bisimulation and Stutter Equivalence)**.** Let as usual $\Sigma = 2^{\mathrm{AP}}$. A *stuttering function* $f : \mathbb{N} \to \mathbb{N}_+$ maps positive integers to strictly positive integers. Let $\sigma = a_0 a_1 \cdots$ be an infinite word of Σ^ω and f a stuttering function, we denote by $\sigma[f]$ the infinite word $a_0^{f(0)} a_1^{f(1)} \cdots$, i.e. where the i-th symbol of σ is repeated $f(i)$ times. A language $L \subseteq \Sigma^\omega$ is *stutter invariant* if, for all words σ in Σ^ω and all stuttering functions f,

$$\sigma \in L \text{ iff } \sigma[f] \in L\ .$$

We saw in Exercises 3 and 4 of TD 3 that LTL(U), the fragment of LTL without the "next" modality, allows to express all the aperiodic stutter invariant languages.

A word $\sigma = a_0 a_1 \cdots$ in Σ^ω is *stutter-free* if, for all i in $\mathbb{N}$, either $a_i \neq a_{i+1}$, or $a_i = a_j$ for all $j \geq i$. For a given infinite word σ, there exists a unique stutter-free infinite word $\mathrm{sf}(\sigma)$ such that $\sigma = \mathrm{sf}(\sigma)[f]$ for some stuttering function f. We note $\mathrm{sf}(L)$ for the set of stutter-free words in a language L. Two Kripke structures are *stutter trace equivalent* if their sets of infinite stutter-free traces are the same. (We consider in the following *total* Kripke structures, i.e. where for any state s, there exists at least one state s' such that (s, s') is a transition in T.)

1. Show that the two systems in Figure 1 are stutter bisimilar but not stutter trace equivalent.

Figure 1: The Kripke structures for Exercise 4.1.

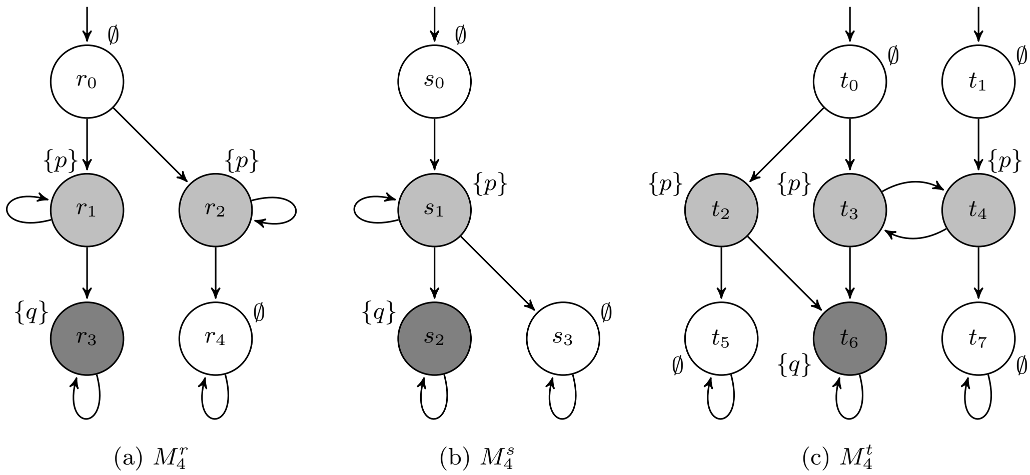


Figure 2: The Kripke structures for Exercise 4.2.

2. Which of the systems in Figure 2 are stutter trace equivalent? Which are stutter bisimilar?

Exercise 5 (Divergence-sensitive Relations)**.** The previous exercise demonstrates that stutter bisimulations might not be the best concept for capturing stutter invariant systems.

Let $M_1 = \langle S_1, T_1, I_1, \mathrm{AP}, \ell_1 \rangle$ and $M_2 = \langle S_2, T_2, I_2, \mathrm{AP}, \ell_2 \rangle$ be two Kripke structures, R a relation on $S_1 \times S_2$, and (s, s') in R. State s' is $R(s)$-*divergent* if there exists an infinite path $\pi = s' s_1 s_2 \cdots$ starting in s' in M such that (s, s_j) is in R for all $j \geq 1$. A relation $R \cup R^{-1}$ is *divergence-sensitive* if for any (s, s') of R, s is $R^{-1}(s')$-divergent iff s' is $R(s)$-divergent.

1. Consider the union M_4 of the two systems of Figure 1 in the previous exercise. Show that the coarsest stutter bisimulation of M_4 is not divergence-sensitive.

2. Two states s_1 and s_2 are *stutter divergent bisimilar*, noted $s_1 \overset{d}{\approx} s_2$, if there exists a divergent-sensitive stutter bisimulation $R \cup R^{-1}$ with $(s_1, s_2) \in R$. Show that

$\overset{d}{\approx}$ is an equivalence on S, and is actually the coarsest divergence-sensitive stutter bisimulation of M.

3. Which of the systems of Figure 2 are stutter divergent bisimilar?

4. Let M be a Kripke structure. Let $\pi_1 = s_{0,1}s_{1,1}s_{2,1}\cdots$ and $\pi_2 = s_{0,2}s_{1,2}s_{2,2}\cdots$ be two infinite paths in M. We say that π_1 and π_2 are *stutter divergent bisimular* if there exists two infinite sequences of indices $0 = i_0 < i_1 < i_2 < \cdots$ and $0 = j_0 < j_1 < j_2 < \cdots$ such that $s_{i,1} \overset{d}{\approx} s_{j,2}$ for all $i_{k-1} \leq i < i_k$ and $j_{k-1} \leq j < j_k$ with $k = 1, 2, \ldots$.

   (a) Show that $\pi \overset{d}{\approx} \pi'$ implies $\mathrm{sf}(\ell(\pi)) = \mathrm{sf}(\ell(\pi'))$.

   (b) Prove that, for any two states s and s', if $s \overset{d}{\approx} s'$, then for all infinite runs $\pi = ss_1s_2\cdots$ starting in s, there exists an infinite run $\pi' = s's'_1s'_2\cdots$ starting in s' such that $\pi \overset{d}{\approx} \pi'$.

Exercise 6 (Logical Characterization)**.** We note CTL*(U) for the class of CTL* formulæ that do not use any "next" modalities, i.e. that follow the following abstract syntax:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathrm{E}\psi \qquad \text{(state formulæ)}$$

$$\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \psi \mathbin{\mathrm{U}} \psi \qquad \text{(path formulæ)}$$

where p ranges over the set AP of atomic propositions. The CTL(U) fragment is defined similarly by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathrm{E}(\varphi \mathbin{\mathrm{U}} \varphi) \mid \mathrm{A}(\varphi \mathbin{\mathrm{U}} \varphi)\ . \qquad \text{(state formulæ)}$$

Consider the Kripke structures M_6^r, M_6^s and M_6^t in Figure 3.

1. Provide a CTL*(U) formula φ_1 such that $r_0 \models \varphi_1$ but $s_0 \not\models \varphi_1$.

2. Consider the union of the two systems M_6^s and M_6^t. Show that $s_0 \overset{d}{\approx} t_0$. Give a CTL(U) formula φ_C for each of the equivalence classes C of $(M_6^s \cup M_6^t)$ by $\overset{d}{\approx}$, such that $[\![\varphi_C]\!] = C$.

3. Let M be a total Kripke structure, s and s' be two states and π and π' two infinite paths in M. Prove the following statements (by simultaneous induction on the structure of CTL*(U) formulæ):

   (a) if $s \overset{d}{\approx} s'$, then for any CTL*(U) state formula φ, $s \models \varphi$ iff $s' \models \varphi$,

   (b) if $\pi \overset{d}{\approx} \pi'$, then for any CTL*(U) path formula ψ, $\pi \models \psi$ iff $\pi' \models \psi$.

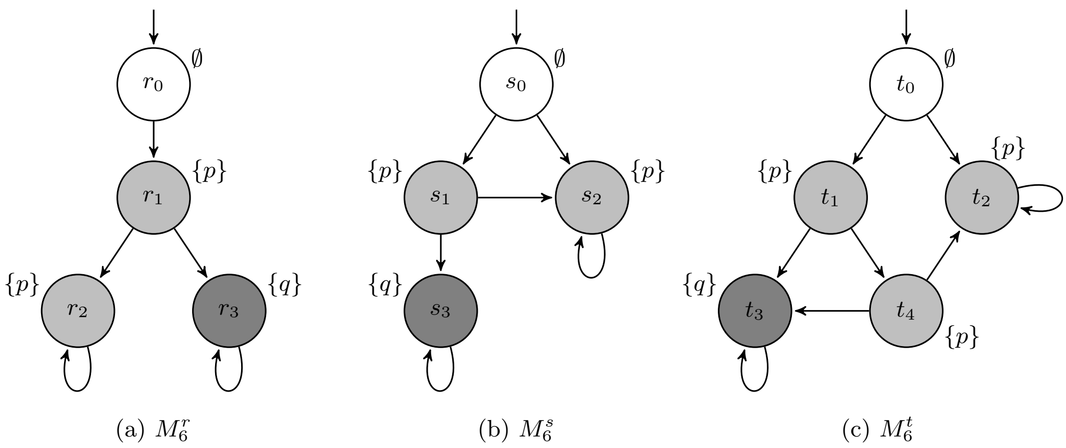


Figure 3: The Kripke structures for Exercise 6.

4. Let M be a total Kripke structure, and s and s' be two states of M. Define the following equivalence relation $\mathcal{F}$ on S:

$$\mathcal{F} = \{(s, s') \mid \forall\varphi \in \text{CTL(U)}, s \models \varphi \text{ iff } s' \models \varphi\} \ .$$

We want to prove that $\mathcal{F}$ is a divergence-sensitive stutter bisimulation for M.

(a) Prove that if (s, s') is in $\mathcal{F}$, then $\ell(s) = \ell(s')$.

(b) We want to prove that $\mathcal{F}$ is a stutter bisimulation. Since $\mathcal{F}$ is an equivalence relation, we can consider its equivalence classes $[s]_\mathcal{F}$ for some state s.

- Show that, for each equivalence class $C = [s]_\mathcal{F}$, there exists a CTL(U) *master formula* φ_C such that $[\![\varphi_C]\!] = C$.
- Show that $\mathcal{F}$ fulfills condition 2b of Definition 1.

(c) Prove that $\mathcal{F}$ is divergence-sensitive.

5. Conclude by proving the following theorem:

Theorem 1 (Logical Characterization of Stutter Divergent Bisimulation)**.** *Let M be a total Kripke structure, and s and s' two states of M. The following three statements are equivalent:*

1. $s \overset{d}{\approx} s'$,

2. s and s' verify the same CTL*(U) state formulæ,

3. s and s' verify the same CTL(U) formulæ.

Exercise 7 (Observational Bisimulation)**.** Consider two (not necessarily different) Kripke structures $M_1 = \langle S_1, T_1, I_1, \mathrm{AP}, \ell_1\rangle$ and $M_2 = \langle S_2, T_2, I_2, \mathrm{AP}, \ell_2\rangle$. An *observational simulation* between M_1 and M_2 is a relation $R \subseteq S_1 \times S_2$ satisfying:

1. for any initial state s_1 in I_1, there exists an initial state s_2 in I_2, such that (s_1, s_2) is in R,
2. for all (s_1, s_2) in R,
   (a) $\ell_1(s_1) = \ell_2(s_2)$,
   (b) if (s_1, s_1') is a transition in T_1, then there exist two integers n and $m \leq n$ in $\mathbb{N}$ and $n+1$ states $u_0, \ldots, u_n$ in S_2 such that
      - $u_0 = s_2$,
      - (s_1', u_n) is in R,
      - for each $0 \leq i < n$, (u_i, u_{i+1}) is a transition in T_2,
      - $\ell_2(u_0) = \ell_2(u_1) = \cdots = \ell_2(u_m)$, and
      - $\ell_2(u_{m+1}) = \ell_2(u_{m+2}) = \cdots = \ell_2(u_n)$.

As usual, an *observational bisimulation* on $S_1 \times S_2 \cup S_2 \times S_1$ between M_1 and M_2 is a union $R \cup R^{-1}$ where R is a observational simulation between M_1 and M_2 and its inverse R^{-1} a observational simulation between M_2 and M_1. Two states s_1 and s_2 (resp. two systems M_1 and M_2) are *observational bisimilar*, noted $s_1 \overset{o}{\approx} s_2$ (resp. $M_1 \overset{o}{\approx} M_2$), if there exists such an observational bisimulation with (s_1, s_2) in R (resp. such an observational bisimulation between M_1 and M_2).

We want to prove that observational bisimulation is coarser than stutter bisimulation.

1. Show that $M_1 \approx M_2$ implies $M_1 \overset{o}{\approx} M_2$.
2. Exhibit two Kripke structures M_1 and M_2 such that $M_1 \overset{o}{\approx} M_2$ but $M_1 \not\approx M_2$.