



Algorithmic Theory of WQOs

S. Schmitz

joint work with D. Figueira, S. Figueira, and
Ph. Schnoebelen

LSV, ENS Cachan & CNRS

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OUTLINE

well quasi orderings (wqo)

generic tools for termination arguments

this talk

beyond termination: complexity upper bounds

contents

WQO Algorithms

Length of Bad Sequences

Subrecursive Hierarchies

A BIT OF HISTORY...

- ▶ First bounds for Dickson's Lemma: McAloon (1984); Clote (1986), using Ramsey-type arguments.
- ▶ Application: finite containment for Petri nets, shown by Mayr and Meyer (1981) (see also Jančar, 2001) to be Ackermann-hard
- ▶ Result: Ackermannian upper bounds



WELL QUASI ORDERINGS

Definition (wqo)

A wqo is a quasi-order (A, \leq) s.t.

$$\forall \mathbf{x} = x_0, x_1, x_2, \dots \in A^\omega, \exists i_1 < i_2, x_{i_1} \leq x_{i_2}.$$

Example (Basic WQO's)

- ▶ (\mathbb{N}, \leq) ,
- ▶ $(\{0, 1, \dots, k\}, \leq)$ for any $k \in \mathbb{N}$,
- ▶ $(\Gamma_p, =)$ for any finite set Γ_p with p elements.



ALGEBRA OF WQO'S

FINITE SEQUENCES

Lemma (Higman's Lemma)

If (A, \leq) is a wqo, then (A^*, \leq_*) is a wqo where \leq_* is the *subword embedding ordering*:

$$a_1 \cdots a_m \leq_* b_1 \cdots b_n \stackrel{\text{def}}{\iff} \begin{cases} \exists 1 \leq i_1 < \cdots < i_m \leq n, \\ \bigwedge_{j=1}^m a_j \leq_A b_{i_j}. \end{cases}$$

Example

$$aba \leq_* baacabbab$$



ALGEBRA OF WQO'S

DISJOINT SUMS

Lemma

If (A_1, \leq_{A_1}) and (A_2, \leq_{A_2}) are two wqo's, then $(A_1 + A_2, \leq_+)$ is a wqo,

where $A_1 + A_2 \stackrel{\text{def}}{=} \{\langle i, a \rangle \mid i \in \{1, 2\} \wedge a \in A_i\}$ and \leq_+ is the **sum ordering**:

$$\langle i, a \rangle \leq_+ \langle j, b \rangle \stackrel{\text{def}}{\iff} i = j \wedge a \leq_{A_i} b.$$



ALGEBRA OF WQO's

CARTESIAN PRODUCTS

Lemma (Dickson's Lemma)

If (A_1, \leq_{A_1}) and (A_2, \leq_{A_2}) are two wqo's, then $(A_1 \times A_2, \leq_{\times})$ is a wqo, where \leq_{\times} is the *product ordering*:

$$\langle a_1, a_2 \rangle \leq_{\times} \langle b_1, b_2 \rangle \stackrel{\text{def}}{\iff} a_1 \leq_{A_1} b_1 \wedge a_2 \leq_{A_2} b_2 .$$



WQO'S FOR TERMINATION

BAD SEQUENCES

- ▶ $\mathbf{x} = x_0, x_1, \dots$ in A^∞ is a **good sequence** if $\exists i_1 < i_2, x_{i_1} \leq x_{i_2}$,
- ▶ a **bad sequence** otherwise,
- ▶ if (A, \leq) is a wqo: every bad sequence is finite



AN EXAMPLE

```
SIMPLE  $(a, b)$   
 $c \leftarrow 1$   
while  $a > 0 \wedge b > 0$   
     $\langle a, b, c \rangle \leftarrow \langle a - 1, b, 2c \rangle$   
    or  
     $\langle a, b, c \rangle \leftarrow \langle 2c, b - 1, 1 \rangle$   
end
```

- ▶ in any run, $\langle a_0, b_0 \rangle, \dots, \langle a_n, b_n \rangle$ is a bad sequence over (\mathbb{N}^2, \leq_x) ,
- ▶ (\mathbb{N}^2, \leq_x) is a wqo: all the runs are finite
- ▶ How long can SIMPLE run?



A COMPUTATION OF SIMPLE(2, 3)

SIMPLE(a, b)

$c \leftarrow 1$

while $a > 0 \wedge b > 0$

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end

$\langle a, b, c \rangle$	loop iterations
$\langle 2, 3, 2^0 \rangle$	0



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$\langle a, b, c \rangle$	loop iterations
$\langle 2, 3, 2^0 \rangle$	0
$\langle 1, 3, 2^1 \rangle$	1
$\langle 2^2, 2, 2^0 \rangle$	2



A COMPUTATION OF SIMPLE(2, 3)

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end

$\langle a, b, c \rangle$	loop iterations
\vdots	\vdots
$\langle 2^2, 2, 2^0 \rangle$	2
\vdots	\vdots
$\langle 1, 2, 2^{2^2-1} \rangle$	$2 + 2^2 - 1$



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$\langle a, b, c \rangle$	loop iterations
\vdots	\vdots
$\langle 2^{2^2}, 1, 1 \rangle$	$2 + 2^2$
\vdots	\vdots
$\langle 1, 1, 2^{2^{2^2}-1} \rangle$	$2 + 2^2 + 2^{2^2} - 1$



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$\langle a, b, c \rangle$	loop iterations
\vdots	\vdots
$\langle 1, 1, 2^{2^{2^2}} - 1 \rangle$	$2 + 2^2 + 2^{2^2} - 1$
$\langle 0, 1, 2^{2^{2^2}} \rangle$	$2 + 2^2 + 2^{2^2}$



A COMPUTATION OF SIMPLE(2, 3)

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- ▶ Non-elementary complexity
- ▶ Derive (matching) upper bounds for termination arguments based on (\mathbb{N}^2, \leq_x) being a wqo



CONTROLLED SEQUENCES

- ▶ bound the length of bad sequences over (A, \leq)



CONTROLLED SEQUENCES

- ▶ bound the length of bad sequences over (A, \leq)
- ▶ but: choose any N , and consider the bad sequence $N, N-1, \dots, 0$ over \mathbb{N}
- ▶ similarly:
 $\langle 3, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 1 \rangle, \langle 3, 0 \rangle, \langle 2, N \rangle, \langle 2, N-1 \rangle, \dots$



CONTROLLED SEQUENCES

- ▶ bound the length of bad sequences over $(A, \leq; |\cdot|_A)$
- ▶ associate a **norm function** $|\cdot|_A : A \rightarrow \mathbb{N}$ to each wqo (A, \leq)
- ▶ assume $|\cdot|_A$ is **proper** $\stackrel{\text{def}}{\iff}$ for all n

$$A_{<n} \stackrel{\text{def}}{=} \{x \in A \mid |x|_A < n\} \text{ is finite}$$

Definition (Normed WQO's)

$$|k|_{\mathbb{N}} \stackrel{\text{def}}{=} k \quad |a_i|_{\Gamma_p} \stackrel{\text{def}}{=} 0 \quad |\langle a, b \rangle|_{A \times B} \stackrel{\text{def}}{=} \max(|a|_A, |b|_B)$$

$$|\langle i, a \rangle|_{A_1 + A_2} \stackrel{\text{def}}{=} |a|_{A_i} \quad |a_1 \cdots a_m|_{A^*} \stackrel{\text{def}}{=} \max(m, |a_1|_A, \dots, |a_m|_A)$$



CONTROLLED SEQUENCES

- ▶ bound the length of **controlled** bad sequences over $(A, \leq ; |\cdot|_A)$
- ▶ fix a **control function** $g : \mathbb{N} \rightarrow \mathbb{N}$
(monotone with $g(x+1) \geq g(x) + 1 \geq x + 2$)
- ▶ $\mathbf{x} = x_0, x_1, \dots$ over A is **(g, n) -controlled** iff

$$\forall i, |x_i|_A < g^i(n)$$



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Example ($\text{SIMPLE}(2, 3)$)

$$A = \mathbb{N}^2, n = 4, g(x) = 2x$$



CONTROLLED SEQUENCES

- ▶ bound the length of **controlled** bad sequences over $(A, \leq ; |\cdot|_A)$
- ▶ for fixed A, g, n , there are **finitely** many bad (g, n) -controlled sequences over A
- ▶ maximal length function

$$L_{A,g}(n)$$



TECHNICAL OVERVIEW

1. **residuals**: inductive definition for $L_{A,g}$
2. **reflections**: approximations to obtain inequalities for $L_{A,g}$ in terms of “simpler” wqo’s
3. **ordinal notations**: associate ordinal terms to wqo’s in order to work with subrecursive hierarchies



RESIDUALS

Definition (Residual)

The **residual** of a wqo (A, \leq) by $a \in A$ is the wqo

$$A/a \stackrel{\text{def}}{=} \{b \in A \mid a \not\leq b\}.$$

Proposition (Descent Equation)

$$L_{A,g}(n) = \max_{a \in A < n} \{1 + L_{A/a,g}(g(n))\}$$

Example

$$\Gamma_{p+1}/a \equiv \Gamma_p$$

$$L_{\Gamma_p,g}(n) = p$$

$$\mathbb{N}/k = \{0, \dots, k-1\}$$

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REFLECTIONS

Definition (Normed Reflection)

A **normed reflection** is a mapping $h : A \rightarrow B$ between two normed wqo's satisfying

$$\begin{aligned} \forall a, b \in A, h(a) \leq_B h(b) \text{ implies } a \leq_A b \\ \forall a \in A, |h(a)|_B \leq |a|_A \end{aligned}$$

Notation

$A \hookrightarrow B$: there exists a normed reflection $h : A \rightarrow B$

Example

$$\{0, \dots, k-1\} \hookrightarrow \Gamma_k \qquad \Gamma_p \hookrightarrow \Gamma_{p+1}$$



REFLECTIONS

Proposition (Monotony of Length Function)

$$A \hookrightarrow B \text{ implies } \forall n, L_{A,g}(n) \leq L_{B,g}(n).$$

Proposition (Reflections for Residuals)

$$(A + B) / \langle 1, a \rangle = (A/a) + B$$

$$(A + B) / \langle 2, b \rangle = A + (B/b)$$

$$(A \times B) / \langle a, b \rangle \hookrightarrow [(A/a) \times B] + [A \times (B/b)]$$

$$\Gamma_{p+1}^* / a_1 \cdots a_m \hookrightarrow \Gamma_m \times (\Gamma_p^*)^m$$



$$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$$

“JULLIEN’S DECOMPOSITION”

Example

$$aba, \overbrace{bbba, bbb, aabb, baa, abb, bb}^{\in \Gamma_2^*/aba}$$



$$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$$

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$$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$$

“JULLIEN’S DECOMPOSITION”

Example

$$aba, \left[\begin{array}{ccc} & \langle bbb, \varepsilon, \varepsilon \rangle, & \langle bb, \varepsilon, \varepsilon \rangle \\ \langle bbb, \varepsilon, \varepsilon \rangle, & & \langle b, a, \varepsilon \rangle, \\ & \langle \varepsilon, a, b \rangle, & \langle \varepsilon, \varepsilon, b \rangle, \end{array} \right] \in \Gamma_3 \times (\Gamma_1^*)^3$$



ORDINAL TERMS

- ▶ **maximal order type**
 $o : WQO \rightarrow \text{CNF}(\omega^{\omega^\omega})$ (de Jongh and Parikh, 1977;
 Hasegawa, 1994)
- ▶ well-founded relations ∂_n over $\text{CNF}(\omega^{\omega^\omega})$
 implement reflection of residuals of norm
 $< n$

Example

$$\begin{array}{ccc}
 \Gamma_2^* & \xrightarrow{\bigcup_{|x|<4} [\cdot/x \hookrightarrow \cdot]} & \Gamma_3 \times (\Gamma_1^*)^3 \\
 o \downarrow & & \downarrow o \\
 \omega^\omega & \xrightarrow{\partial_4} & \omega^3 \cdot 3
 \end{array}$$



MAIN INEQUALITY

$$L_{o^{-1}(\alpha),g}(n) \leq \max_{\alpha' \in \partial_n \alpha} \{1 + L_{o^{-1}(\alpha'),g}(g(n))\} .$$



A BOUNDING FUNCTION

$$M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha} \{1 + M_{\alpha',g}(g(n))\}.$$

- ▶ Then for all α and n

$$L_{A,g}(n) \leq M_{o(A),g}(n)$$

- ▶ find the **functional complexity** of M



SUBRECURSIVE HIERARCHIES

Hierarchies of functions (and function classes)
indexed by **ordinal terms**.



FUNDAMENTAL SEQUENCES

Subrecursive hierarchies are defined through an assignment of **fundamental sequences** $(\lambda_x)_{x < \omega}$ for limit ordinal terms λ , s.t. $\lambda_x < \lambda$ and $\lambda = \sup_x \lambda_x$: e.g.

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot (x + 1)$$

$$(\gamma + \omega^\lambda)_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x},$$

Example

$$\omega_x = x + 1$$

$$(\omega^{\omega^{p+1}})_x = \omega^{\omega^p \cdot (x+1)}$$



FAST GROWING HIERARCHY: $(F_\alpha)_\alpha$

(LÖB AND WAINER, 1970)

$$F_0(x) \stackrel{\text{def}}{=} x + 1, \quad F_{\alpha+1}(x) \stackrel{\text{def}}{=} F_\alpha^{x+1}(x), \quad F_\lambda \stackrel{\text{def}}{=} F_{\lambda_x}(x).$$

Example

$$F_1(x) = 2x + 1$$

$$F_2(x) = (x + 1) \cdot 2^{x+1} - 1$$

F_3 is non elementary

F_ω is non primitive-recursive

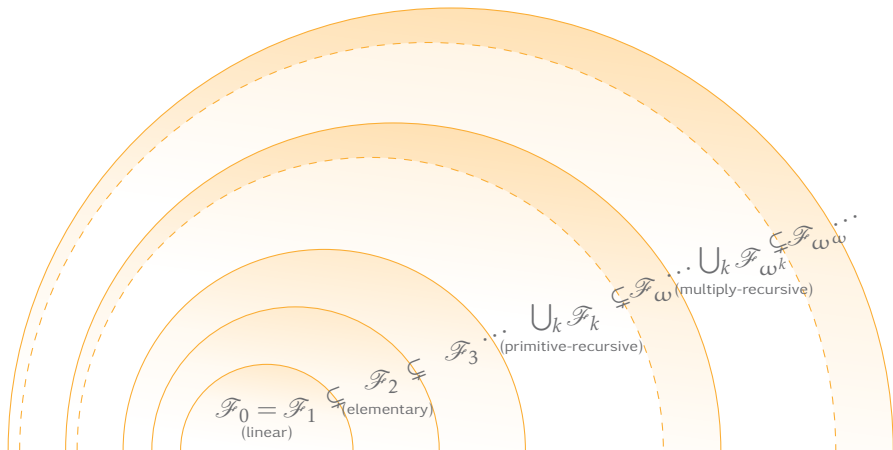
F_{ω^ω} is non multiply-recursive



EXTENDED GRZEGORCZYK HIERARCHY: $(\mathcal{F}_\alpha)_\alpha$

(LÖB AND WAINER, 1970)

Elementary-recursive closure of the $(F_\alpha)_\alpha$





COMPARISON WITH $(\mathcal{F}_\alpha)_\alpha$

Theorem (Higman's Lemma)

Let $p \geq 2$ and g primitive-recursive. Then $L_{\Gamma_p^*, g}$ is bounded by a function in $\mathcal{F}_{\omega^{p-1}}$.

Theorem (Dickson's Lemma)

Let $k, \gamma \geq 1$ and g in \mathcal{F}_γ . Then $L_{\mathbb{N}^k, g}$ is bounded by a function in $\mathcal{F}_{\gamma+k}$.



CONCLUDING REMARKS

- ▶ practical applications of wqo's yield upper bounds!
- ▶ out-of-the-box upper bounds
- ▶ “essentially” matching lower bounds



F_ω -COMPLETE PROBLEMS

Decision of problems on

- ▶ well-structured counter systems (Finkel and Schnoebelen, 2001), e.g.
 - ▶ finite VASS containment (Mayr and Meyer, 1981; Jančar, 2001)
 - ▶ lossy systems (Schnoebelen, 2010),
- ▶ transition invariants in \mathbb{N} (Podelski and Rybalchenko, 2004),
- ▶ relevance logics (Urquhart, 1999),
- ▶ data logics (Demri and Lazić, 2009; Figueira and Segoufin, 2009), ...



F_{ω^ω} -COMPLETE PROBLEMS

Decision of problems on

- ▶ lossy channel systems (Chambart and Schnoebelen, 2008),
- ▶ Post embedding problem PEP^{reg} (Chambart and Schnoebelen, 2007),
- ▶ 1-clock alternating timed automata (Lasota and Walukiewicz, 2008),
- ▶ Metric temporal logic (Ouaknine and Worrell, 2007),
- ▶ finite concurrent programs under weak (TSO/PSO) memory models (Atig et al., 2010)
- ▶ alternating register automata over ordered domains (Figueira et al., 2010), . . .



FUTURE WORK

- ▶ full algebra based on $+$, \times , $*$,
- ▶ more algebraic operations: set, multisets,...
- ▶ applications
 - ▶ lower bounds for timed Petri nets,
 - ▶ upper bounds for Petri net reachability



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PROGRAM TERMINATION PROOFS

(PODELSKI AND RYBALCHENKO, 2004)

Monolithic Termination Argument

- ▶ prove that the program's transition relation R is **well-founded**
- ▶ **ranking function** ρ from program configurations $\mathbf{x} = x_0, x_1, \dots$ into a wqo s.t.
 $R \subseteq \{(x_i, x_j) \mid \rho(x_i) \not\leq \rho(x_j)\}$
- ▶ for SIMPLE: $\rho(a, b, c) = \omega \cdot b + a$



PROGRAM TERMINATION PROOFS

(PODELSKI AND RYBALCHENKO, 2004)

Disjunctive Termination Argument

- ▶ find well-founded relations T_1, \dots, T_k on program configurations
- ▶ prove $R^+ \subseteq T_1 \cup \dots \cup T_k$
- ▶ for SIMPLE:

$$T_1 = \{(\langle a, b, c \rangle, \langle a', b', c' \rangle) \mid a > 0 \wedge a' < a\}$$

$$T_2 = \{(\langle a, b, c \rangle, \langle a', b', c' \rangle) \mid b > 0 \wedge b' < b\}$$

- ▶ at the heart of the TERMINATOR tool



TERMINATION BY DICKSON'S LEMMA

- ▶ each T_j shown well-founded thanks to a ranking function ρ_j into a wqo (S_j, \leq_j)
- ▶ map any sequence of program configurations

$$\mathbf{x} = x_0, x_1, \dots$$

to a sequence of tuples

$$\mathbf{y} = \langle \rho_1(x_0), \dots, \rho_k(x_0) \rangle, \langle \rho_1(x_1), \dots, \rho_k(x_1) \rangle, \dots$$

in $S_1 \times \dots \times S_k$

- ▶ \mathbf{y} is **bad**: if $i_1 < i_2$, there exists j s.t.

$$(x_{i_1}, x_{i_2}) \in R^+ \cap T_j \text{ but } \rho_j(x_{i_1}) \not\leq \rho_j(x_{i_2})$$



TERMINATION BY DICKSON'S LEMMA

- ▶ each T_j shown well-founded thanks to a ranking function ρ_j into a wqo (S_j, \leq_j)
- ▶ map any sequence of program configurations

$$\mathbf{x} = x_0, x_1, \dots$$

to a sequence of tuples

$$\mathbf{y} = \langle \rho_1(x_0), \dots, \rho_k(x_0) \rangle, \langle \rho_1(x_1), \dots, \rho_k(x_1) \rangle, \dots$$

in $S_1 \times \dots \times S_k$

- ▶ \mathbf{y} is **bad**: if $i_1 < i_2$, there exists j s.t.

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BOUNDS ON PROGRAM COMPLEXITY

Make some assumptions:

- ▶ complexity bound f on atomic program operations
 - ▶ for instance linear
- ▶ complexity bound ρ on ranking functions into \mathbb{N}
 - ▶ for instance linear
- ▶ \mathbf{y} controlled by $f \circ \rho$ in some \mathcal{F}_γ
 - ▶ in this case a linear function in \mathcal{F}_1
- ▶ time complexity in $\mathcal{F}_{\gamma+k}$
 - ▶ in this case \mathcal{F}_{k+1}
- ▶ matches the lower bound (expand SIMPLE to dimension k instead of 2)



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$$\mathbb{N}^2 / \langle 2, 2 \rangle \hookrightarrow \mathbb{N}/2 \times \mathbb{N} + \mathbb{N}/2 \times \mathbb{N}$$

Example

$$\mathbf{x} = \langle 2, 2 \rangle, \langle 1, 5 \rangle, \langle 4, 0 \rangle, \langle 1, 1 \rangle, \langle 0, 100 \rangle, \langle 0, 99 \rangle, \langle 3, 0 \rangle$$

$$\langle 2, 2 \rangle, \left[\begin{array}{ccc} \langle 1, 5 \rangle, & \langle 1, 1 \rangle, \langle 0, 100 \rangle, \langle 0, 99 \rangle, & (\{0, 1\} \times \mathbb{N}) \\ & \langle 4, 0 \rangle, & \langle 3, 0 \rangle (\{0, 1\} \times \mathbb{N}) \end{array} \right]$$



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ORDINAL TERMS FOR WQO'S

QUICK REMINDER

Definition (Ordinal Terms)

$$\alpha ::= 0 \mid \omega^\alpha \mid \alpha + \alpha$$

Definition (Cantor Normal Form)

$$\alpha = \omega^{\beta_1} + \dots + \omega^{\beta_m}$$

with $\alpha > \beta_1 \geq \dots \geq \beta_m \geq 0$ with each β_i in CNF itself.



ORDINAL TERMS FOR WQO'S

OPERATIONS ON CNF

Definition (Natural Sum)

$$\sum_{i=1}^m \omega^{\beta_i} \oplus \sum_{j=1}^n \omega^{\beta'_j} \stackrel{\text{def}}{=} \sum_{k=1}^{m+n} \omega^{\gamma_k},$$

where $\gamma_1 \geq \dots \geq \gamma_{m+n}$ is a rearrangement of $\beta_1, \dots, \beta_m, \beta'_1, \dots, \beta'_n$.

Definition (Natural Product)

$$\sum_{i=1}^m \omega^{\beta_i} \otimes \sum_{j=1}^n \omega^{\beta'_j} \stackrel{\text{def}}{=} \bigoplus_{i=1}^m \bigoplus_{j=1}^n \omega^{\beta_i \oplus \beta'_j}.$$



ORDINAL TERMS FOR WQO'S

MAXIMAL ORDER TYPE (DE JONGH AND PARIKH, 1977; HASEGAWA, 1994)

A bijection between normed WQO's with $+$, \times , and $*$ over finite sets, and $\text{CNF}(\omega^{\omega^\omega})$:

$$o(\Gamma_p) \stackrel{\text{def}}{=} p, \quad o(\Gamma_0^*) \stackrel{\text{def}}{=} \omega^0, \quad o(\Gamma_{p+1}^*) \stackrel{\text{def}}{=} \omega^{\omega^p},$$

$$o(A + B) \stackrel{\text{def}}{=} o(A) \oplus o(B), \quad o(A \times B) \stackrel{\text{def}}{=} o(A) \otimes o(B).$$

Example

$$o((\Gamma_{p+2}^*)^{k+1} \times \Gamma_{q+1}) = \omega^{\omega^{p+1} \cdot (k+1)} \cdot (q+1).$$



ORDINAL TERMS FOR WQO'S

- ▶ translate reflections of residuals into **derivative** ordinal terms
- ▶ $\forall n$, we define a well-founded relation ∂_n over $\text{CNF}(\omega^{\omega^\omega})$ s.t. if $a \in A_{<n}$, then $\exists \alpha' \in \partial_n o(A)$ s.t. $A/a \hookrightarrow o^{-1}(\alpha')$.

Example

$$\partial_n 0 = \emptyset, \quad \partial_n 1 = \{0\}, \quad \partial_n \omega = \{n-1\},$$

$$\begin{aligned} \partial_n(\omega^{\omega^{p+1} \cdot (k+1)} \cdot (q+1)) = & \{\omega^{\omega^{p+1} \cdot (k+1)} \cdot q \\ & + \omega^{[\omega^{p+1} \cdot k + \omega^p \cdot (n-1)]} \cdot (k+1)(n-1)\}. \end{aligned}$$



FUNDAMENTAL SEQUENCES

Subrecursive hierarchies are defined through an assignment of **fundamental sequences** $(\lambda_x)_{x < \omega}$ for limit ordinal terms λ , s.t. $\lambda_x < \lambda$ and $\lambda = \sup_x \lambda_x$: e.g.

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot (x + 1)$$

$$(\gamma + \omega^\lambda)_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x},$$

Example

$$\omega_x = x + 1$$

$$(\omega^{\omega^{p+1}})_x = \omega^{\omega^p \cdot (x+1)}$$



WELL-STRUCTURED TRANSITION SYSTEMS

- ▶ transition systems (Q, \rightarrow, q_0) with a wqo \leq on Q compatible with transitions:

$$\forall p, q, p' \in Q, (p \xrightarrow{a} q \wedge p \leq p') \Rightarrow \exists q', (q \leq q' \wedge p' \xrightarrow{a} q')$$

- ▶ a generic framework for decidability results: safety, termination, EF model checking, ...
- ▶ many classes of concrete systems are WSTS:
 - ▶ over (\mathbb{N}^k, \leq_x) : vector addition systems, resets/transfer Petri nets, increasing counter



EXAMPLE: (NON) TERMINATION

- ▶ given (Q, \rightarrow, q_0) , decide whether there exists an infinite run $q_0 \rightarrow q_1 \rightarrow \dots$
- ▶ holds iff there exists $q_i \leq q_j$ with $q_0 \rightarrow^* q_i \rightarrow^+ q_j$
- ▶ thanks to wqo, termination is both r.e. and co-r.e.
- ▶ what is the complexity?



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- ▶ **what is the complexity?**



EXAMPLE: LOSSY CHANNEL SYSTEMS

- ▶ $\langle Q, M, C, \delta \rangle$
- ▶ Q finite set of q states, M size- m message alphabet, C set of c channels, $\delta \subseteq Q \times (\{\text{nop}\} \uplus (\{!, ?\} \times M))^C \times Q$ set of transitions,
- ▶ configurations in $A = Q \times (M^*)^C$,
- ▶ WSTS for $(s, w_1, \dots, w_{|C|}) \rightarrow (s', w'_1, \dots, w'_{|C|})$ iff $\exists (s, e_1, \dots, e_{|C|}, s') \in \delta$ s.t. $\forall i$

$$\left\{ \begin{array}{ll} w'_i \leq_* w_i & \text{if } e_i = \text{nop} \\ \exists w \in M^*, w \leq_* w_i \wedge w'_i \leq_* aw & \text{if } e_i = !a \\ \exists w \in M^*, wa \leq_* w_i \wedge w'_i \leq_* w & \text{if } e_i = ?a \end{array} \right.$$



EXAMPLE: LOSSY CHANNEL SYSTEMS

- ▶ $o(A) = \omega^{\omega^{m-1} \cdot c} \cdot q$
- ▶ linear control by $g(x) = x + 1$ in \mathcal{F}_1
- ▶ non-terminating run from $s_{\text{init}} = (s_0, w_1, \dots, w_{|C|})$ iff there exists a run of length $L_{A,g}(|s_{\text{init}}|)$
- ▶ non termination in $\mathcal{F}_{\omega^{m-1} \cdot c}$



HARDY HIERARCHY: $(h^\alpha)_\alpha$

Fix $h : \mathbb{N} \rightarrow \mathbb{N}$:

$$h^0(x) \stackrel{\text{def}}{=} x, \quad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^\alpha(h(x)), \quad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda_x}(x).$$

Example

For $h(x) = x + 1$:

$$H^\omega(x) = H^{x+1}(x) = 2x + 1 \quad H^{\omega \cdot 2}(x) = H^{\omega+x+1}(x) = 4x + 3$$

Lemma

For all $r < \omega$, α , and x ,

$$h^{\omega^{\alpha \cdot r}}(x) = f_\alpha^r(x).$$



LENGTH HIERARCHY: $(h_\alpha)_\alpha$

Fix $h : \mathbb{N} \rightarrow \mathbb{N}$:

$$h_0(x) \stackrel{\text{def}}{=} 0, \quad h_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + h_\alpha(h(x)), \quad h_\lambda(x) \stackrel{\text{def}}{=} h_{\lambda_x}(x).$$

Lemma

For all α, x

$$h_\alpha(x) \leq h^\alpha(x) - x$$

Lemma

Define the *predecessor* at x of $\alpha > 0$ as

$$P_x(\alpha + 1) \stackrel{\text{def}}{=} \alpha, \quad P_x(\lambda) \stackrel{\text{def}}{=} P_x(\lambda_x)$$

$$\text{Then} \quad h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x)).$$



MONOTONICITY MATTERS

Lemma

For h monotone with $h(x) \geq x$ and any α ,

$$x < y \text{ implies } h^\alpha(x) \leq h^\alpha(y).$$

But: for $x < n$,

$$H^\omega(x) = 2x + 1 < x + n + 1 = H^{n+1}(x), \text{ i.e.}$$

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MONOTONICITY MATTERS

Definition (Pointwise Ordering)

For all x , \prec_x is the smallest transitive relation s.t.

$$\alpha \prec_x \alpha + 1$$

$$\lambda_x \prec_x \lambda.$$

Lemma

For h monotone with $h(x) \geq x$ and any x ,

$$\alpha \prec_x \beta \text{ implies } h^\alpha(x) \leq h^\beta(x).$$



COMPARISON WITH $(h_\alpha)_\alpha$

Contrast $M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha} \{1 + M_{\alpha',g}(g(n))\}$
 with $h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x))$:

Proposition

For all α in $\text{CNF}(\omega^{\omega^\omega})$, there is a constant k s.t.
 for all $n > 0$, $M_{\alpha,g}(n) \leq h_\alpha(kn)$ where
 $h(x) \stackrel{\text{def}}{=} x \cdot g(x)$.

Example (Higman's Lemma)

For bad (g,n) -controlled sequences in Γ_p^* :

$$L_{\Gamma_p^*,g}(n) \leq h_{\omega^{\omega^{p-1}}}((p-1)n) \quad \text{where } h(x) \stackrel{\text{def}}{=} x \cdot g(x).$$



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LOWER BOUND

Specific sequence, bad for $(\mathbb{N}^k, \leq_{\text{lex}})$, of length $\ell_{k,f}(t)$.

Example

$k = 2, t = 1, f(x) = x + 3$:

i	0	1	2	3	4	5	...	10	11	12	13	...	26	27	28	29	...	58	59
$x_i[1]$	3	3	3	3	2	2	...	2	2	1	1	...	1	1	0	0	...	0	0
$x_i[2]$	3	2	1	0	7	6	...	1	0	15	14	...	1	0	31	30	...	1	0
$f(i+t)$	4	5	6	7	8	9	...	14	15	16	17	...	30	31	32	33	...	62	63

$$5 = 1 + 4 = 1 + \ell_{1,f}(1)$$

$$13 = 5 + 8 = 5 + \ell_{1,f}(5)$$

$$29 = 16 + 13 = 13 + \ell_{1,f}(13)$$



LOWER BOUND

Specific sequence, bad for $(\mathbb{N}^k, \leq_{\text{lex}})$, of length $\ell_{k,f}(t)$. In general, on the $k+1$ th coordinate:

$$\underbrace{f(t)-1, f(t)-1, \dots, f(t)-1}_{\ell_{k,f}(t) \text{ times}} \quad \underbrace{f(t)-2, f(t)-2, \dots, f(t)-2}_{\ell_{k,f}(o_{k,f}(t)) \text{ times}}$$

$$\dots \quad \underbrace{0, 0, \dots, 0}_{\ell_{k,f}(o_{k,f}^{f(t)-1}(t)) \text{ times}}$$

$$o_{k,f}(t) \stackrel{\text{def}}{=} t + \ell_{k,f}(t)$$

$$\ell_{k+1,f}(t) = \sum_{j=1}^{f(t)} \ell_{k,f}(o_{k,f}^{j-1}(t))$$



LOWER BOUND

Specific sequence, bad for $(\mathbb{N}^k, \leq_{\text{lex}})$, of length $\ell_{k,f}(t)$. One can have $\ell_{k,f}(t) < L(\omega^k, t)$: let $f(x) = 2x$ and $t = 1$,

$\langle 1, 1 \rangle, \langle 1, 0 \rangle, \langle 0, 5 \rangle, \langle 0, 4 \rangle, \langle 0, 3 \rangle, \langle 0, 2 \rangle, \langle 0, 1 \rangle, \langle 0, 0 \rangle$
 $\langle 1, 1 \rangle, \langle 0, 3 \rangle, \langle 0, 2 \rangle, \langle 0, 1 \rangle, \langle 9, 0 \rangle, \langle 8, 0 \rangle, \langle 7, 0 \rangle, \langle 6, 0 \rangle, \langle 5, 0 \rangle, \dots, \langle 0, 0 \rangle$

$$\ell_{2,f}(1) = 8$$

$$L_{\omega^2,f}(1) \geq 14$$