Ideal Decompositions
for Vector Addition Systems

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OUTLINE

- **vector addition systems (VAS)** and their reachability problem
- **ideals** of well-quasi-orders
- a counter-example guided abstraction refinement (CEGAR) procedure
- the **KLMST decomposition algorithm** named after ?, ?, ?, and ?
Vector Addition Systems (VAS)

Syntax

- dimension $d \in \mathbb{N}$

- finite set $A \subseteq_{\text{fin}} \mathbb{Z}^d$ of actions $a \in A$

Semantics

- configurations $u, v, \ldots \in \mathbb{N}^d$

- transitions $u \xrightarrow{a} v \in \mathbb{N}^d \times A \times \mathbb{N}^d$ with $v = u + a$
**Vector Addition Systems (VAS)**

(?)

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Example VAS

d = 2

A = \{ \text{data} \}

\begin{align*}
(0, 2) &\rightarrow (2, 4) \\
(2, 4) &\rightarrow (3, 5) \\
(3, 5) &\rightarrow (4, 6) \\
(4, 6) &\rightarrow (3, 4) \\
(3, 4) &\rightarrow (2, 2) \\
(2, 2) &\rightarrow (0, 1)
\end{align*}
**Example VAS**

**Example**

\[ d = 2 \]

\[ A = \{ \text{fig:arrow} \} \]

\[ x = (0, 2) \]
\[ d = 2 \quad \text{and} \quad A = \left\{ \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 2 \end{pmatrix} \right\} \]

\[ x = (0, 2) \xrightarrow{\begin{pmatrix} -1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}} \in \mathbb{N}^2 \]
Example VAS

Example

d = 2

A = \{ (0,2) \rightarrow (1,3) \}

x = (0,2) \rightarrow (1,3)
**Example VAS**

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\[ d = 2 \]

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Example VAS

\[ d = 2 \]

\[ A = \{ \ldots \} \]

\[ x = (0,2) \rightarrow (1,3) \rightarrow (2,4) \rightarrow (3,5) \]
**Example VAS**

\[ d = 2 \quad A = \{ \text{example} \} \]

\[ x = (0,2) \rightarrow (1,3) \rightarrow (2,4) \rightarrow (3,5) \rightarrow (4,6) \]
**Example VAS**

\[ d = 2 \]

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Example VAS

\[ d = 2 \]

\[ A = \left\{ \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \right\} \]

\[ x = (0,2) \rightarrow (1,3) \rightarrow (2,4) \rightarrow (3,5) \rightarrow (4,6) \rightarrow (3,4) \rightarrow (2,2) \rightarrow (0,1) = y \]
**Runs and Preruns**

**Definition (Prerun)**

A **prerun** is an element

\[(u, (u_1, a_1, v_1) \cdots (u_k, a_k, v_k), v)\]

from \(\text{PreRuns}_A \overset{\text{def}}{=} \mathbb{N}^d \times (\mathbb{N}^d \times A \times \mathbb{N}^d)^* \times \mathbb{N}^d\)

**Definition (Run)**

A prerun is **connected** (is a run) if

- (source) \(u = u_1\)
- (transitions) \(\forall 1 \leq j \leq k, u_j + a_j = v_j\)
- (contiguity) \(\forall 1 < j \leq k, v_{j-1} = u_j\)
- (target) \(v_k = v\)
The Reachability Problem

\( \text{Runs}_A(x, y) \overset{\text{def}}{=} \{ \rho \in \text{PreRuns}_A \mid \rho \text{ is a run with source } x \text{ and target } y \} \)

**VAS Reachability**

input \( A \subseteq_{\text{fin}} \mathbb{Z}^d, x, y \in \mathbb{N}^d \)

question Is \( y \) reachable from \( x \) in \( A \)?

l.e., is \( \text{Runs}_A(x, y) \neq \emptyset \)?

**Theorem** (?; ?; ?; ?; ?; ?; ?; ?)

VAS Reachability is decidable.

- by the KLMST decomposition algorithm (???)
- by Presburger invariants (?)
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**DECOMPOSITION THEOREM**

**Theorem (?,?,?)**

The KLMST decomposition algorithm computes the ideal decomposition of

\[ \downarrow \text{Runs}_A(x,y) \overset{\text{def}}{=} \{ \rho' \in \text{PreRuns}_A \mid \exists \rho \in \text{Runs}_A(x,y). \rho' \preceq \rho \} \]

- entails decidability of VAS Reachability:

  \[ \text{Runs}_A(x,y) = \emptyset \text{ iff } \downarrow \text{Runs}_A(x,y) = \emptyset \]

**Upcoming**

- definition of a wqo over preruns (??)

- wqo ideals (??)
**Decomposition Theorem**

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**Well-Quasi-Orders (wqo)**

**Definition**

A quasi-order \((X, \leq)\) is a wqo if in any infinite sequence \(x_0, x_1, \ldots\) of elements of \(X\), \(\exists i < j \text{ s.t. } x_i \leq x_j\).

**Example**

- finite sets with equality \((X, =)\)
- natural numbers \((\mathbb{N}, \leq)\)
- Dickson’s Lemma: if \((A, \leq_A)\) and \((B, \leq_B)\) are wqos, then \((A \times B, \leq_x)\) is a wqo, where \((a, b) \leq_x (a', b')\) iff \(a \leq_A a'\) and \(b \leq_B b'\)
- Higman’s Lemma: if \((A, \leq)\) is a wqo, then \((A^*, \leq_*)\) is a wqo, where \(u \leq_* v\) iff \(u = a_1 \cdots a_k\) and \(v = v_0 b_1 v_1 \cdots v_{k-1} b_k v_k\) with \(v_0, \ldots, v_k \in A^*\) and \(\forall 1 \leq j \leq k. a_j \leq b_j \in A\).
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Prerun Embeddings

- \((\mathbb{N}^d, \preceq)\) is a wqo for the componentwise ordering
  - \((\mathbb{N}^d \times A \times \mathbb{N}^d, \preceq)\) is a wqo, where 
    \((u, a, v) \preceq (u', b, v')\) if \(u \leq u', a = b\), and \(v \leq v'\)
  - \(((\mathbb{N}^d \times A \times \mathbb{N}^d)^*, \preceq_*)\) is a wqo
  - ?: \((\text{PreRuns}_A, \preceq)\) is a wqo, where 
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Characterising WQOs

Upward closure: $\uparrow S \overset{\text{def}}{=} \{ x \in X \mid \exists s \in S . s \leq x \}.$

**Lemma (Minimal Basis Property)**

A qo $(X, \leq)$ is a wqo iff every non-empty subset $S \subseteq X$ has a finite set of minimal elements $\min_{\leq} S.$

**Lemma (Ascending Chain Property)**

A qo $(X, \leq)$ is a wqo iff every ascending chain $U_0 \subsetneq U_1 \subsetneq \cdots$ of upward-closed sets is finite.

Template for many algorithms: represent the sets $U_n$ as $\uparrow (\min_{\leq} U_n)$ using finitely many elements.
**Characterising WQOs**

Downward closure: \( \downarrow S \overset{\text{def}}{=} \{ x \in X \mid \exists s \in S . x \leq s \} \).

**Lemma (Minimal Basis Property)**

A qo \((X, \leq)\) is a wqo iff every non-empty subset \(S \subseteq X\) has a finite set of minimal elements \(\text{min}< S\).

**Lemma (Descending Chain Property)**

A qo \((X, \leq)\) is a wqo iff every descending chain \(D_0 \supsetneq D_1 \supsetneq \cdots\) of downward-closed sets is finite.

Template for many algorithms: represent the sets \(U_n\) as \(\uparrow(\text{min}< U_n)\) using finitely many elements.
Ideals as Canonical Bases

Downward closure: $\downarrow S \overset{\text{def}}{=} \{ x \in X \mid \exists s \in S . x \leq s \}$.

**Lemma (Canonical Ideal Decomposition; ?, ?)**

Every downward-closed subset $D \subseteq X$ of a wqo $(X, \leq)$ is the union of a unique finite family of incomparable (for the inclusion) ideals.

**Lemma (Descending Chain Property)**

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IDEALS

(??)

- Directed set $\Delta$
  non-empty and $\forall x_1, x_2 \in I$, $\exists x. x_1 \leq x$ and $x_2 \leq x$
- Ideal $I$
downwards-closed and directed
- Examples
  - $\downarrow x \in \text{Idl}(X)$ for any $x$ in $X$
  - $\mathbb{N} \in \text{Idl}(\mathbb{N})$
  - $\{a, b\}^* \in \text{Idl}(\{a, b, c\}^*)$
- Canonical Decompositions
  if $D \subseteq X$ is downwards-closed, then $D = I_1 \cup \cdots \cup I_n$
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- **Canonical Decompositions**
  - If \( D \subseteq X \) is downwards-closed, then \( D = I_1 \cup \cdots \cup I_n \)
**Effectivity**

- represent canonical decompositions $D = I_1 \sqcup \cdots \sqcup I_k$ where the $I_j$’s are **maximal** for inclusion

- must allow effective operations over ideals: $I \subseteq J$, $I \cap J$, $I \setminus \uparrow x$ for $x \in X$

- ???: effective representations exist for all the wqos in this talk

- for Cartesian products:
  $\text{Idl}(A \times B) = \{I \times J \mid I \in \text{Idl}(A) \text{ and } J \in \text{Idl}(B)\}$

- for finite sequences: $\text{Idl}(X^*)$ are **products** defined by:
  
  $$P ::= \varepsilon \mid A \cdot P$$  \hspace{1cm} (products)

  $$A ::= (I + \varepsilon) \mid (I_1 \sqcup \cdots \sqcup I_n)^*$$  \hspace{1cm} (atoms)

  where $I, I_1, \ldots, I_n$ range over $\text{Idl}(X)$
Effectivity

- represent canonical decompositions $D = I_1 \sqcup \cdots \sqcup I_k$ where the $I_j$’s are maximal for inclusion
- must allow effective operations over ideals: $I \subseteq J$, $I \cap J$, $I \setminus \uparrow x$ for $x \in X$
- ??: effective representations exist for all the wqos in this talk
- for Cartesian products:
  $\text{Idl}(A \times B) = \{I \times J \mid I \in \text{Idl}(A) \text{ and } J \in \text{Idl}(B)\}$
- for finite sequences: $\text{Idl}(X^*)$ are products defined by:
  \[
P ::= \varepsilon \mid A \cdot P \hspace{10cm} (\text{products})
  \]
  \[
A ::= (I + \varepsilon) \mid (I_1 \sqcup \cdots \sqcup I_n)^* \hspace{10cm} (\text{atoms})
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**Effectivity**

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where $I, I_1, \ldots, I_n$ range over $\text{Idl}(X)$
An Abstraction Refinement Procedure (CEGAR)

Build a sequence $D_0 \supseteq D_1 \supseteq \cdots$ of $\downarrow$-closed sets s.t.

$$\forall n. \downarrow \text{Runs}_A(x, y) \subseteq D_n$$

initially $D_0 \overset{\text{def}}{=} \text{PreRuns}_A$

$$\forall n \quad \text{if } D_n = I \sqcup D \text{ and }$$
$$\exists p \in I \setminus \downarrow \text{Runs}_A(x, y),$$

$$D_{n+1} \overset{\text{def}}{=} D \cup (I \setminus \uparrow p)$$

Otherwise stop:

$$D_n = \downarrow \text{Runs}_A(x, y)$$

terminates by Descending Chain Property
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**Containment Oracles**

**Ideal Containment (into VAS Runs) Problem**

**Input** \( A \subseteq_{\text{fin}} \mathbb{Z}^d, x, y \in \mathbb{N}^d, I \in \text{Idl}(\text{PreRuns}_A) \)

**Question** \( \exists \rho \in I \setminus \downarrow \text{Runs}_A(x, y) \)?

**Proposition**

VAS Reachability reduces to Ideal Containment.

**Proof.**

Because \( \downarrow (0, \epsilon, 0) \subseteq \downarrow \text{Runs}_A(x, y) \) iff \( \text{Runs}_A(x, y) \neq \emptyset \).

**Proposition**

Ideal Containment is decidable.

**Proof.**

Consequence of the Decomposition Theorem.
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**Proof.**

Consequence of the Decomposition Theorem.
Adherence Oracles

Adherence (of VAS Runs) Membership Problem

input $A \subseteq_{\text{fin}} \mathbb{Z}^d, x, y \in \mathbb{N}^d, I \in \text{Idl(PreRuns}_A)$

question $\exists \Delta \subseteq \text{Runs}_A(x, y)$ directed s.t. $\downarrow\Delta = I$?

Claim
In the context of the CEGAR procedure, containment checks are equivalent to adherence membership checks.

Theorem
Adherence Membership is undecidable.

Proof Idea.
By a reduction from Boundedness in Lossy Counter Machines.
### Adherence Oracles

**Adherence (of VAS Runs) Membership Problem**

**Input**

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**Question**

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In the context of the CEGAR procedure, containment checks are equivalent to adherence membership checks.

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How to Salvage the CEGAR Procedure?

- both containment and adherence miss a crucial point: if $\downarrow \text{Runs}_A(x, y) = D_n = I \sqcup D$, then $I$ is some maximal ideal of $\downarrow \text{Runs}_A(x, y)$

- find ‘nice’ invariants of such ideals:

  initially $D_0 \overset{\text{def}}{=} \text{PreRuns}_A$ is nice

  $\forall n$ if $D_n = I \sqcup D$ and $\exists \rho \in I \setminus \downarrow \text{Runs}_A(x, y)$, which is decidable:

  $D_{n+1} \overset{\text{def}}{=} D \cup (I \uparrow \rho)$

  otherwise stop:

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- template for the KLMST decomposition algorithm
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- template for the KLMST decomposition algorithm
Run Embeddings

Fix $\rho = c_0 \xrightarrow{a_1} c_1 \cdots c_{k-1} \xrightarrow{a_k} c_k$ from $\text{Runs}_A(x,y)$

If $\rho' \succeq \rho$ is a run, $\exists v_0, \ldots, v_{k+1} \in \mathbb{N}^d$ and $\sigma_0, \ldots, \sigma_k \in A^*$:

$$\rho' = (v_0 + c_0) \xrightarrow{\sigma_0} (v_1 + c_0) \xrightarrow{a_1} (v_1 + c_1) \cdots (v_k + c_{k-1}) \xrightarrow{a_k} (v_k + c_k) \xrightarrow{\sigma_k} (v_{k+1} + c_k)$$

Lemma (Run Amalgamation) 

If $\rho \preceq \rho_1, \rho_2$ are runs, then there exists a run $\rho' \succeq \rho_1, \rho_2$. 
**Run Embeddings**

\[
(3,3) \rightarrow (2,1) \rightarrow (3,2) \rightarrow (2,0) \rightarrow (3,1)
\]

Fix \( \rho = c_0 \xrightarrow{a_1} c_1 \cdots c_{k-1} \xrightarrow{a_k} c_k \) from \( \text{Runs}_A(x,y) \)

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\[
\rho' = (v_0+c_0) \xrightarrow{\sigma_0} (v_1+c_0) \xrightarrow{a_1} (v_1+c_1) \cdots (v_k+c_{k-1}) \xrightarrow{a_k} (v_k+c_k) \xrightarrow{\sigma_k} (v_{k+1}+c_k)
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**Lemma (Run Amalgamation)**

If \( \rho \preceq \rho_1, \rho_2 \) are runs, then there exists a run \( \rho' \succeq \rho_1, \rho_2 \).
**Run Embeddings**

Fix $\rho = c_0 \xrightarrow{a_1} c_1 \cdots c_{k-1} \xrightarrow{a_k} c_k$ from $\text{Runs}_A(x,y)$

If $\rho' \geq \rho$ is a run, $\exists v_0, \ldots, v_{k+1} \in \mathbb{N}^d$ and $\sigma_0, \ldots, \sigma_k \in A^*$:

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**Lemma (Run Amalgamation)**

*If $\rho \preceq \rho_1, \rho_2$ are runs, then there exists a run $\rho' \supseteq \rho_1, \rho_2$.***
Maximal Run Ideals (1/2)

Since $\leq$ is a wqo, $B \overset{\text{def}}{=} \min_{\leq} \text{Runs}_A(x, y)$ is finite:

$$\downarrow \text{Runs}_A(x, y) = \bigcup_{\rho \in B} \downarrow (\uparrow \rho \cap \text{Runs}_A(x, y))$$

For any run $\rho$, $\downarrow (\uparrow \rho \cap \text{Runs}_A(x, y))$ is

- non-empty: it contains at least $\rho$
- directed by run amalgamation
- downward-closed by definition

Proposition

The maximal ideals of $\downarrow \text{Runs}_A(x, y)$ are the ideals of the form $\downarrow (\uparrow \rho \cap \text{Runs}_A(x, y))$ for $\rho \in \text{Runs}_A(x, y)$. 
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Maximal Run Ideals (2/2)

Transformer Relations

- $\mathbb{A} \triangleq \{(u, v) \mid \exists \sigma \in A^* . u + c \xrightarrow{\sigma} v + c\}

- $\mathbb{A}$ is periodic: it contains 0, and if $u \mathbb{A} v$ and $u' \mathbb{A} v'$, then $u + u' \mathbb{A} v + v'$

Decomposition of $\uparrow \rho \cap \text{Runs}_A(x, y)$

- let $\rho = c_0 \xrightarrow{a_1} c_1 \cdots c_{k-1} \xrightarrow{a_k} c_k$

- consider all the $(k+1)$-tuples $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ s.t. $v_0 \mathbb{A} v_1 \mathbb{A} \cdots \mathbb{A} v_k$

- every projection $P_j \triangleq \{(v_j, v_{j+1}) \mid \ldots\}$ is also periodic

- define $\Omega_j$ as the set of runs $v_j + c_j \xrightarrow{\sigma_j} v_{j+1} + c_j$ for each $j$
Maximal Run Ideals (2/2)

Transformer Relations

- $\overset{c}{\rightsquigarrow} \overset{\text{def}}{=} \{(u, v) \mid \exists \sigma \in A^* . u + c \overset{\sigma}{\Rightarrow} v + c\}$

- $\overset{c}{\rightsquigarrow}$ is periodic: it contains 0, and if $u \overset{c}{\rightsquigarrow} v$ and $u' \overset{c}{\rightsquigarrow} v'$, then $u + u' \overset{c}{\rightsquigarrow} v + v'$

Decomposition of $\uparrow \rho \cap \text{Runs}_A(x, y)$

- Let $\rho = c_0 \overset{a_1}{\rightarrow} c_1 \cdots c_{k-1} \overset{a_k}{\rightarrow} c_k$

- Consider all the $(k + 1)$-tuples $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ s.t. $v_0 \overset{c_0}{\rightsquigarrow} v_1 \overset{c_1}{\rightsquigarrow} \cdots \overset{c_k}{\rightsquigarrow} v_k$

- Every projection $P_j \overset{\text{def}}{=} \{(v_j, v_{j+1}) \mid \ldots\}$ is also periodic

- Define $\Omega_j$ as the set of runs $v_j + c_j \overset{\sigma_j}{\Rightarrow} v_{j+1} + c_j$ for each $j$
**Marked Witness Graphs**

**Example**

\[ A = \{a, b\} \text{ where } a = (1, 1, -1) \quad b = (-1, 0, 1) \]

\[ c_j = (1, 0, 1) \quad P_j = \{((0, 0, 0), (0, n, 0)) \mid n \in \mathbb{N}\} \]

\[ \Omega_j = \{c_j \xrightarrow{w_1 \cdots w_n} c_j + (0, n, 0) \mid n \in \mathbb{N}, w_i \in \{ab, ba\}\} \]
MARKED WITNESS GRAPHS

Each $\Omega_j$ can be represented as a finite marked witness graph $M_j$.

**Example**

$A = \{a, b\}$ where $a = (1, 1, -1)$, $b = (-1, 0, 1)$

$c_j = (1, 0, 1)$, $P_j = \{(0, 0, 0), (0, n, 0)\} | n \in \mathbb{N}$

$\Omega_j = \{c_j \xrightarrow{w_1\ldots w_n} c_j + (0, n, 0) | n \in \mathbb{N}, w_i \in \{ab, ba\}\}$
Marked Witness Graph Sequences

Back to $\rho = c_0 \xrightarrow{a_1} c_1 \cdots c_{k-1} \xrightarrow{a_k} c_k$:

- $\uparrow \rho \cap \text{Runs}_A(x,y)$ can be represented using a sequence of marked witness graphs and actions from $A$:

$$\xi = M_0, a_1, M_1, \ldots, a_k, M_k$$

- Conversely, each such sequence defines an associated set of runs $\Omega_\xi$ and an associated prerun ideal $I_\xi$.

- Conditions on such sequences:
  - Consistent markings (？)
  - θ condition (？)
  - Perfectness condition (？)

Lemma (Perfectness implies Adherence Membership)

If $\xi$ is perfect then $I_\xi = \downarrow \Omega_\xi$. 
Marked Witness Graph Sequences

Back to $\rho = c_0 \xrightarrow{a_1} c_1 \cdots c_{k-1} \xrightarrow{a_k} c_k$:

- $\uparrow \rho \cap \text{Runs}_A(x, y)$ can be represented using a sequence of marked witness graphs and actions from $A$:

\[ \xi = M_0, a_1, M_1, \ldots, a_k, M_k \]

- Conversely, each such sequence defines an associated set of runs $\Omega_\xi$ and an associated prerun ideal $I_\xi$.

- Conditions on such sequences:
  - Consistent markings (?)
  - $\emptyset$ condition (?)
  - Perfectness condition (?)

**Lemma (Perfectness implies Adherence Membership)**

*If $\xi$ is perfect then $I_\xi = \downarrow \Omega_\xi$.**
Marked Witness Graph Sequences

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- Perfectness condition on such sequences

Lemma (Perfectness implies Adherence Membership)

If $\xi$ is perfect then $I_\xi = \downarrow \Omega_\xi$.

Theorem

There exists a finite set $\Xi$ of perfect marked witness graph sequences s.t. $\downarrow \text{Runs}_A(x,y) = \bigcup_{\xi \in \Xi} I_\xi$. 
KLMST Algorithm (Schematically)

Construct a sequence $\Xi_0, \Xi_1, \ldots$ of finite sets of marked witness graph sequences with $\forall n$

$$D_n \overset{\text{def}}{=} \bigcup_{\xi \in \Xi_n} I_{\xi} \supset \downarrow \text{Runs}_A(x, y)$$

initially $\Xi_0$ is s.t. $D_0 = \text{PreRuns}_A$

$\forall n \quad$ if $\Xi_n = \{\xi\} \cup \Xi$ and
$\xi$ is not perfect, which is decidable,

$\Xi_{n+1} \overset{\text{def}}{=} \Xi \cup (\text{decompose}(\xi))$

$\quad$ otherwise stop:

$D_n = \downarrow \text{Runs}_A(x, y)$

terminates via a ranking function argument
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\[\forall n \quad \text{if } \Xi_n = \{\xi\} \cup \Xi \text{ and } \xi \text{ is not perfect, which is decidable, then }\]

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**Concluding Remarks**

- ideals as an *algorithmic* tool to work with downward-closed sets

- new *understanding* of the KLMST decomposition
  extension to other models (BVASS, PDVAS,…)?

- complexity of VAS Reachability:
  - PSPACE-complete with states if $d = 2$ (?)
  - EXPSPACE-hard (?) and in $F_{\omega^3}$ (?) in general

- to learn more: references in the next slide and 
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