Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory
Part I: Basics of WQO Theory

Philippe Schnoebelen

LSV, CNRS & ENS Cachan

Chennai Mathematical Institute, Jan. 2017

Based on joint work with Sylvain Schmitz, Prateek Karandikar, K. Narayan Kumar, Alain Finkel, ..

or via www.lsv.ens-cachan.fr/~phs
MOTIVATIONS FOR THE COURSE

▶ Well-quasi-orderings (wqo’s) proved to be a powerful tool for decidability/termination in logic, AI, program verification, etc. NB: they can be seen as a version of well-orderings with more flexibility.

▶ In program verification, wqo’s are prominent in well-structured transition systems (WSTS’s), a generic framework for infinite-state systems with good decidability properties.

▶ Analysing the complexity of wqo-based algorithms is still one of the dark arts . . .

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OUTLINE OF THE COURSE

▶ (This) Lecture 1 = Basics of WQO’s. Rather basic material: explaining and illustrating the definition of wqo’s. Building new wqo’s from simpler ones.

▶ Lecture 2 = WQO-based reasoning. Well-Structured Transition Systems (WSTS’s), termination proofs, decidable logics, etc.

▶ Lecture 3 = Fast-growing complexity I. The Fast-growing hierarchy, Length function theorems for proving upper bounds.

▶ Lecture 4 = Fast-growing complexity II. Hardy computations for proving lower bounds.

▶ Lecture 5 = Ideals of WQO’s. Basic concepts, Effective representations, Algorithms.
**Def.** A non-empty \((X, \leq)\) is a **quasi-ordering** (qo) \(\overset{\text{def}}{\implies} \leq\) is a reflexive and transitive relation.

\(\approx\) a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent

**Examples.** \((\mathbb{N}, \leq)\), also \((\mathbb{R}, \leq)\), \((\mathbb{N} \cup \{ \omega \}, \leq)\), \ldots

divisibility: \((\mathbb{Z}, \_ \mid \_)\) where \(x \mid y \overset{\text{def}}{\implies} \exists a : a.x = y\)
tuples: \((\mathbb{N}^3, \leq_{\text{prod}})\), or simply \((\mathbb{N}^3, \leq_X)\), where \((0, 1, 2) <_X (10, 1, 5)\) and \((1, 2, 3) \#_X (3, 1, 2)\).

words: \((\Sigma^*, \leq_{\text{pref}})\) for some alphabet \(\Sigma = \{a, b, \ldots\}\) and \(ab \leq_{\text{pref}} abba\). 
\((\Sigma^*, \leq_{\text{lex}})\) with e.g. \(abba \leq_{\text{lex}} abc\) (NB: this assumes \(\Sigma\) is linearly ordered: \(a < b < c\))

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(Recalls) Ordered Sets

**Def.** A non-empty \((X, \leq)\) is a quasi-ordering \((qo)\) if \(\leq\) is a reflexive and transitive relation.

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Def. \((X, \leq)\) is linear, a.k.a. total, if for any \(x, y \in X\) either \(x \leq y\) or \(y \leq x\). (i.e., there is no \(x \# y\).)

Def. \((X, \leq)\) is well-founded if there is no infinite strictly decreasing sequence \(x_0 > x_1 > x_2 > \cdots\)

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WELL-QUASI-ORDERING (WQO)

**Def1.** \((X, \leq)\) is a wqo \(\overset{\text{def}}{\iff}\) any infinite sequence \(x_0, x_1, x_2, \ldots\) contains an **increasing pair**: \(x_i \leq x_j\) for some \(i < j\).

**Def2.** \((X, \leq)\) is a wqo \(\overset{\text{def}}{\iff}\) any infinite sequence \(x_0, x_1, x_2, \ldots\) contains an **infinite increasing subsequence**: \(x_{n_0} \leq x_{n_1} \leq x_{n_2} \leq \ldots\)

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**Fact.** These three definitions are equivalent. Clearly, Def2 \(\Rightarrow\) Def1 and Def1 \(\Rightarrow\) Def3. But the reverse implications are non-trivial.

To show Def3 \(\Rightarrow\) Def2, we first recall the **Infinite Ramsey Theorem**: “Let \(X\) be some countably infinite set and colour the elements of \(X^{(n)}\) (the subsets of \(X\) of size \(n\)) in \(c\) different colours. Then there exists some infinite subset \(M\) of \(X\) s.t. the size \(n\) subsets of \(M\) all have the same colour.”
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**Proving Def3 \(\Rightarrow\) Def2**

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What color?
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**Blue**: infinite strictly decreasing sequence, contradicts WF
Proving Def3 $\Rightarrow$ Def2

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Red $\Rightarrow$ infinite antichain, contradicts FAC
Infinite Ramsey Theorem:
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Must be green \( \Rightarrow \) infinite increasing sequence! QED
## Spot the WQO’s

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More generally

**Fact.** For **linear** qo’s: well-founded \(\iff\) wqo.

**Cor.** Any ordinal is wqo.
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\((\mathbb{Z}, |\cdot|)\): The prime numbers \(\{2, 3, 5, 7, 11, \ldots\}\) are an infinite antichain.
### Spot the WQO’s

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More generally

**Generalized Dickson’s lemma.** If \((X_1, \leq_1), \ldots, (X_n, \leq_n)\)’s are wqo’s, then \(\prod_{i=1}^{n} X_i, \leq \times\) is wqo.

**Proof.** Easy with Def2. Otherwise, an application of the Infinite Ramsey Theorem.

**Usual Dickson’s Lemma.** \((\mathbb{N}^k, \leq \times)\) is wqo for any \(k\).
**Spot the WQO’s**

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<td>$\times$</td>
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$(\Sigma^*, \leq_{\text{pref}})$ has an infinite antichain

$b, ab, aab, aaab, \ldots$

$(\Sigma^*, \leq_{\text{lex}})$ is not well-founded:

$b >_{\text{lex}} ab >_{\text{lex}} aab >_{\text{lex}} aaab >_{\text{lex}} \ldots$
### Spot the WQO’s

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\((\Sigma^*, \leq_*) \) is WQO by Higman’s Lemma (see next slide).

We can get some feeling by trying to build a bad sequence, i.e., some \( \omega_0, \omega_1, \omega_2, \ldots \) without an increasing pair \( \omega_i \leq_* \omega_j \).
**Higman’s Lemma**

**Def.** The **sequence extension** of a qo \((X, \leq)\) is the qo \((X^*, \leq_{*})\) of finite sequences over \(X\) ordered by embedding:

\[
\omega = x_1 \ldots x_n \leq_{*} y_1 \ldots y_m = \nu \iff x_1 \leq y_{l_1} \land \ldots \land x_n \leq y_{l_n}
\]

for some \(1 \leq l_1 < l_2 < \ldots < l_n \leq m\)

\[
\iff \omega \leq_{\times} \nu' \text{ for a length-}n \text{ subsequence} \nu' \text{ of} \nu
\]

**Higman’s Lemma.** \((X^*, \leq_{*})\) is a wqo iff \((X, \leq)\) is.

With \((\Sigma^*, \leq_{*})\), we are considering the sequence extension of \((\Sigma, =)\) which is finite, hence necessarily wqo.

Later we’ll consider the sequence extension of more complex wqo’s, e.g., \(\mathbb{N}^2:\)

| 0 | 2 | 0 | \leq_{*} | 2 | 0 | 0 | 2 | 2 | 2 | 0 | 1 |
**Higman’s Lemma**

**Def.** The sequence extension of a qo \((X, \leq)\) is the qo \((X^*, \leq_*)\) of finite sequences over \(X\) ordered by embedding:

\[ w = x_1 \ldots x_n \leq_* y_1 \ldots y_m = v \iff x_1 \leq y_{l_1} \land \ldots \land x_n \leq y_{l_n} \]

for some \(1 \leq l_1 < l_2 < \ldots < l_n \leq m \)

\[ \iff w \leq_\times v' \] for a length-\(n\) subsequence \(v'\) of \(v\)

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\[
\begin{array}{ccc}
0 & 2 & 0 \\
1 & 0 & 2 \\
\end{array} \leq_* ? \begin{array}{ccc}
2 & 0 & 2 \\
0 & 2 & 2 \\
0 & 2 & 0 & 1 \\
\end{array}
\]
PROOF OF HIGMAN’S LEMMA

Let \((X, \leq)\) be wqo and assume by way of contradiction that \((X^*, \leq^*)\) admits infinite bad sequences (sequences with no increasing pairs).

Let \(w_0 \in X^*\) be a shortest word that can start an infinite bad sequence.

Let \(w_1 \in X^*\) be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with \(w_0, w_1\).

Continue. This way we pick an infinite sequence \(S = w_0, w_1, w_2, w_3, \ldots\)

**Claim.** \(S\) too is bad

Write \(w_i\) under the form \(w_i = x_i v_i\). Since \(X\) is wqo, there is an infinite increasing sequence \(x_{n_0} \leq x_{n_1} \leq x_{n_2} \leq \cdots\) (here we use Def2)

Now consider \(S' \overset{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots\)

It cannot be bad (otherwise \(w_{n_0}\) would not have been shortest).

But an increasing pair like \(v_n \leq^* v_m\) in \(S'\) leads to \(x_n v_n \leq^* x_m v_m\), i.e., \(w_n \leq^* w_m\), a contradiction.
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MORE WQO’S

- Finite Trees ordered by embeddings (Kruskal’s Tree Theorem)
Proof of Kruskal’s Tree Theorem

Let \((X, \leq)\) be wqo and assume, b.w.o.c., that \((\mathcal{T}(X), \sqsubseteq)\) is not wqo.

We pick a “minimal” bad sequence \(S = t_0, t_1, t_2, \ldots\)

Write every \(t_i\) under the form \(t_i = f_i(u_{i,1}, \ldots, u_{i,k_i}).\)

Claim. The set \(U = \{u_{i,j}\}\) of the immediate subterms is wqo.
(Indeed, an infinite bad sequence \(u_{i_0,j_0}, u_{i_1,j_1}, \ldots\) could be used to show that \(t_{i_0}\) was not “shortest”).

Since \(U\) is wqo, and using Higman’s Lemma on \(U^*\), there is some
\((u_{n_1,1}, \ldots, u_{n_1,k_{n_1}}) \leq^* (u_{n_2,1}, \ldots, u_{n_2,k_{n_2}}) \leq^* (u_{n_3,1}, \ldots, u_{n_3,k_{n_3}}) \leq^* \ldots\)

Further extracting some \(f_{n_1} \leq f_{n_2} \leq \ldots\) exhibits an infinite increasing subsequence \(t_{n_1} \sqsubseteq t_{n_2} \sqsubseteq \ldots\) in \(S\), a contradiction.
PROOF OF KRUSKAL’S TREE THEOREM

Let $(X, \preceq)$ be wqo and assume, b.w.o.c., that $(\mathcal{T}(X), \sqsubseteq)$ is not wqo.

We pick a “minimal” bad sequence $S = t_0, t_1, t_2, \ldots$

Write every $t_i$ under the form $t_i = f_i(u_{i,1}, \ldots, u_{i,k_i})$.

Claim. The set $U = \{u_{i,j}\}$ of the immediate subterms is wqo.
(Indeed, an infinite bad sequence $u_{i_0,j_0}, u_{i_1,j_1}, \ldots$ could be used to show that $t_{i_0}$ was not “shortest”).

Since $U$ is wqo, and using Higman’s Lemma on $U^*$, there is some

$(u_{n_1,1}, \ldots, u_{n_1,k_{n_1}}) \preceq^* (u_{n_2,1}, \ldots, u_{n_2,k_{n_2}}) \preceq^* (u_{n_3,1}, \ldots, u_{n_3,k_{n_3}}) \preceq^* \ldots$

Further extracting some $f_{n_{i_1}} \preceq f_{n_{i_2}} \preceq \ldots$ exhibits an infinite increasing subsequence $t_{n_{i_1}} \sqsubseteq t_{n_{i_2}} \sqsubseteq \ldots$ in $S$, a contradiction.
Let \((X, \leq)\) be wqo and assume, b.w.o.c., that \((\mathcal{T}(X), \sqsubseteq)\) is not wqo. We pick a “minimal” bad sequence \(S = t_0, t_1, t_2, \ldots\)

Write every \(t_i\) under the form \(t_i = f_i(u_{i,1}, \ldots, u_{i,k_i})\).

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MORE WQO’S

- Finite Trees ordered by embeddings (Kruskal’s Tree Theorem)

- Finite Graphs ordered by minor (Robertson-Seymour Theorem)

\[ C_n \preceq_{\text{minor}} K_n \text{ and } C_n \preceq_{\text{minor}} C_{n+1} \]

- \((X^\omega, \preceq_*)\) for \(X\) linear wqo.

- \((\mathcal{P}_f(X), \sqsubseteq_H)\) for \(X\) wqo, where

\[ U \sqsubseteq_H V \iff \forall x \in U : \exists y \in V : x \preceq y \]
MORE WQO’S

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▶ Finite Graphs ordered by minor (Robertson-Seymour Theorem)

\[ C_n \leq_{\text{minor}} K_n \text{ and } C_n \leq_{\text{minor}} C_{n+1} \]

▶ \( (X^\omega, \leq_*) \) for \( X \) linear wqo.

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\[
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\[ U \subseteq_H V \iff \forall x \in U : \exists y \in V : x \leq y \]
FINITE-BASIS CHARACTERIZATION

**Defn.** $(X, \leq)$ is a wqo $\overset{\text{def}}{\iff}$ every non-empty subset $V$ of $X$ has at least one and at most finitely many (non-equivalent) minimal elements.

Say $V \subseteq X$ is **upward-closed** if $x \geq y \in V$ implies $x \in V$. (There is a similar notion of downward-closed sets).

For $B \subseteq X$, the **upward-closure** $\uparrow B$ of $B$ is $\{x \mid x \geq b \text{ for some } b \in B\}$. Note that $\uparrow(\bigcup_i B_i) = \bigcup_i \uparrow B_i$, and that $V$ is upward-closed iff $V = \uparrow V$.

**Cor1.** Any upward-closed $U \subseteq X$ has a finite basis, i.e., $U$ is some $\uparrow\{m_1, \ldots, m_k\}$.

**Cor2.** Any downward-closed $V \subseteq X$ can be defined by a finite set of excluded minors:

$$x \in V \iff m_1 \not\leq x \land \cdots \land m_k \not\leq x$$
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**FINITE-BASIS CHARACTERIZATION**

**Defn.** \((X, \leq)\) is a wqo \(\text{def} \iff\) every non-empty subset \(V\) of \(X\) has at least one and at most finitely many (non-equivalent) minimal elements.

Say \(V \subseteq X\) is **upward-closed** if \(x \geq y \in V\) implies \(x \in V\). (There is a similar notion of downward-closed sets).

For \(B \subseteq X\), the **upward-closure** \(\uparrow B\) of \(B\) is \(\{x \mid x \geq b\ \text{for some } b \in B\}\). Note that \(\uparrow(\bigcup_i B_i) = \bigcup_i \uparrow B_i\), and that \(V\) is upward-closed iff \(V = \uparrow V\).

**Cor1.** Any upward-closed \(U \subseteq X\) has a **finite basis**, i.e., \(U\) is some \(\uparrow\{m_1, \ldots, m_k\}\).

**Cor2.** Any downward-closed \(V \subseteq X\) can be defined by a finite set of excluded minors:

\[ x \in V \iff m_1 \not\leq x \land \cdots \land m_k \not\leq x \]

E.g, **Kuratowksi Theorem**: a graph is planar iff it does not contain \(K_5\) or \(K_{3,3}\).

Gives polynomial-time characterization of closed sets.
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Cor3. Any sequence \(\uparrow V_0 \subseteq \uparrow V_1 \subseteq \uparrow V_2 \subseteq \cdots\) of upward-closed subsets converges in finite-time: \(\exists m : (\bigcup_i \uparrow V_i) = \uparrow V_m = \uparrow V_{m+1} = \cdots\)
For \((X, \leq)\), we consider \((\mathcal{P}(X), \subseteq_S)\) defined with

\[ U \subseteq_S V \iff \forall y \in V : \exists x \in U : x \leq y \quad (\overset{\text{def}}{\iff} \uparrow U \supseteq \uparrow V) \]

**Fact.** \(\mathcal{P}(X)\) is well-founded iff \(X\) is wqo  

---Defn’

**NB.** \(X\) well-founded \(\Rightarrow\) \(\mathcal{P}(X)\) well-founded

**Question.** Does \(X\) wqo \(\Rightarrow\) \(\mathcal{P}(X)\) wqo? (Equivalently \(\mathcal{P}_f(X)\) wqo?)
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X \overset{\text{def}}{=} \{(a, b) \in \mathbb{N}^2 \mid a < b\}
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(a, b) < (a', b') \overset{\text{def}}{=} \begin{cases} 
  a = a' \text{ and } b < b' \\
  \text{or } b < a'
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**BEYOND WQO’S**

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**Thm. 1.** \((\mathcal{P}_f(X), \subseteq_S)\) is not wqo: rows are incomparable

2. \((\mathcal{P}(Y), \subseteq_S)\) is wqo iff \(Y\) does not contain \(X\)