Expressiveness of Temporal Logics

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Branching time

With Branching-time Temporal logics, formulas are interpreted over states of a tree-like structure (or a Kripke structure).

A state may have several successors !

Thus properties may express that a state has a successor satisfying a given formula and another one satisfying another formula.

Example

When specifying reactive system (or a program):

- with linear-time temporal logic, the system is seen as a set of executions.
- with branching-time temporal logic, the system is seen as a Kripke structure.

(here: we only consider discrete time !)

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Branching-time temporal logics

In this lecture, we will consider the following questions:

- Which kind of properties can we express with BT-TL ?
- What is the difference between BT-TL and linear-time TL ?
- There are a lot of logics in BT... Why ?
- What is the relationship of BT TL with automata theory ?
- ...

Tree-like structure

Definition

A tree is a set $T \subseteq \mathbb{N}^*_{>0}$ such that if $x \cdot c \in T$ with $x \in \mathbb{N}^*_{>0}$ and $c \in \mathbb{N}_{>0}$, we have:

- $x \in T$
- for all $1 \leq c' < c, \quad x \cdot c' \in T$

A tree is as a partially ordered set of nodes s.t. the set of predecessors of any node is finite, totally ordered and with a common minimal element (the root ε).

Let Σ be an alphabet.

Definition

A Σ-labeled tree is a pair $\langle T, l \rangle$ where:

- T is a tree
- $l : T \to \Sigma$ maps each node with a letter in Σ.

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Kripke structure

Let AP be a set of atomic propositions.

Definition

A Kripke structure is a tuple $\mathcal{S} = \langle Q, q_{\text{init}}, R, l \rangle$ where

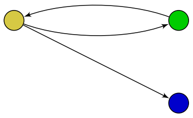
- Q is a set of *states*,
- $q_{\text{init}} \in Q$ is the initial state,
- $R \subseteq Q \times Q$ is a total *transition relation*,
- $l \colon Q \to 2^{\text{AP}}$ labels every state with the propositions it satisfies.

($+$ fairness constraints)

(qRq' is usually denoted $q \to q'$)

Branching time

A Kripke structure. . .



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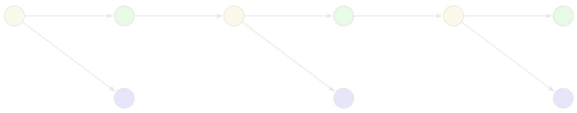

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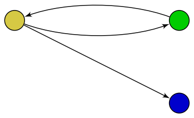

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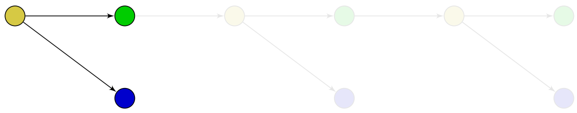

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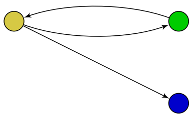

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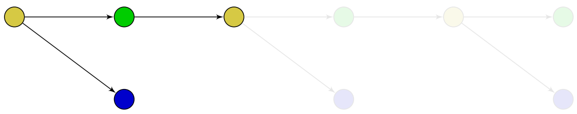

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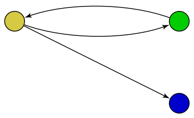

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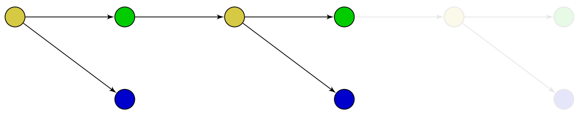

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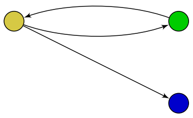

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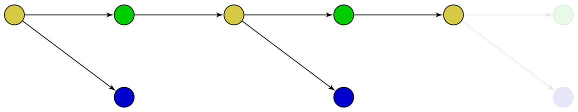

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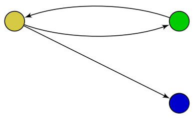

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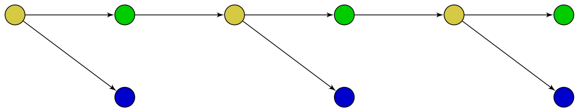

Computation Tree Logic – *CTL*

Definition (Clarke & Emerson, 1981)

$$\varphi,\psi ::= P_1 \mid P_2 \mid \ldots \mid \neg\varphi \mid \varphi\wedge\psi \mid \mathbf{EX}\,\varphi \mid \mathbf{AX}\,\varphi \mid \mathbf{E}\varphi\,\mathbf{U}\,\psi \mid \mathbf{A}\varphi\,\mathbf{U}\,\psi$$

$P_1, P_2, \ldots \in \mathsf{AP}$.

Notation: $\mathcal{B}(\mathbf{X}, \mathbf{U})$.

Formulas are interpreted over states of a structure $\mathcal{S}$.

$\mathsf{Exec}(q)$ denotes the set of (infinite) executions from q.
Given an execution $\rho = q_0 \to q_1 \to q_2 \ldots$, we have:

- $\rho(i)$ denotes the i-th state (*i.e.* q_i),
- $\rho_{|i}$ denotes the prefix $q_1 \to \ldots \to q_i$,
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Computation Tree Logic – *CTL*

$\mathcal{S} = \langle Q, q_{\text{init}}, R, l\rangle$ or $\mathcal{S} = (T, l)$

Definition

These clauses define when a state q in $\mathcal{S}$ satisfies a formula φ, written $q \models_{\mathcal{S}} \varphi$:

$$\begin{array}{lll}
q \models_{\mathcal{S}} P & \text{iff} & P \in l(q) \\
q \models_{\mathcal{S}} \varphi \wedge \psi & \text{iff} & q \models_{\mathcal{S}} \varphi \text{ and } q \models_{\mathcal{S}} \psi \\
q \models_{\mathcal{S}} \neg\varphi & \text{iff} & q \not\models_{\mathcal{S}} \varphi \\
 & & \\
q \models_{\mathcal{S}} \mathbf{EX}\,\varphi & \text{iff} & \exists\rho \in \mathsf{Exec}(q) \text{ s.t. } \rho(1) \models_{\mathcal{S}} \varphi \\
q \models_{\mathcal{S}} \mathbf{AX}\,\varphi & \text{iff} & \forall\rho \in \mathsf{Exec}(q) \text{ we have } \rho(1) \models_{\mathcal{S}} \varphi \\
q \models_{\mathcal{S}} \mathbf{E}\varphi\,\mathbf{U}\,\psi & \text{iff} & \exists\rho \in \mathsf{Exec}(q) \text{ s.t. } \exists i \geq 0, \rho(i) \models_{\mathcal{S}} \psi \text{ and} \\
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q \models_{\mathcal{S}} \mathbf{A}\varphi\,\mathbf{U}\,\psi & \text{iff} & \forall\rho \in \mathsf{Exec}(q), \exists i \geq 0, \text{s.t. } \rho(i) \models_{\mathcal{S}} \psi \text{ and} \\
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\end{array}$$

Abbreviations

$\top$, $\bot$, $\varphi \vee \psi$, $\varphi \Rightarrow \psi$ . . .

- $\mathbf{EF}\,\varphi \overset{\text{def}}{=} \mathbf{E}\top\,\mathbf{U}\,\varphi$: "It is possible to reach a state satisfying φ"
- $\mathbf{AF}\,\varphi \overset{\text{def}}{=} \mathbf{A}\top\,\mathbf{U}\,\varphi$: "Along any path, there exists a state satisfying φ"
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We write $\mathcal{S} \models \varphi$ when $q_{\text{init}} \models_{\mathcal{S}} \varphi$.

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Examples of *CTL* formulae

Example

- "It is possible that the door is open at the first floor while the cabin is at the second floor"

$$\mathbf{EF}\,\Big(\mathit{open}_1 \;\wedge\; \mathit{cabin}_2\Big)$$

- "Any request is eventually served"

$$\bigwedge_{i=1\ldots n} \mathbf{AG}\,\Big(\mathit{call}_i \;\Rightarrow\; \mathbf{AF}\,(\mathit{cabin}_i \wedge \mathit{open}_i)\Big)$$

- "If a request for the i-th floor is done when the cabin is at the i-th floor, the request is satisfied immediately"

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1. Distinguishing power
2. Comparison with *LTL*
3. *CTL**
4. Expressivity of the fragments of *CTL**
    - *UB* and *UB*$^+$
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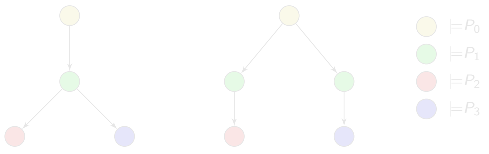
Two Kripke structures satisfy the same *LTL* formulas iff they have the same set of executions (*i.e.* they are trace-equivalent).



$$\forall\Phi \in LTL, \quad \mathcal{S} \models \Phi \text{ iff } \mathcal{S}' \models \Phi$$

But:

$$\mathcal{S} \models \mathbf{EX}\Big(\mathbf{EX}\, P_2 \wedge \mathbf{EX}\, P_3\Big) \text{ and } \mathcal{S}' \not\models \mathbf{EX}\Big(\mathbf{EX}\, P_2 \wedge \mathbf{EX}\, P_3\Big)$$

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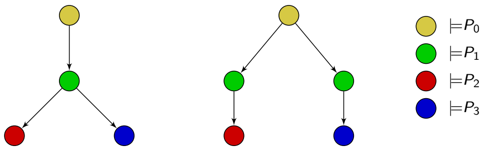

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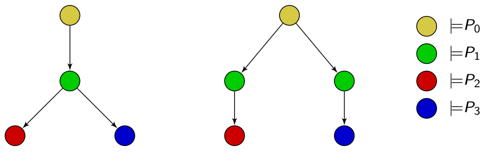

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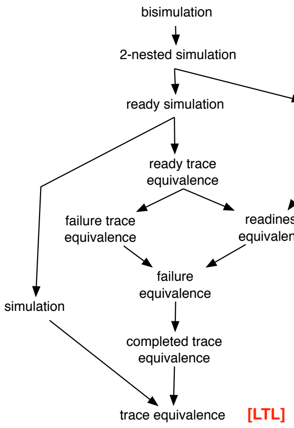
Behavioral equivalences (Van Glabbeek, 1990)



Distinguishing power

Definition

Let $\mathcal{S}_1 = \langle Q_1, q^1_{\text{init}}, R_1, l_1\rangle$ and $\mathcal{S}_2 = \langle Q_2, q^2_{\text{init}}, R_2, l_2\rangle$ be two Kripke structures.
A relation $\mathcal{R} \subseteq Q_1 \times Q_2$ is a bisimulation iff $q_1 \mathcal{R} q_2$ implies:

- $l_1(q_1) = l_2(q_2)$,
- $\forall q_1 \rightarrow q_1'$, $\exists q_2 \rightarrow q_2'$ such that $q_1' \mathcal{R} q_2'$,
- $\forall q_2 \rightarrow q_2'$, $\exists q_1 \rightarrow q_2'1$ such that $q_1' \mathcal{R} q_2'$.

$\mathcal{S}_1$ and $\mathcal{S}_2$ are bisimilar (written $\mathcal{S}_1 \approx \mathcal{S}_2$) iff there exists a bisimulation $\mathcal{R}$ s.t. $q^1_{\text{init}} \mathcal{R} q^2_{\text{init}}$.

Bisimulation vs *CTL*(Hennessy, 1980)

$q \equiv_{CTL} r \overset{\text{def}}{\equiv} \left(\forall \varphi \in CTL, q \models \varphi \Leftrightarrow r \models \varphi\right)$

Proposition

$$q \approx r \quad \Rightarrow \quad q \equiv_{CTL} r$$

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For Kripke structures with finite branching, we have:

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NB: the modality **EX** is sufficient ($\mathcal{B}(\mathbf{X})$).

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Characteristic formulas for *finite* KS

Theorem (Browne, 1988)

Given a finite KS $\mathcal{S}$, there is a CTL formula $\Phi_{\mathcal{S}}$ such that for any finite KS $\mathcal{S}'$, we have:

$$\mathcal{S}' \models \Phi_{\mathcal{S}} \quad \textit{iff} \quad \mathcal{S} \approx \mathcal{S}'$$

- Given a state q of $\mathcal{S}$ and $n \in \mathbb{N}$, we define $\Psi^n(q)$ as follows:
- For any q and q' in $\mathcal{S}$, $q' \models \Psi^n(q)$ iff the computation trees of depth n rooted in q and q' *correspond.*

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- There exists a number c for $\mathcal{S}$ s.t. for any state q and q' in $\mathcal{S}$, we have:

$$q' \models \Psi^c(q) \quad \Leftrightarrow \quad q \approx q'$$

- We define $\Phi_{\mathcal{S}}$ as follows:

Definition

$$\Phi_{\mathcal{S}} \overset{\text{def}}{=} \Psi^c(q_{\text{init}}) \wedge \bigwedge_{q \in Q} \mathbf{AG}\Big(\Psi^c(q) \Rightarrow \bigwedge_{q \to q'} \mathbf{EX}\ \Psi^c(q') \wedge \mathbf{AX} \bigvee_{q \to q'} \Psi^c(q')\Big)$$

Clearly $\mathcal{S}' \models \Phi_{\mathcal{S}} \quad \Leftrightarrow \quad \mathcal{S} \approx \mathcal{S}'$

(the same approach can be used to characterize equivalence w.r.t. stuttering with *CTL* $\setminus$ **X**)

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(the same approach can be used to characterize equivalence w.r.t. stuttering with $CTL \smallsetminus \mathbf{X}$)

Outline

1. Distinguishing power
2. Comparison with *LTL*
3. *CTL**
4. Expressivity of the fragments of *CTL**
    - *UB* and *UB*$^+$
    - *CTL* vs *CTL*$^+$
    - *ECTL* vs *CTL*
    - *ECTL*$^+$, *CTL**, and beyond

Comparison with *LTL*

For a $\Phi \in LTL$, we write $\mathcal{S} \models \Phi$ iff $\rho \models \Phi\ \forall \rho \in \text{Exec}(\mathcal{S})$ (*i.e.* $\mathcal{S} \models \mathbf{A}\Phi$).

Proposition

LTL is NOT as expressive as CTL.

The previous *CTL*-formulas can distinguish KS that verify the same *LTL* formulae. Another ex.: $\mathbf{A\,G\ E\,F}\,P$

Proposition

There is no CTL formula equivalent to $\mathbf{A\,F\,G}\ P$.

(proof in next section)
NB: $\mathbf{AF\ AG}\ P \Rightarrow \mathbf{A\,F\,G}\ P$ but $\mathbf{A\,F\,G}\ P \not\Rightarrow \mathbf{AF\ AG}\ P$.

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LTL and CTL are uncomparable.

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Definition of CTL^*(Emerson & Halpern, 1986)

Idea: merging linear-time TL and branching-time TL...

Definition

$$CTL^* \ni \varphi,\psi \quad ::= \quad P_1 \mid P_2 \mid \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{E}\varphi_p \mid \mathbf{A}\varphi_p$$

$$CTL^*_p \ni \varphi_p,\psi_p \quad ::= \quad \varphi \mid \neg\varphi_p \mid \varphi_p \wedge \psi_p \mid \mathbf{X}\,\varphi_p \mid \varphi_p\,\mathbf{U}\,\psi_p$$

with $P \in \mathrm{AP}$

State formulae (CTL^*) are interpreted over states of a KS.

Path formulae (CTL^*_p) are interpreted over executions in a KS.

Semantics of CTL^*

Definition

$$\begin{array}{lll}
q \models_S P & \text{iff} & P \in l(q) \\
q \models_S \varphi \wedge \psi & \text{iff} & q \models_S \varphi \text{ and } q \models_S \psi \\
q \models_S \neg\varphi & \text{iff} & q \not\models_S \varphi \\
 & & \\
q \models_S \mathbf{E}\varphi_p & \text{iff} & \exists \rho \in \mathrm{Exec}(q) \text{ s.t. } \rho \models_S \varphi_p \\
q \models_S \mathbf{A}\varphi_p & \text{iff} & \forall \rho \in \mathrm{Exec}(q) \text{ we have } \rho \models_S \varphi_p \\
 & & \\
\rho \models_S P & \text{iff} & P \in l(\rho(0)) \\
\rho \models_S \varphi_p \wedge \psi_p & \text{iff} & \rho \models_S \varphi \text{ and } \rho \models_S \psi \\
\rho \models_S \neg\varphi_p & \text{iff} & \rho \not\models_S \varphi_p \\
\rho \models_S \varphi_p\,\mathbf{U}\,\psi_p & \text{iff} & \exists i \geq 0, \rho^i \models_S \psi_p \text{ and} \\
 & & \forall 0 \leq j < i, \text{ we have } \rho^j \models_S \varphi_p \\
\rho \models_S \mathbf{X}\,\varphi_p & \text{iff} & \rho^1 \models_S \varphi_p
\end{array}$$

Fragments of CTL^* – 1

Definition (Ben-Ari, Manna & Pnueli, 1983)

$$UB \ni \varphi,\psi \quad ::= \quad P_1 \mid P_2 \mid \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{E}\varphi_p \mid \mathbf{A}\varphi_p$$

$$UB_p \ni \varphi_p \quad ::= \quad \mathbf{X}\,\varphi \mid \mathbf{F}\,\varphi$$

Definition

$$UB^+ \ni \varphi,\psi \quad ::= \quad P_1 \mid P_2 \mid \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{E}\varphi_p \mid \mathbf{A}\varphi_p$$

$$UB^+_p \ni \varphi_p,\psi_p \quad ::= \quad \neg\varphi_p \mid \varphi_p \wedge \psi_p \mid \mathbf{X}\,\varphi \mid \mathbf{F}\,\varphi$$

$\mathbf{A}(\mathbf{F}\,P_1 \Rightarrow \mathbf{F}\,P2)$ is in UB^+ but not in UB.

Fragments of CTL^* – 2

Definition (Clarke & Emerson, 1981)

$$CTL \ni \varphi,\psi \quad ::= \quad P_1 \mid P_2 \mid \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{E}\varphi_p \mid \mathbf{A}\varphi_p$$

$$CTL_p \ni \varphi_p \quad ::= \quad \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\psi$$

Definition (Emerson & Halpern, 1985)

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$\mathbf{A}(P_1\,\mathbf{U}\,P_2 \Rightarrow P_3\,\mathbf{U}\,P_4)$ is in CTL^+ but not in CTL.

Fragments of CTL^* – 3

"infinitely often" : $\overset{\infty}{\mathbf{F}} \overset{\text{def}}{=} \mathbf{G}\,\mathbf{F}$

Definition (Emerson & Emerson, 1986)

$$ECTL \ni \varphi,\psi \quad ::= \quad P_1 \mid P_2 \mid \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{E}\varphi_p \mid \mathbf{A}\varphi_p$$

$$ECTL_p \ni \varphi_p \quad ::= \quad \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\psi \mid \overset{\infty}{\mathbf{F}}\,\varphi$$

Definition

$$ECTL^+ \ni \varphi,\psi \quad ::= \quad P_1 \mid \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{E}\varphi_p \mid \mathbf{A}\varphi_p$$

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$\mathbf{E}P_1\,\mathbf{U}\,(P_2 \wedge \mathbf{E}\overset{\infty}{\mathbf{F}}\,P_3)$ is in $ECTL$.
$\mathbf{E}(P_1\,\mathbf{U}\,P_2 \wedge \overset{\infty}{\mathbf{F}}\,P_3 \wedge \overset{\infty}{\mathbf{F}}\,P_4)$ is in $ECTL^+$ but not in $ECTL$.

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   - UB and UB^+
   - CTL vs CTL^+
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   - $ECTL^+$, CTL^*, and beyond

Syntactic inclusions

$L \longrightarrow L'$: "L is included in L'"

$$\begin{tikzcd}
UB \arrow[rr] \arrow[dr] & & CTL \arrow[rr] \arrow[dr] & & ECTL \arrow[rr] \arrow[dr] & & CTL^* \\
& UB^+ \arrow[rr] & & CTL^+ \arrow[rr] & & ECTL^+ \arrow[ur] &
\end{tikzcd}$$

And all these logics have the same distinguishing power for structures with finite branching.

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Expressivity of UB

UB formulas are built from atomic propositions, boolean operators and the following modalities: **EX**, **AX**, **EF**, **AF**, **EG** and **AG**.

As we have:

$$\begin{aligned} \mathbf{AX}\ \varphi &\equiv \neg\,\mathbf{EX}\ \neg\varphi \\ \mathbf{EG}\ \varphi &\equiv \neg\,\mathbf{AF}\ \neg\varphi \\ \mathbf{AG}\ \varphi &\equiv \neg\,\mathbf{EF}\ \neg\varphi \end{aligned}$$

we only consider modalities **EX**, **EF** and **AF**.

But UB^+ does not contain a finite number of modalities.
For ex.: $\mathbf{E}(\mathbf{F}\ P_1 \wedge \mathbf{F}\ P_2 \wedge \mathbf{G}\ \neg P_3) \in UB^+$

UB^+ and UB have the same distinguishing power...
But UB^+ is more expressive than UB...

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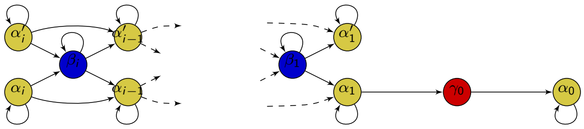
UB less expr. than UB^+ (Emerson & Halpern, 1985)

● $\models P_1 \wedge \neg P_2$ ● $\models \neg P_1 \wedge \neg P_2$ ● $\models P_1 \wedge P_2$



$$\alpha_i \models \mathbf{E}\Big(\mathbf{G}\ P_1 \ \wedge\ \mathbf{F}\ P_2\Big) \text{ but } \alpha'_i \not\models \mathbf{E}\Big(\mathbf{G}\ P_1 \ \wedge\ \mathbf{F}\ P_2\Big)$$

Lemma

$$\forall\varphi \in UB,\ |\varphi| \leq i \ \Rightarrow\ \Big(\alpha_i \models \varphi \Leftrightarrow \alpha'_i \models \varphi\Big)$$

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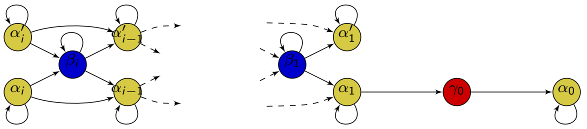

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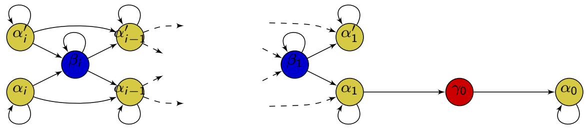

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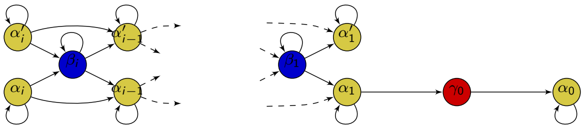

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Proof: by induction on $|\varphi|$.
Direct for atomic propositions and boolean operators.

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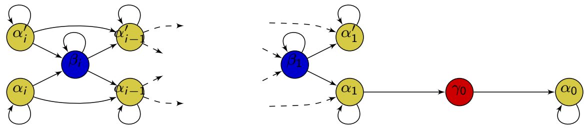

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Proof: by induction on $|\varphi|$.
$\varphi = \mathbf{E}\,\mathbf{X}\,\psi$.
If $\alpha_i \models \mathbf{E}\,\mathbf{X}\,\psi$ then either $\beta_i \models \psi$, $\alpha_i \models \psi$ or $\alpha_{i-1} \models \psi$.
Induction hypothesis allows to deduce $\alpha'_i \models \mathbf{E}\,\mathbf{X}\,\psi$.

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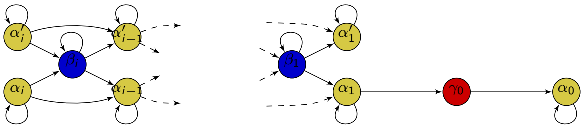

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$\varphi = \mathbf{E}\,\mathbf{F}\,\psi$.
If $\alpha_i \models \mathbf{E}\,\mathbf{F}\,\psi$, then either $\alpha_i \models \psi$ (and the i.h. can be applied) or another state (also reachable from α'_i) satisfies ψ.

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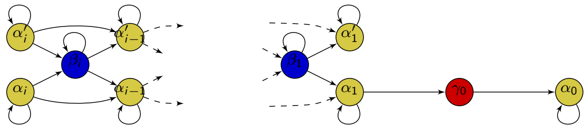

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Proof: by induction on $|\varphi|$.
$\varphi = \mathbf{A}\,\mathbf{F}\,\psi$.
If $\alpha_i \models \mathbf{A}\,\mathbf{F}\,\psi$, then $\alpha_i \models \psi$ and from the i.h. we have $\alpha'_i \models \psi$ and then $\alpha'_i \models \mathbf{A}\,\mathbf{F}\,\psi$.

UB less expr. than UB^+ (Emerson & Halpern, 1985)

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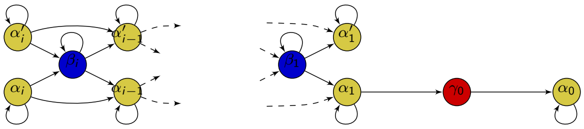

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Lemma

$$\forall \varphi \in UB,\ |\varphi| \leq i \;\Rightarrow\; \Big(\alpha_i \models \varphi \;\Leftrightarrow\; \alpha'_i \models \varphi\Big)$$

Conclusion: there is no UB formula equivalent to $\mathbf{E}(\mathbf{G}\, P_1 \wedge \mathbf{F}\, P_2)$.

UB^+ vs CTL – (Emerson & Halpern, 1985)

Theorem

CTL is strictly more expressive than UB^+!

- Let $\Phi \overset{\text{def}}{=} \mathbf{E}\, P_1 \;\mathbf{U}\; P_2$.
- Assume there exists an UB^+ formula Ψ equivalent to Φ.
- Let Ψ' be the formula Ψ where any path quantifiers **E** and **A** have been removed : $\Psi' \in \mathcal{L}(\mathbf{F}, \mathbf{G}, \mathbf{X})$.
- For any path ρ, we clearly have $\rho(1) \models \Psi$ iff $\rho \models \Psi'$.
- Then Ψ' is equivalent to $P_1 \;\mathbf{U}\; P_2$.
- But $\mathcal{L}(\mathbf{F}, \mathbf{G}, \mathbf{X}) < \mathcal{L}(\mathbf{U}, \mathbf{X})$!!

There is no UB^+ formula equivalent to $\mathbf{E}\, P_1 \;\mathbf{U}\; P_2$.

From the expressivity point of view, we have:

$$UB < UB^+ < CTL$$

UB^+ vs CTL – (Emerson & Halpern, 1985)

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UB^+ vs CTL – (Emerson & Halpern, 1985)

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Expressivity of *CTL*

$\mathit{CTL} = \mathcal{B}(\mathbf{X}, \mathbf{U})$.

CTL formulae are built from atomic propositions, boolean operators and the following modalities: **EX**, **EU** and **AU**.

Theorem

$\mathbf{A}\varphi\ \mathbf{U}\ \psi \equiv \mathbf{AF}\,\psi \wedge \neg\mathbf{E}\neg\psi\ \mathbf{U}\,(\neg\psi \wedge \neg\varphi)$

$\Rightarrow$ *The TL based on the modalities* **EX**, **EU** *and* **AF** *is as expressive as CTL.*

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Theorem (Laroussinie, 1995)

It is not possible to express $\mathbf{E}P_1\ \mathbf{U}\ P_2$ *with* **EX**, **AU** *and* **EF**.

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$$\mathbf{E}\Big(P_1\ \mathbf{U}\ P_2\ \wedge\ P_3\ \mathbf{U}\ P_4 \wedge \mathbf{G}\,P_5\Big) \equiv$$

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But:

Theorem (Wilke, 1999 and Adler & Immerman, 1999)

CTL^+ is exponentially more succinct than CTL.

Any CTL formula equivalent to $\Phi_n = \mathbf{E}\big(\mathbf{F}\, P_1 \wedge \ldots \wedge \mathbf{F}\, P_n\big)$ is of length $\Omega(2^n/\sqrt{n})$.

- For any $\varphi \in CTL$, there exist an Alternating Tree Aut. A_φ recognizing the $Mod(\varphi)$ s.t. $|A_\varphi|$ is linear in $|\varphi|$.
- Every ATA recognizing $Mod(\Phi_n)$ has at least $\binom{n}{\lceil n/2 \rceil}$ states

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$$\mathbf{A}\overset{\infty}{\mathbf{F}}\,\varphi \overset{def}{=} \mathbf{A\,G\,F}\,\varphi \equiv \mathbf{A\,G\,A\,F}\,\varphi$$

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$\mathbf{E}\overset{\infty}{\mathbf{F}}$ *P* and *CTL*(Emerson & Halpern, 1986)

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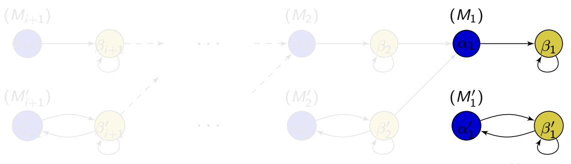

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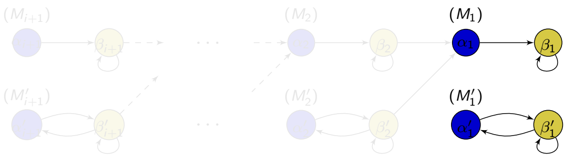

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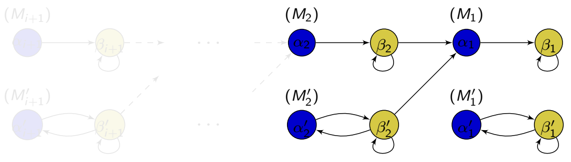

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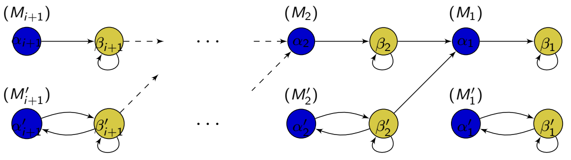

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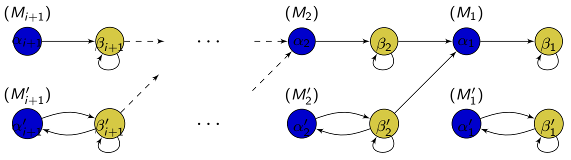

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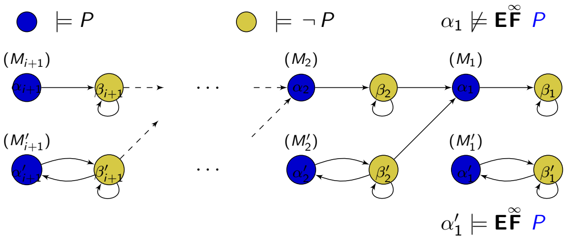

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ECTL$^+$, *CTL** and beyond (Emerson, 1990)

Theorem (Emerson & Halpern, 1986)

$\mathbf{E}(\overset{\infty}{\mathbf{F}}\ P_1 \ \wedge\ \overset{\infty}{\mathbf{F}}\ P_2)$ *cannot be expressed with ECTL.*
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$\mathbf{E\,G}(P_1 \ \vee\ \mathbf{X}\ P_2)$ *cannot be expressed with ECTL*$^+$*.*
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Theorem (Wolper, 1983)

*"P holds for every even state" cannot be expressed in CTL***.*
$\Rightarrow$ *Propositional* μ*-calculus is strictly more expressive than CTL***.*
$(\nu Y.P \wedge \mathbf{EX}\ \mathbf{EX}\ Y)$

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Theorem (Wolper, 1983)

*"P holds for every even state" cannot be expressed in CTL***.*
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$(\nu Y.P \wedge \mathbf{EX}\ \mathbf{EX}\ Y)$

ECTL$^+$, *CTL** and beyond (Emerson, 1990)

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