# Expressiveness of Temporal Logics

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# Outline of today's lecture

1. LTL+Past and the $\mu$-calculus
2. LTL+Past and Büchi automata
   - Büchi automata
   - From LTL+Past to Büchi automata
   - Büchi automata are more expressive
   - Alternating Büchi automata
   - Application: Succinctness of LTL+Past
3. Stuttering
   - The stuttering principle
   - The generalized stuttering principle
4. Ehrenfeucht-Fraïssé games
   - The rules of the game
   - EF games and the Until-Since hierarchy

# Introduction to the second lecture

- In this second course, we focus on discrete time.
- as is usual in that case, we use the following definitions:
  - LTL $= \mathcal{L}(\mathbf{U}, \mathbf{X})$,
  - LTL+Past $= \mathcal{L}(\mathbf{U}, \mathbf{S}, \mathbf{X}, \mathbf{X}^{-1})$.

  where $\mathbf{U}$ and $\mathbf{S}$ have their *non-strict* meaning, e.g.:

$\varphi$ $\mathbf{U}$ $\psi$ holds iff

- either $\psi$ holds,
- or $\varphi$ holds, and $\varphi$ $\mathbf{U}$ $\psi$ holds in the next location,

In other words, we have the following equivalence:

$$\varphi \ \mathbf{U} \ \psi \equiv \psi \vee (\varphi \wedge \mathbf{X}(\varphi \ \mathbf{U} \ \psi)).$$

- linear structures are seen as (infinite) words over the alphabet $2^{\mathsf{AP}}$.

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# LTL+Past and the $\mu$-calculus

## Definition

The *linear-time $\mu$-calculus* is an extension of $\mathcal{L}(\mathbf{X})$ with fixpoint operators:

$$\mu\text{-calculus} \ni \varphi, \psi ::= \top \mid p \mid \neg p \mid Z \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \mathbf{X}\,\psi \mid \mathbf{X}^{-1}\,\psi \mid \mu Z\,\varphi \mid \nu Z\,\varphi$$

where $p$ ranges over AP and $Z$ ranges over a finite set of variables.

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where $p$ ranges over AP and $Z$ ranges over a finite set of variables.

## Example

$$\mu Z\,(\texttt{green} \vee \mathbf{X}\,Z)$$

# LTL+Past and the $\mu$-calculus

**Theorem (Knaster, 1928 & Tarski, 1955)**

*Let $T$ be a linear structure. Given a formula $\varphi(Z)$, the set of positions satisfying $\mu Z\, \varphi(Z)$ is*

$$\bigcap \{U \subseteq T \mid \varphi(U) \subseteq U\}$$

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$$\bigcap \{U \subseteq T \mid \varphi(U) \subseteq U\}$$

Moreover, fixpoints can be computed iteratively:

**Theorem (Knaster, 1928 & Tarski, 1955)**

*The set of positions satisfying $\mu Z\, \varphi(Z)$ is the limit of the following sequence:*

$$\begin{aligned} [\![\mu Z\, \varphi(Z)]\!]_0 &= \varnothing \\ [\![\mu Z\, \varphi(Z)]\!]_{i+1} &= \{t \in T \mid \langle T, t\rangle \models \varphi([\![\mu Z\, \varphi(Z)]\!]_i)\} \end{aligned}$$

# LTL+Past and the $\mu$-calculus

## Example

Consider the following linear struture (represented as a word):

$$T = \texttt{g}\ \texttt{r}\ \texttt{r}\ \texttt{g}\ \texttt{g}\ \texttt{b}\ \texttt{r}\ \texttt{b}\ \texttt{g}\ \texttt{r}\ \texttt{r}\ \texttt{g}\ \texttt{r}\ \texttt{r}\ \texttt{r}\ \texttt{r}\ \texttt{r}\ \texttt{r} \ldots$$

and the following formula:

$$\mu Z\,(\texttt{green} \vee \mathbf{X}\, Z).$$

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and the following formula:

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Then:

$$[\![\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z)]\!]_0 = \varnothing$$

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and the following formula:

$$\mu Z\,(\mathtt{green} \vee \mathbf{X}\, Z).$$

Then:

$$\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z) \equiv_0 \bot$$

# LTL+Past and the $\mu$-calculus

## Example

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and the following formula:

$$\mu Z\,(\mathtt{green} \vee \mathbf{X}\, Z).$$

Then:

$$[\![\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z)]\!]_1 = \{t \in T \mid \langle T, t\rangle \models \mathtt{g} \vee \mathbf{X} \bot\}$$

# LTL+Past and the $\mu$-calculus

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and the following formula:

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Then:

$$\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z) \equiv_1 \mathtt{g}$$

# LTL+Past and the $\mu$-calculus

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and the following formula:

$$\mu Z\,(\mathtt{green} \vee \mathbf{X}\, Z).$$

Then:

$$[\![\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z)]\!]_2 = \{t \in T \mid \langle T, t\rangle \models \mathtt{g} \vee \mathbf{X}\, \mathtt{g}\}$$

# LTL+Past and the $\mu$-calculus

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and the following formula:

$$\mu Z\,(\mathtt{green} \vee \mathbf{X}\, Z).$$

Then:

$$\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z) \equiv_2 \mathtt{g} \vee \mathbf{X}\, \mathtt{g}$$

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and the following formula:

$$\mu Z\,(\mathtt{green} \vee \mathbf{X}\, Z).$$

Then:

$$\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z) \equiv_n \mathtt{g} \vee \mathbf{X}\,\mathtt{g} \vee \mathbf{X}\,\mathbf{X}\,\mathtt{g} \vee \ldots \vee \mathbf{X}^{n-1}\mathtt{g}$$

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and the following formula:

$$\mu Z\,(\mathtt{green} \vee \mathbf{X}\, Z).$$

Then:

$$\mu Z\,(\mathtt{g} \vee \mathbf{X}\, Z) \equiv \mathbf{F}\, \mathtt{g}$$

# LTL+Past and the $\mu$-calculus

$$\varphi \ \mathbf{U} \ \psi \equiv \psi \vee (\varphi \wedge \mathbf{X}(\varphi \ \mathbf{U} \ \psi)).$$

From this equivalence, we get:

$$\varphi \ \mathbf{U} \ \psi \equiv \mu Z \, (\psi \vee (\varphi \wedge \mathbf{X} \, Z))$$

# LTL+Past and the $\mu$-calculus

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From this equivalence, we get:

$$\varphi \ \mathbf{U} \ \psi \equiv \mu Z \, (\psi \vee (\varphi \wedge \mathbf{X} \, Z))$$

Thus:

**Proposition**

*$\mu$-calculus is at least as expressive as LTL+Past.*

# LTL+Past and the $\mu$-calculus

In fact:

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*$\mu$-calculus is strictly more expressive than LTL+Past.*

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*Proof.*

The negation of

"`green` occurs at every even position"

is

"$\neg$ `green` occurs at some even position"

i.e.

$$\neg \texttt{green} \vee \mathbf{X}\,\mathbf{X}\,\neg \texttt{green} \vee \mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\neg \texttt{green} \vee \ldots$$

# LTL+Past and the $\mu$-calculus

In fact:

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*$\mu$-calculus is strictly more expressive than LTL+Past.*

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i.e.

$$\neg \texttt{green} \vee \mathbf{X}\,\mathbf{X}\,\neg \texttt{green} \vee \mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\neg \texttt{green} \vee \ldots$$

It can be written as

$$\mu Z\,(\neg \texttt{green} \vee \mathbf{X}\,\mathbf{X}\,Z)$$

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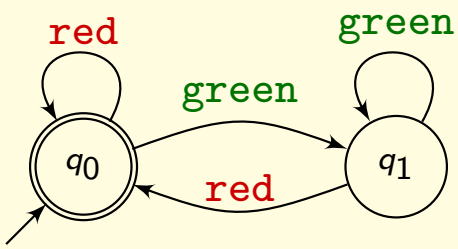
# LTL+Past and Büchi automata

Finite-state automata are a powerful formalism for defining languages.

## Example

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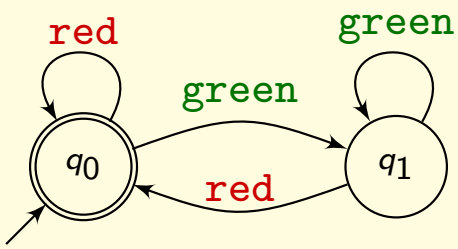


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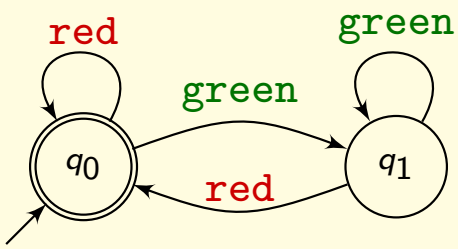


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What is the relationship between automata (on words) and (linear-time) temporal logics?

# Büchi automata

## Definition

A Büchi automaton is a 5-tuple $\mathcal{B} = \langle Q, Q_0, \Sigma, \rightarrow, F \rangle$ where

- $Q$ is the set of states (or locations) of the automaton,
- $Q_0 \subseteq Q$ is the set of initial states,
- $\Sigma$ is the alphabet,
- $\rightarrow \subseteq Q \times \Sigma \times Q$ is the transition relation,
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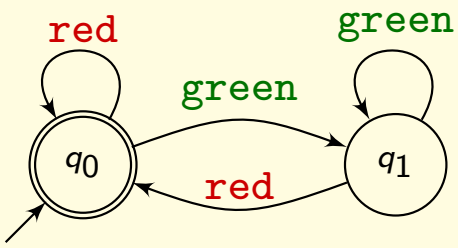


$$
\begin{array}{rcl}
Q & = & \{q_0, q_1\}, \quad Q_0 = \{q_0\}, \\
\Sigma & = & \{\texttt{green}, \texttt{red}\}, \\
\rightarrow & = & \{(q_0, \texttt{green}, q_1), (q_1, \texttt{green}, q_1), \\
 & & \ (q_1, \texttt{red}, q_0), (q_0, \texttt{red}, q_0)\}, \\
F & = & \{q_0\}.
\end{array}
$$

# Büchi automata

## Definition

An (infinite) word $w_0\, w_1 \ldots$ is *accepted* by a Büchi automaton $\mathcal{B}$ if there exists an infinite sequence $\pi = (\ell_0, \ell_1, \ \ldots)$ of states s.t.:

- $\ell_0 \in Q_0$,
- for each $i$, $(\ell_i, w_i, \ell_{i+1}) \in \rightarrow$;
- at least one state in $F$ occurs infinitely often in $\pi$.

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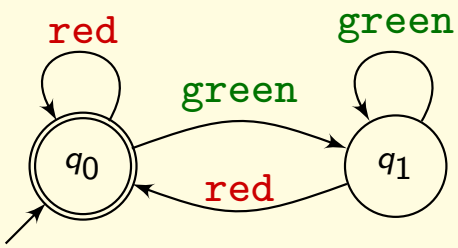


$$\texttt{green} \cdot \texttt{red}^\omega \in \mathcal{L}(\mathcal{B}),$$

$$\texttt{green} \cdot \texttt{red} \cdot \texttt{green}^\omega \notin \mathcal{L}(\mathcal{B}).$$

# Outline of today's lecture

# From LTL+Past to Büchi automata

**Theorem (Lichtenstein, Pnueli, Zuck, 1985)**

*Let $\varphi$ a formula in LTL+Past. There exists a Büchi automaton $\mathcal{B}_\varphi$ s.t.*

$$\forall w \in (2^{AP})^\omega. \qquad w \in \mathcal{L}(\mathcal{B}_\varphi) \;\Leftrightarrow\; w, 0 \models \varphi.$$

*Sketch of proof.*

- each state of the automaton corresponds to a set of subformulas of $\varphi$ (and negations thereof),
- if a word $w$ is accepted from a location $q_0$, then any subformula represented by that state holds initially along $w$.

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# From LTL+Past to Büchi automata

## Definition

The closure of $\varphi$, denoted by $\mathrm{Cl}(\varphi)$, is the smallest set of formulas containing $\varphi$ and closed under the following rules:

- $\top$ and $\bot$ are in $\mathrm{Cl}(\varphi)$,
- $\neg\psi \in \mathrm{Cl}(\varphi)$ iff $\psi \in \mathrm{Cl}(\varphi)$ (identifying $\neg\neg\psi$ with $\psi$),
- if $\psi_1 \wedge \psi_2$ or $\psi_1 \vee \psi_2$ is in $\mathrm{Cl}(\varphi)$, then $\psi_1 \in \mathrm{Cl}(\varphi)$ and $\psi_2 \in \mathrm{Cl}(\varphi)$,
- if $\mathbf{X}\,\psi_1$ is in $\mathrm{Cl}(\varphi)$, then so $\psi_1$,
- if $\psi_1\ \mathbf{U}\ \psi_2$ is in $\mathrm{Cl}(\varphi)$, then so are $\psi_1$, $\psi_2$, and $\mathbf{X}(\psi_1\ \mathbf{U}\ \psi_2)$,
- if $\mathbf{X}^{-1}\,\psi_1$ is in $\mathrm{Cl}(\varphi)$, then so $\psi_1$,
- if $\psi_1\ \mathbf{S}\ \psi_2$ is in $\mathrm{Cl}(\varphi)$, then so are $\psi_1$, $\psi_2$, and $\mathbf{X}^{-1}(\psi_1\ \mathbf{S}\ \psi_2)$.

# From LTL+Past to Büchi automata

**Proposition**

*The size of* $\mathrm{Cl}(\varphi)$ *is at most* $4|\varphi|$.

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- if $\varphi = \psi_1 \wedge \psi_2$ or $\varphi = \psi_1 \vee \psi_2$, then

$$\mathrm{Cl}(\varphi) = \mathrm{Cl}(\psi_1) \cup \mathrm{Cl}(\psi_2) \cup \{\varphi, \neg\varphi\}.$$

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- if $\varphi = \psi_1 \ \mathbf{U} \ \psi_2$, then

$$\mathrm{Cl}(\varphi) = \mathrm{Cl}(\psi_1) \cup \mathrm{Cl}(\psi_2) \cup \{\varphi, \neg\varphi, \mathbf{X}\,\varphi, \neg\,\mathbf{X}\,\varphi\}.$$

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- the other cases are similar.

# From LTL+Past to Büchi automata

## Example

Consider formula $\varphi = \mathbf{G}(\texttt{green} \Rightarrow (\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green}))$.
Then:

$$
\begin{aligned}
\mathrm{Cl}(\varphi) = \{\varphi, \neg\varphi,\\
&\texttt{green} \Rightarrow (\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green}),\\
&\neg(\texttt{green} \Rightarrow (\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green})),\\
&\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green},\\
&\neg(\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green}),\\
&\mathbf{F}\,\texttt{red}, \neg\,\mathbf{F}\,\texttt{red}, \mathbf{X}\,\mathbf{F}\,\texttt{red}, \neg\,\mathbf{X}\,\mathbf{F}\,\texttt{red},\\
&\mathbf{G}^{-1}\,\texttt{green}, \neg\,\mathbf{G}^{-1}\,\texttt{green},\\
&\mathbf{X}^{-1}\,\mathbf{G}^{-1}\,\texttt{green}, \neg\,\mathbf{X}^{-1}\,\mathbf{G}^{-1}\,\texttt{green},\\
&\texttt{green}, \neg\,\texttt{green}, \texttt{red}, \neg\,\texttt{red}, \top, \bot\}.
\end{aligned}
$$

# From LTL+Past to Büchi automata

## Definition

A subset $S$ of $\mathrm{Cl}(\varphi)$ is *maximal consistent* if:

- $\top \in S$,
- for any $\psi \in \mathrm{Cl}(\varphi)$, $\psi \in S$ iff $\neg\psi \notin S$,
- for any $\psi = \psi_1 \wedge \psi_2 \in \mathrm{Cl}(\varphi)$: $\psi \in S$ iff $\psi_1 \in S$ and $\psi_2 \in S$,
- for any $\psi = \psi_1 \vee \psi_2 \in \mathrm{Cl}(\varphi)$: $\psi \in S$ iff $\psi_1 \in S$ or $\psi_2 \in S$,
- for any $\psi = \psi_1 \ \mathbf{U} \ \psi_2 \in \mathrm{Cl}(\varphi)$:
  $\psi \in S$ iff $\psi_2 \in S$, or both $\psi_1$ and $\mathbf{X}(\psi_1 \ \mathbf{U} \ \psi_2)$ are in $S$,
- for any $\psi = \psi_1 \ \mathbf{S} \ \psi_2 \in \mathrm{Cl}(\varphi)$:
  $\psi \in S$ iff $\psi_2 \in S$, or both $\psi_1$ and $\mathbf{X}^{-1}(\psi_1 \ \mathbf{S} \ \psi_2)$ are in $S$.

# From LTL+Past to Büchi automata

**Example**

The set

$$\{\varphi, \neg(\texttt{green} \Rightarrow (\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green})),$$
$$\neg(\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green}),$$
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$$\texttt{green}, \neg\,\texttt{red}\}$$

is maximal consistent.

# From LTL+Past to Büchi automata

**Example**

The set

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**Proposition**

*There are at most* $2^{4|\varphi|}$ *maximal consistent subsets of* $\mathrm{Cl}(\varphi)$.

# From LTL+Past to Büchi automata

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**Proposition**

*There are at most* $2^{4|\varphi|}$ *maximal consistent subsets of* $\mathrm{Cl}(\varphi)$.

Maximal consistent subsets are the states of our Büchi automaton.

# From LTL+Past to Büchi automata

Given two maximal consistent subsets $S$ and $T$ of $\mathrm{Cl}(\varphi)$, and a "letter" $\sigma \subseteq \mathsf{AP}$, there is a transition $(S, \sigma, T)$ iff:

- for any $p \in \mathsf{AP}$, we have $p \in S$ iff $p \in \sigma$,
- for any subformula $\mathbf{X}\,\varphi_1 \in \mathrm{Cl}(\varphi)$:
$$\mathbf{X}\,\varphi_1 \text{ is in } S \quad \text{iff} \quad \varphi_1 \in T,$$
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# From LTL+Past to Büchi automata

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**Example**

$S$:
...
$\neg\,\mathbf{X}\,\mathbf{F}\,\mathtt{red}$
$\neg\,\mathbf{G}^{-1}\,\mathtt{green}$
$\mathtt{green}$

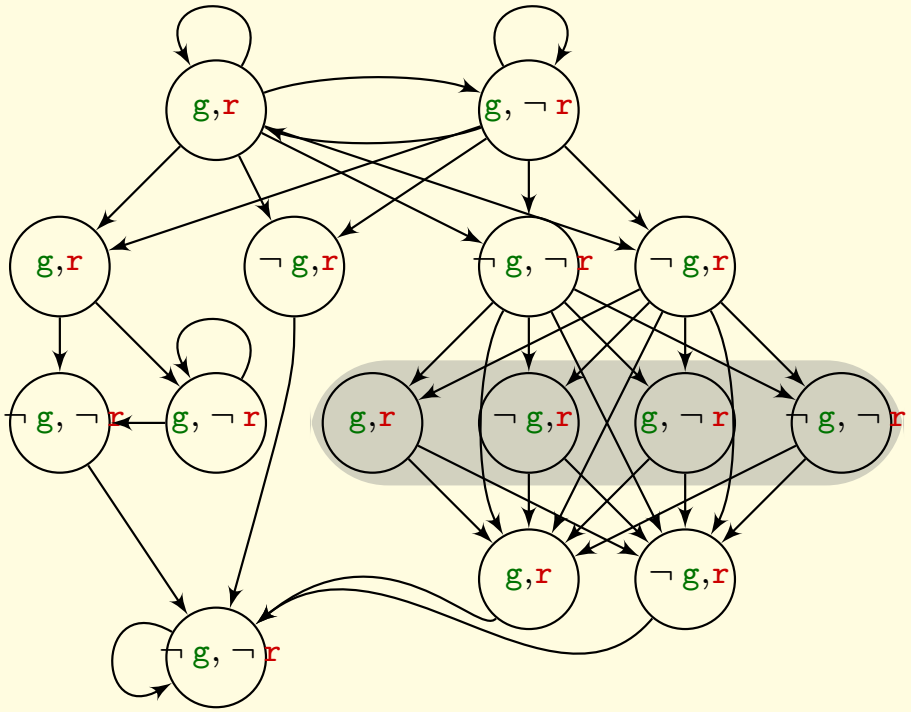
$\longrightarrow$

$T$:
...
$\neg\,\mathbf{F}\,\mathtt{red}$
$\neg\,\mathbf{X}^{-1}\,\mathbf{G}^{-1}\,\mathtt{green}$
$\neg\,\mathtt{red}$

# From LTL+Past to Büchi automata

## Example

For formula $\mathbf{G}(\texttt{green} \Rightarrow (\mathbf{F}\,\texttt{red} \vee \mathbf{G}^{-1}\,\texttt{green}))$, we get (only "useful" states are displayed):



# From LTL+Past to Büchi automata

We use (generalized) Büchi acceptance condition is used to enforce that eventualities eventually occur:

- For each subformula $\psi = \varphi_1 \ \mathbf{U} \ \varphi_2$, we write

$$F_\psi = \{I \in Q \mid \varphi_2 \in I \text{ or } \psi \in I\}$$

- a word is accepted if it has a trajectory whose repeated states intersect $F_\psi$ for each $\mathbf{U}$-subformula $\psi$.

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- initial states are those where all $\mathbf{X}^{-1}$-subformulas are false.

# From LTL+Past to Büchi automata

## Example

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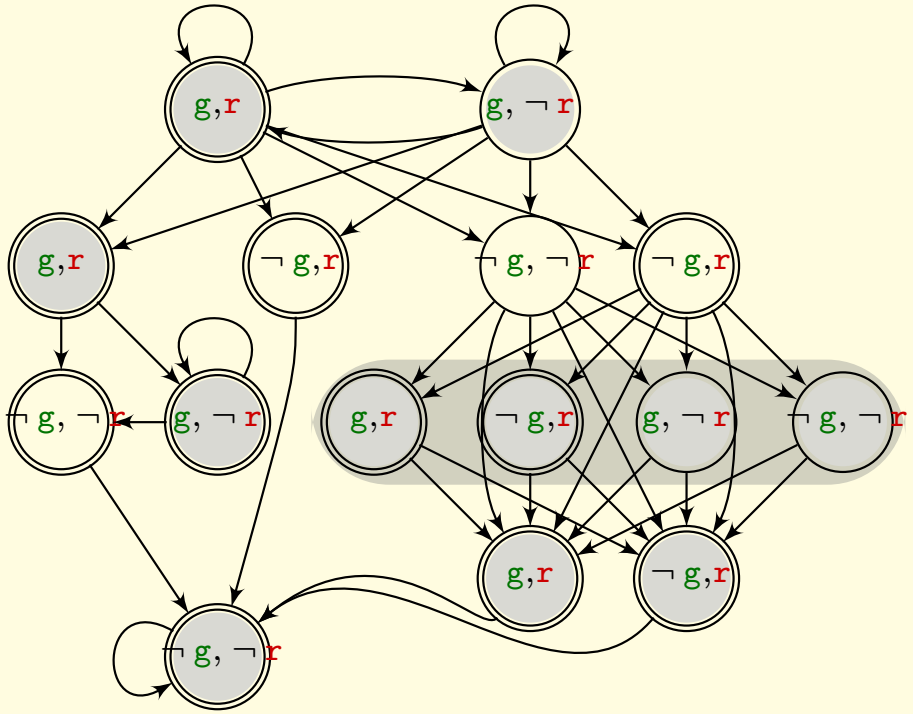


# From LTL+Past to Büchi automata

**Theorem**

*For any LTL+Past formula $\varphi$, there exists a Büchi automaton $\mathcal{A}$ s.t.*

- *a word is accepted by $\mathcal{A}$ iff if satisfies $\varphi$;*
- *$\mathcal{A}$ has at most $2^{4|\varphi|}$ states.*

This result is extremely important in computer science: it provides a nice way of verifying that an automaton satisfies an LTL+Past formula. In particular:

**Theorem**

*Satisfiability of an LTL+Past formula is PSPACE-complete.*

# Outline of today's lecture

# Büchi automata are more expressive

**Theorem**

*Büchi automata are strictly more expressive than LTL+Past.*

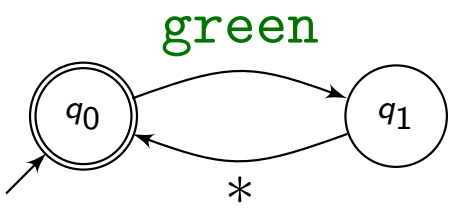
# Büchi automata are more expressive

**Theorem**

*Büchi automata are strictly more expressive than LTL+Past.*

*Proof.*

The following Büchi automaton accepts words where `green` holds (at least) at even positions:



# Outline of today's lecture

# Alternating Büchi automata

## Alternation

Alternating automata are automata in which non-deterministic "choices" can be both disjunctive (as usual) and conjunctive.

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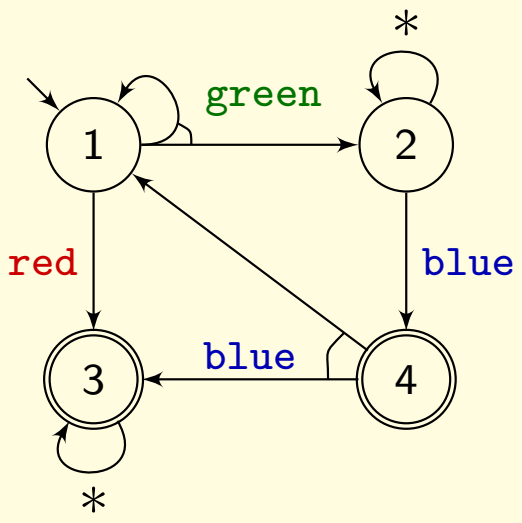

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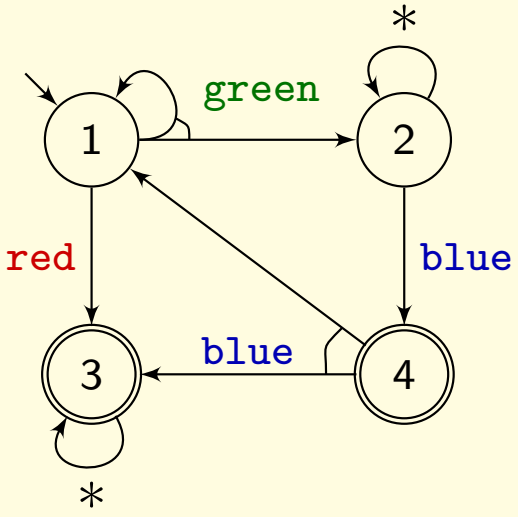


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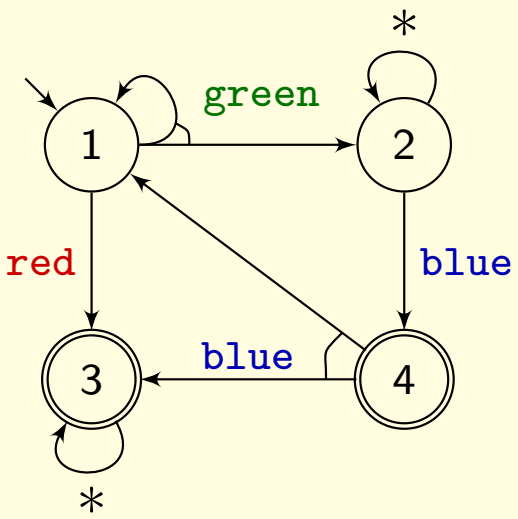


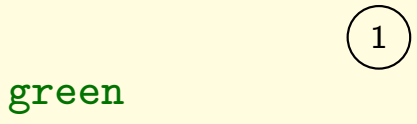

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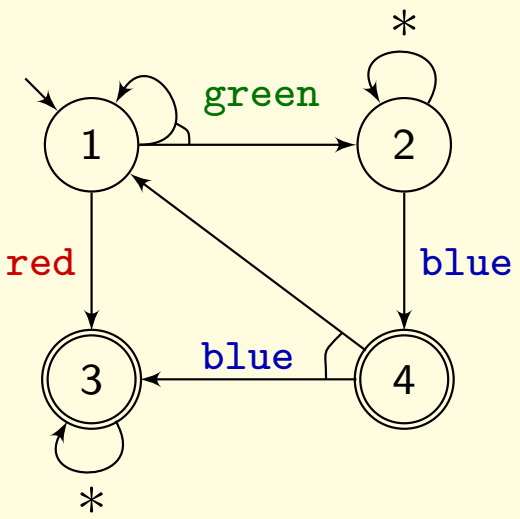


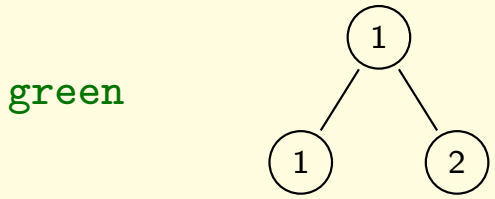

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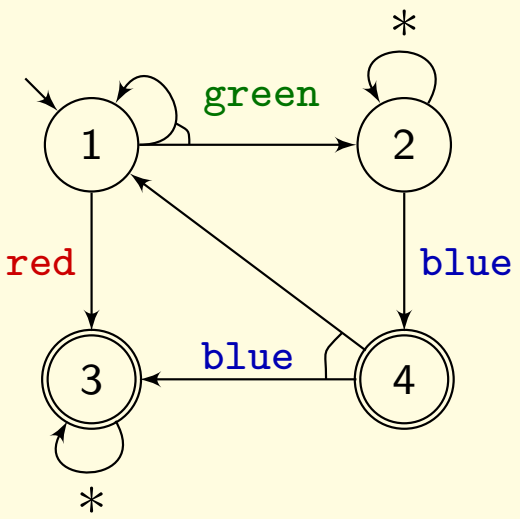
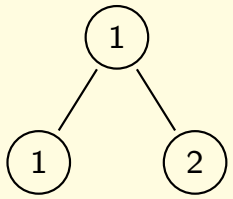



green

green



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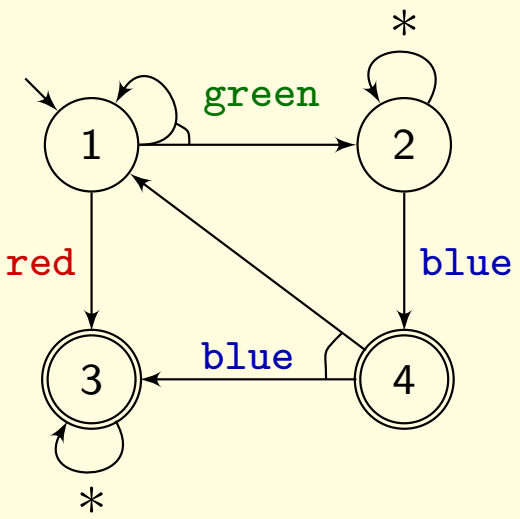


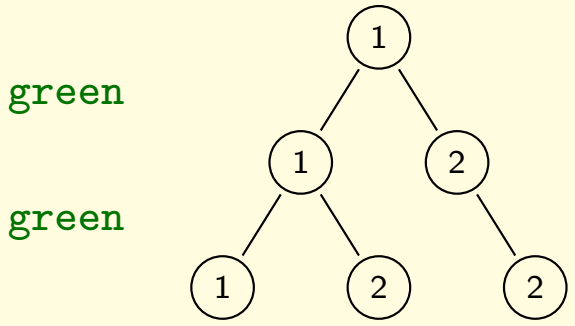

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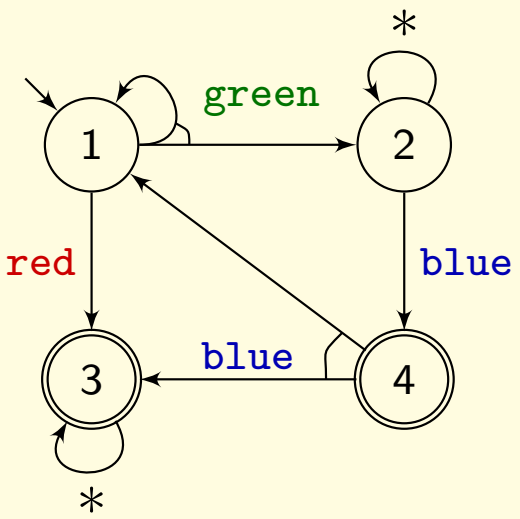


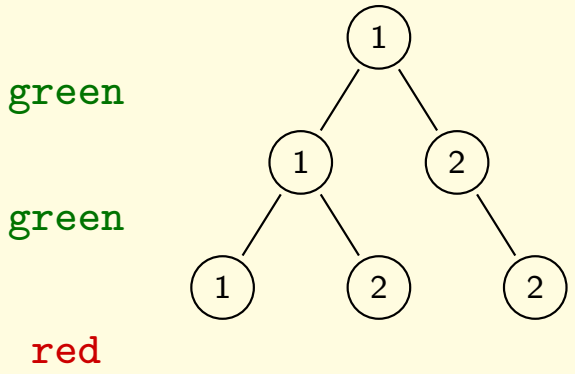

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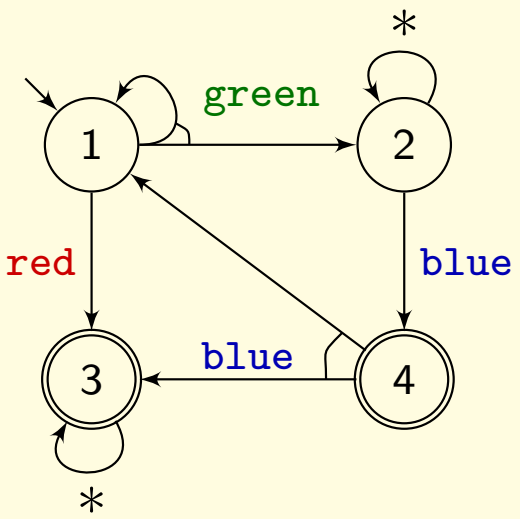


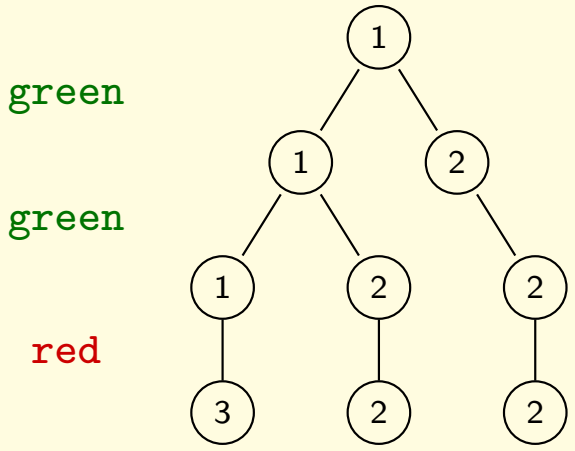

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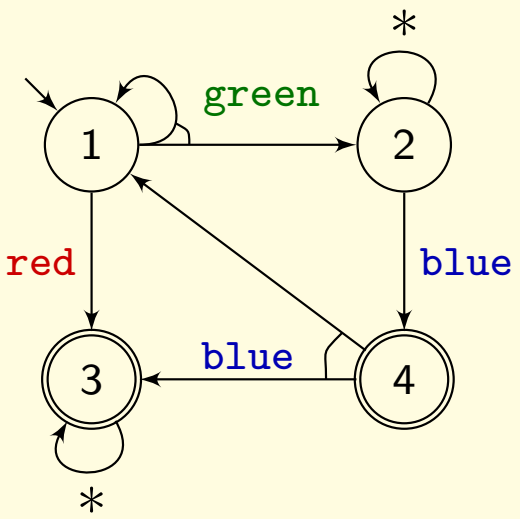


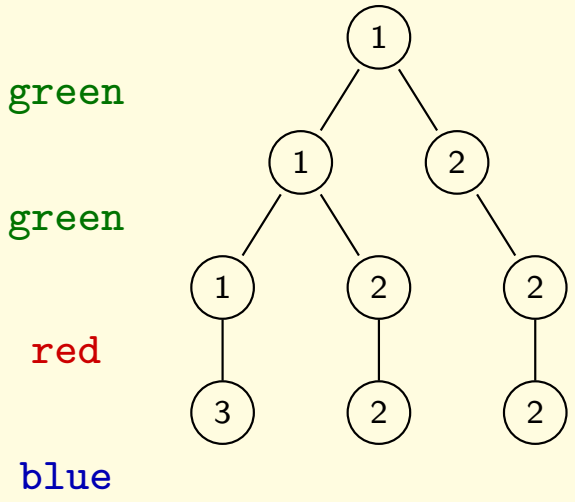

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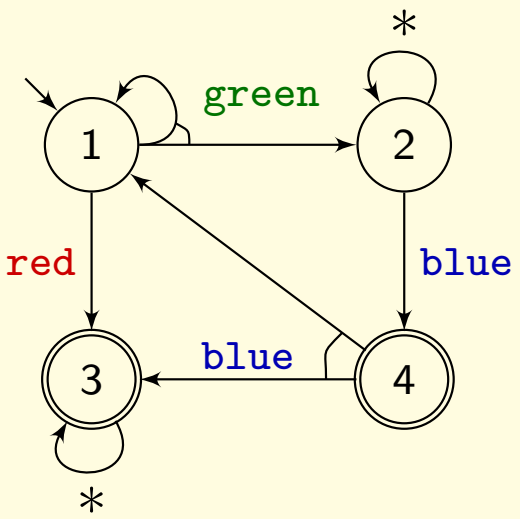


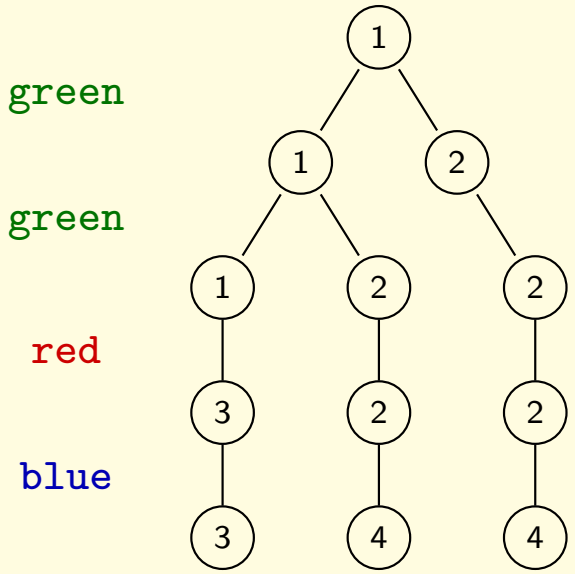

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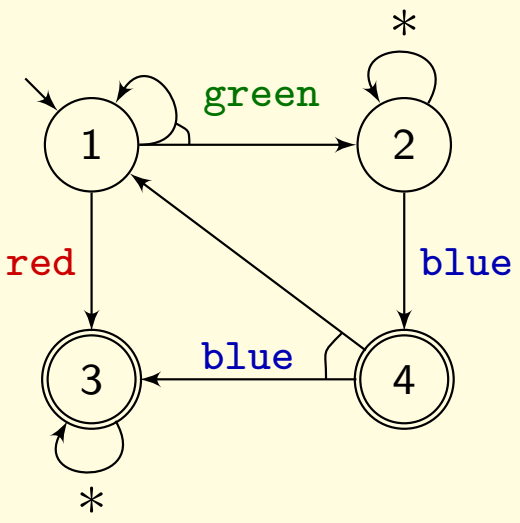


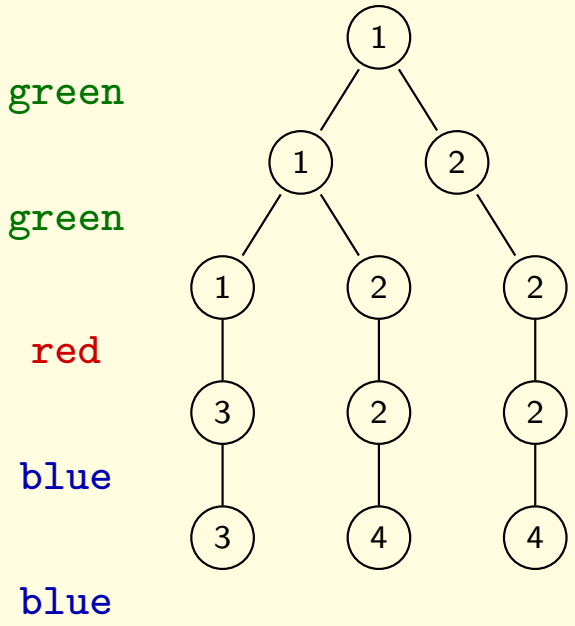

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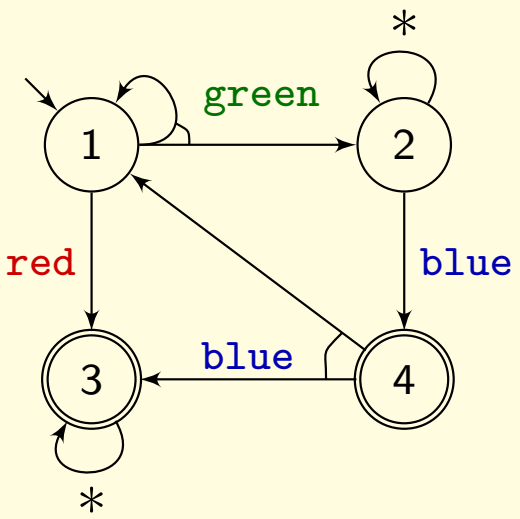


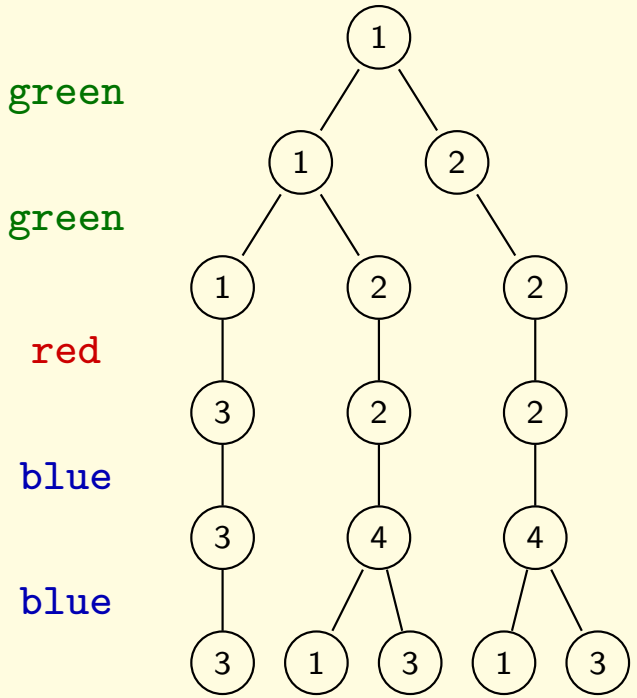

# Alternating Büchi automata

**Definition**

An alternating automaton is 1-*weak* if there exists a total order on its set of states such that transitions are always "decreasing".

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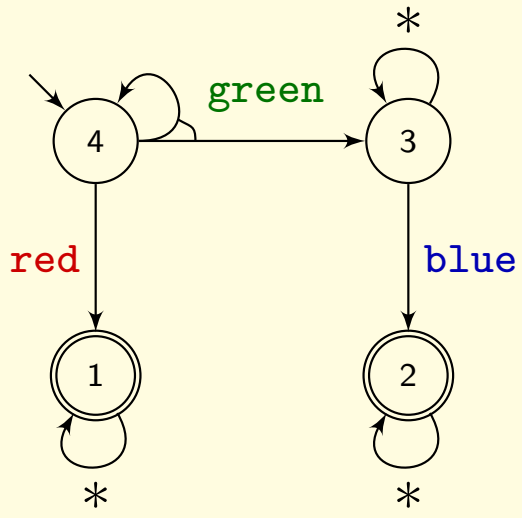

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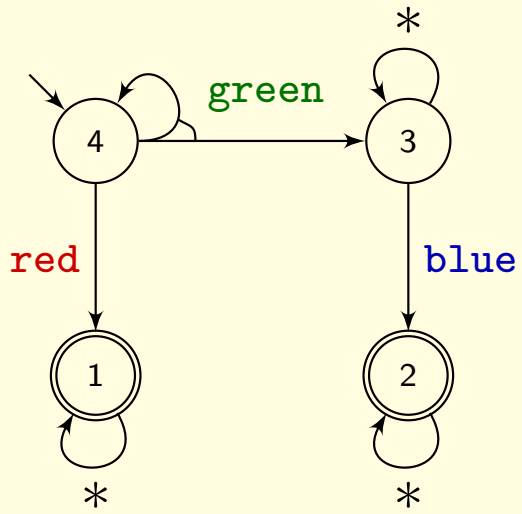


$$(\texttt{green} \wedge \mathbf{F}\,\texttt{blue})\ \mathbf{U}\ \texttt{red}$$

# Alternating Büchi automata

**Theorem (Vardi, 1994)**

*Any LTL formula $\varphi$ can be transformed into a 1-weak alternating Büchi automaton $\mathcal{B}_\varphi$ s.t.*

- *a word is accepted by $\mathcal{B}_\varphi$ iff it satisfies $\varphi$,*
- *$\mathcal{B}_\varphi$ has at most $|\varphi|$ states.*

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Conversely:

**Theorem (Rohde, 1997)**

*A 1-weak alternating Büchi automaton $\mathcal{B}$ can be transformed into an LTL formula $\varphi_\mathcal{B}$ s.t.*

- *a word is accepted by $\varphi$ iff it satisfies $\varphi_\mathcal{B}$,*
- *$\varphi_\mathcal{B}$ has size exponential in the size of $\mathcal{B}$.*

# Outline of today's lecture

# Succinctness of LTL+Past

Consider the following property, built on $\mathrm{AP} = \{p_0, \ldots, p_n\}$:

($\mathcal{P}$): any two states that agree on propositions $p_1$ to $p_n$ also agree on proposition $p_0$.

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It can be expressed in LTL by enumerating the possible valuations for $p_0$ to $p_n$:

$$\bigwedge_{(b_0,\ldots,b_n)\in\{\top,\bot\}^{n+1}} \left( \mathbf{F}\Big(\bigwedge_{i\geq 0} p_i = b_i\Big) \Rightarrow \mathbf{G}\Big(\big(\bigwedge_{i\geq 1} p_i = b_i\big) \Rightarrow p_0 = b_0\Big)\right)$$

The size of this formula is exponential in $n$.

# Succinctness of LTL+Past

$(\mathcal{P})$: any two states that agree on propositions $p_1$ to $p_n$ also agree on proposition $p_0$.

Let $\mathcal{A}$ be a Büchi automaton corresponding to property $(\mathcal{P})$.

Let $\Sigma = \{a_0, a_1, ..., a_{2^n-1}\}$ be the subsets of $\{p_1, ..., p_n\}$.

# Succinctness of LTL+Past

($\mathcal{P}$): any two states that agree on propositions $p_1$ to $p_n$ also agree on proposition $p_0$.

For each $K \subseteq \{0, ..., 2^n - 1\}$, we define $w_K = b_0 ... b_{2^n - 1}$ with

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**Lemma**

*There are $2^{2^n}$ different such words.*

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**Lemma**

*For any $K \neq K'$, the word $w_{K'} \cdot w_K^\omega$ is not accepted by $\mathcal{A}$.*

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**Theorem**

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**Corollary**

*Any LTL formula expressing property ($\mathcal{P}$) has size at least $2^{n-1}$.*

# Succinctness of LTL+Past

Consider now the following property, slightly different:

($\mathcal{P}'$): any state that agrees on propositions $p_1$ to $p_n$ with the initial state also agrees on proposition $p_0$.

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This can be expressed in LTL+Past by the following (polynomial-size) formula:

$$\mathbf{G}\Big(\big(\bigwedge_{i\geq 1} p_i \Leftrightarrow \mathbf{F}^{-1}\,\mathbf{G}^{-1}\,p_i\big) \Rightarrow \big(p_0 \Leftrightarrow \mathbf{F}^{-1}\,\mathbf{G}^{-1}\,p_0\big)\Big).$$

# Succinctness of LTL+Past

Consider now the following property, slightly different:

($\mathcal{P}'$): any state that agrees on propositions $p_1$ to $p_n$ with the initial state also agrees on proposition $p_0$.

This can be expressed in LTL+Past by the following (polynomial-size) formula:

$$\mathbf{G}\Big(\big(\bigwedge_{i\geq 1} p_i \Leftrightarrow \mathbf{F}^{-1}\,\mathbf{G}^{-1}\,p_i\big) \Rightarrow \big(p_0 \Leftrightarrow \mathbf{F}^{-1}\,\mathbf{G}^{-1}\,p_0\big)\Big).$$

Let $\varphi$ be an LTL formula expressing property ($\mathcal{P}'$). Then $\mathbf{G}\,\varphi$ precisely expresses property ($\mathcal{P}$), and thus has size at least $2^{n-1}$.

# Outline of today's lecture

# The stuttering principle

**Definition**

Let $w = w_0\, w_1 \ldots$ be a word over AP. A letter $a = w_i$ is *stuttering* in $w$ if it appears several consecutive times, i.e., if $a = w_i = w_{i+1}$.

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**Example**

$$w = \mathtt{g} \cdot \mathtt{r} \cdot \underbrace{\mathtt{g} \cdot \mathtt{g} \cdot \mathtt{g} \cdot \mathtt{g}} \cdot \mathtt{b} \cdot \mathtt{r} \cdots$$

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**Definition**

The relation $\preccurlyeq$ is defined as follows:

$$w \preccurlyeq w' \quad \Leftrightarrow \quad w \text{ is obtained from } w' \text{ by removing one copy of a stuttering letter.}$$

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**Example**

$$\mathtt{g} \cdot \mathtt{r} \cdot \underbrace{\mathtt{g} \cdot \mathtt{g} \cdot \mathtt{g}} \cdot \mathtt{b} \cdot \mathtt{r} \cdots \;\preccurlyeq\; \mathtt{g} \cdot \mathtt{r} \cdot \underbrace{\mathtt{g} \cdot \mathtt{g} \cdot \mathtt{g} \cdot \mathtt{g}} \cdot \mathtt{b} \cdot \mathtt{r} \cdots$$

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**Definition**

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**Example**

The words $\mathtt{g} \cdot (\mathtt{b} \cdot \mathtt{b} \cdot \mathtt{r})^\omega$ and $\mathtt{g} \cdot (\mathtt{b} \cdot \mathtt{r} \cdot \mathtt{r})^\omega$ are stuttering equivalent.

# The stuttering principle

**Definition**

*Stuttering equivalence* is the least equivalence relation that subsumes $\preccurlyeq$.

**Theorem**

*Two words are stuttering-equivalent iff they cannot be distinguished by any formula of* $\mathcal{L}(\mathbf{U})$.

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**Definition**

*Stuttering equivalence* is the least equivalence relation that subsumes $\preccurlyeq$.

**Theorem**

*Two words are stuttering-equivalent iff they cannot be distinguished by any formula of* $\mathcal{L}(\mathbf{U})$.

**Corollary**

$\mathcal{L}(\mathbf{U}, \mathbf{X})$ *has strictly more distinguishing power than* $\mathcal{L}(\mathbf{U})$.

# The generalized stuttering principle

**Definition**

A subword $w[i, j]$ of a word $w$ is $(m, n)$-redundant if the subword $w[i + j, mj - m + 1 + n]$ is a prefix of $w[i, j]^\omega$.

# The generalized stuttering principle

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## Example

r r g g r g r r g r r g r r g r g g r

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## Example

r r g g $\boxed{\text{r g r}}$ r g r r g r r g r g g r

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## Example

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| r | r | g | g | r | g | r | r | g | r | r | g | r | r | g | r | g | g | r |

$w[5,3]^\omega =$

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| r | g | r | r | g | r | r | g | r | r | g | r | r | g | r |

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| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| r | r | g | g | r | g | r | r | g | r | r | g | r | r | g | r | g | g | r |

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| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| r | g | r | r | g | r | r | g | r | r | g | r | r | g | r |

$w[5,3]$ is $(0,8)$-redundant, but also $(3,2)$-redundant.

# The generalized stuttering principle

**Definition**

Given two words $w$ and $w'$, and two integers $m$ and $n$, we define:

$$w \preccurlyeq_{m,n} w' \quad \Leftrightarrow \quad w \text{ is obtained from } w' \text{ by removing } (m,n)\text{-redundant subwords.}$$

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**Example**

$$\texttt{rrgg rgr rgr ggr}\ldots \preccurlyeq_{3,2} \texttt{rrgg rgr rgr rgr ggr}\ldots$$

# The generalized stuttering principle

**Definition**

We write $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^n)$ for the fragment of LTL where nesting identical modalities is bounded (by $m$ and $n$ for $\mathbf{U}$ and $\mathbf{X}$, respectively).

# The generalized stuttering principle

## Example

The following formula is in $\mathcal{L}(\mathbf{U}^2, \mathbf{X}^4)$:

$$\texttt{green}\ \mathbf{U}\ (\mathbf{X}(\texttt{red}\ \mathbf{U}\ \texttt{green} \wedge \mathbf{X}\ \texttt{blue})) \vee \mathbf{X}\,\mathbf{X}\,\mathbf{X}\,\mathbf{X}\ \texttt{red}$$

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The following formula is in $\mathcal{L}(\mathbf{U}^2, \mathbf{X}^4)$:

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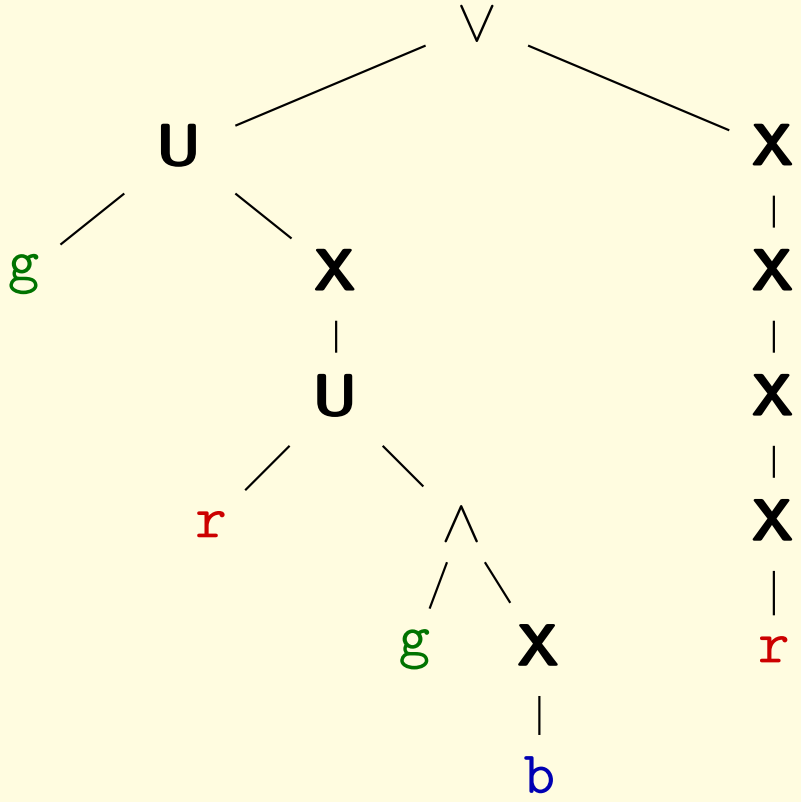

# The generalized stuttering principle

**Theorem (Kučera, Strejček, 2005)**

*If two words are $(m, n)$-stuttering equivalent, then they can't be distinguished by formulas of $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^n)$.*

# The generalized stuttering principle

**Theorem (Kučera, Strejček, 2005)**

*If two words are $(m, n)$-stuttering equivalent, then they can't be distinguished by formulas of $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^n)$.*

**Corollary**

*For any $m$ and $n$, there exists formulas in $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^n)$ that can be expressed neither in $\mathcal{L}(\mathbf{U}^{m-1}, \mathbf{X})$ nor in $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^{n-1})$.*

# The generalized stuttering principle

## Example

We illustrate the case where $m = 3$ and $n = 2$: let

$$\varphi = \mathbf{F}(\texttt{green} \wedge \mathbf{F}(\texttt{red} \wedge \mathbf{F}(\texttt{blue} \wedge \mathbf{X}\,\mathbf{X}\,\texttt{green}))).$$

and

$$w = \texttt{brg}\ \texttt{brg}\ \texttt{brg}\ \texttt{y}^\omega$$
$$w' = \texttt{brg}\ \texttt{brg}\ \texttt{y}^\omega$$

Then $\quad w, 0 \models \varphi \quad$ and $\quad w', 0 \not\models \varphi$.

$w$ and $w'$ are $(2,2)$- and $(3,1)$-stutter equivalent, and thus can be distinguished neither by formulas in $\mathcal{L}(\mathbf{U}^2, \mathbf{X}^2)$ nor by formulas in $\mathcal{L}(\mathbf{U}^3, \mathbf{X}^1)$.

# The generalized stuttering principle

**Theorem**

*The family $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^n)$ form a strict hierarchy w.r.t. expressive power.*

# The generalized stuttering principle

**Theorem**

*The family $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^n)$ form a strict hierarchy w.r.t. expressive power.*

It seems natural to conjecture that

> if a property can be expressed in $\mathcal{L}(\mathbf{U}^{m+1}, \mathbf{X}^n)$ and in $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^{n+1})$, then it can be expressed in $\mathcal{L}(\mathbf{U}^m, \mathbf{X}^n)$.

This is false: for instance, $\mathbf{F}(\texttt{green} \wedge \mathbf{F} \neg \texttt{green})$ and $\mathbf{F}(\texttt{green} \wedge \mathbf{X} \neg \texttt{green})$ are equivalent, but cannot be expressed in $\mathcal{L}(\mathbf{U}^1)$.

# Outline of today's lecture

# Ehrenfeucht-Fraïssé games

EF games are 2-player games used to show that two linear structures $T$ and $U$ can (or cannot) be distinguished by a logic.

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A *configuration* of the game is a couple $\langle t, u \rangle$ where $t \in T$ and $u \in U$.

# Ehrenfeucht-Fraïssé games

EF games are 2-player games used to show that two linear structures $T$ and $U$ can (or cannot) be distinguished by a logic.

**Definition**

A *configuration* of the game is a couple $\langle t, u \rangle$ where $t \in T$ and $u \in U$.

**Definition**

From a configuration $\langle t_0, u_0 \rangle$, the rules of a $k$-round EF game is defined recursively as follows:

- when $k = 0$, player $\mathcal{A}$ wins if $t_0$ and $u_0$ are labeled by exactly the same atomic propositions;
- when $k \geq 1$, two cases may arise:
  - if $t_0$ and $u_0$ are not labeled by the same atomic propositions, then the game stops and player $\mathcal{A}$ is declared the winner;
  - otherwise, player $\mathcal{A}$ plays an **U**-move or a **S**-move.

# Ehrenfeucht-Fraïssé games

EF games are 2-player games used to show that two linear structures $T$ and $U$ can (or cannot) be distinguished by a logic.

## Definition

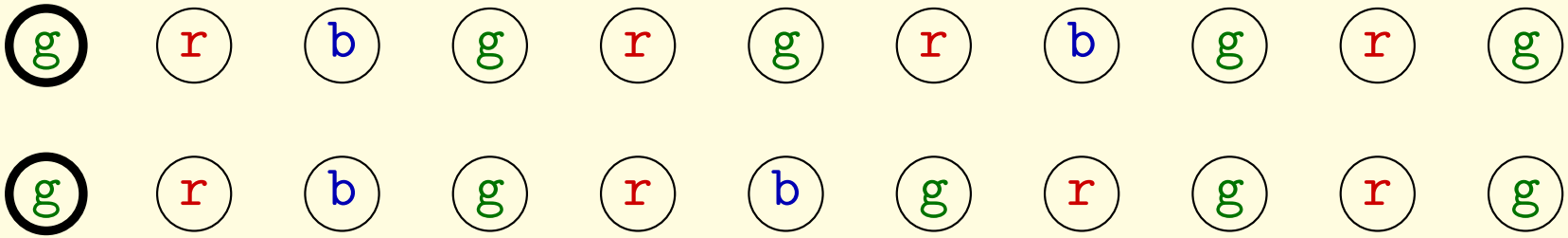
An **U**-move from configuration $\langle t, u \rangle$ is played as follows:

- player $\mathcal{A}$ selects the structure he wants to play on (say $T$), and an element $t'$ of that structure s.t. $t \leq t'$;
- player $\mathcal{B}$ responds by choosing an element $u'$ in the other structure s.t. $u \leq u'$;
- player $\mathcal{A}$ has then two choices:
  - either he sets the new configuration to $\langle t', u' \rangle$;
  - or he selects a position $u''$ in $U$ s.t. $u \leq u'' < u'$, player $\mathcal{B}$ chooses $t''$ in $T$ with $t \leq t'' < t'$, and the new configuration is $\langle t'', u'' \rangle$.

**S**-moves are symmetric.

# Ehrenfeucht-Fraïssé games

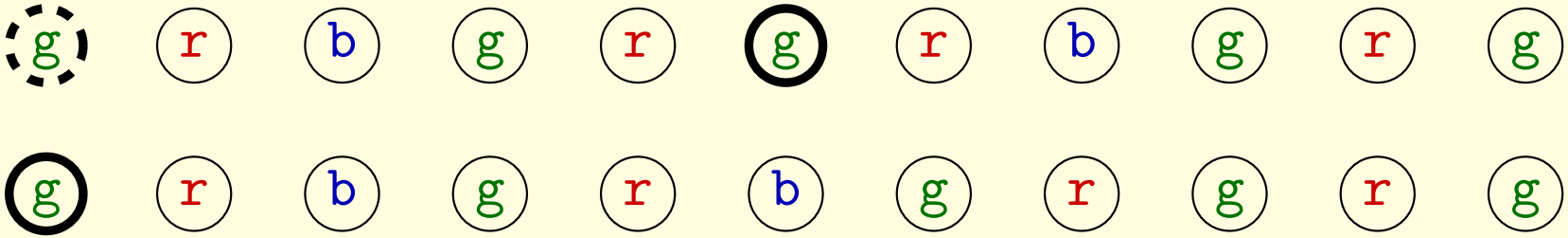
## Example




Initial configuration

# Ehrenfeucht-Fraïssé games

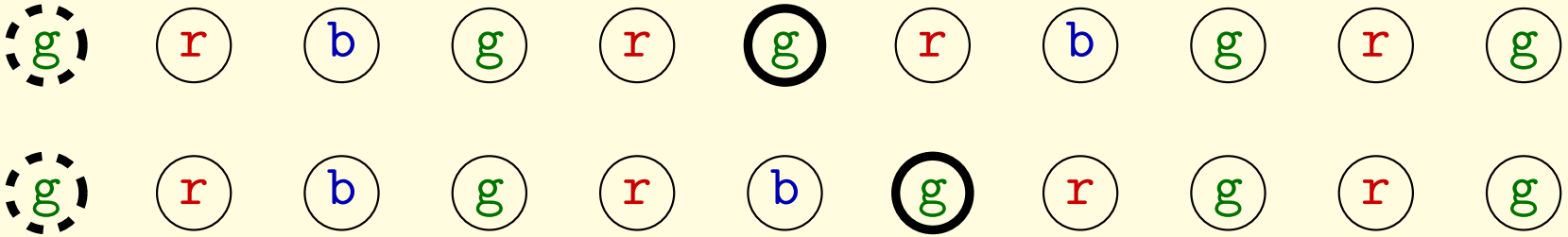
## Example




Player $\mathcal{A}$ plays an **U**-move on the first structure.

# Ehrenfeucht-Fraïssé games

## Example




Player $\mathcal{B}$ responds on the other structure.

# Ehrenfeucht-Fraïssé games

## Example

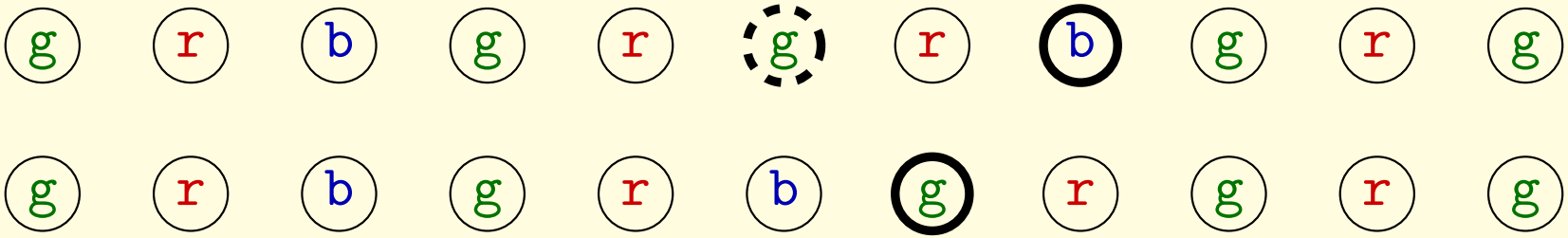
g r b g r g r b g r g

g r b g r b g r g r g

This is the new configuration.

# Ehrenfeucht-Fraïssé games

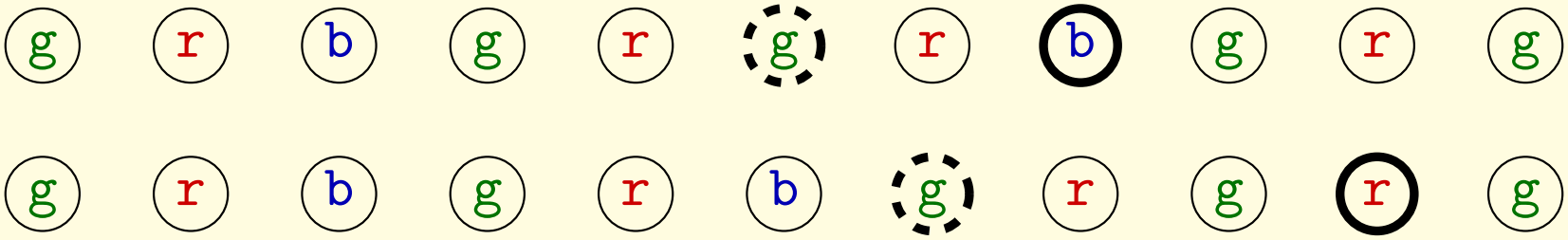
## Example




Player $\mathcal{A}$ plays an **U**-move on the first structure.

# Ehrenfeucht-Fraïssé games

## Example




Player $\mathcal{B}$ responds on the other structure...

# Ehrenfeucht-Fraïssé games

## Example

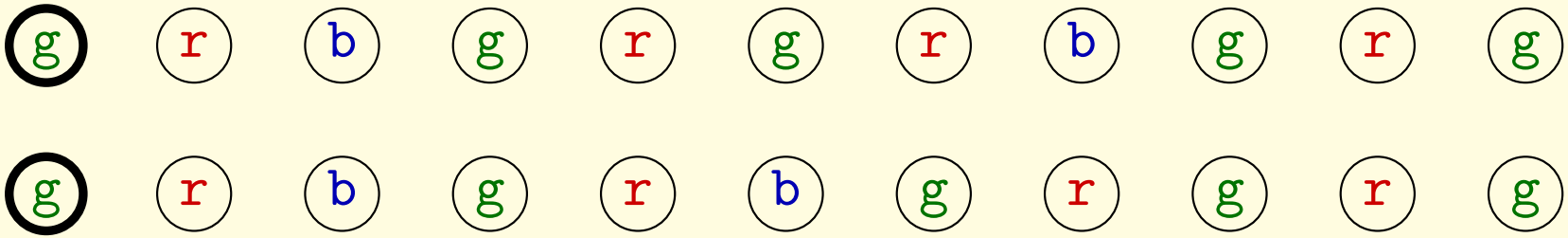
g r b g r g r **b** g r g

g r b g r b g r g **r** g

Player $\mathcal{B}$ responds on the other structure... and loses.

# Ehrenfeucht-Fraïssé games

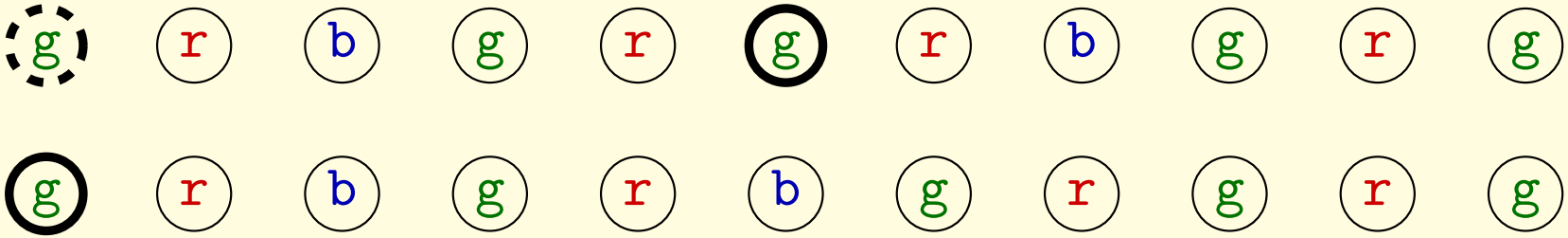
## Example




Initial configuration

# Ehrenfeucht-Fraïssé games

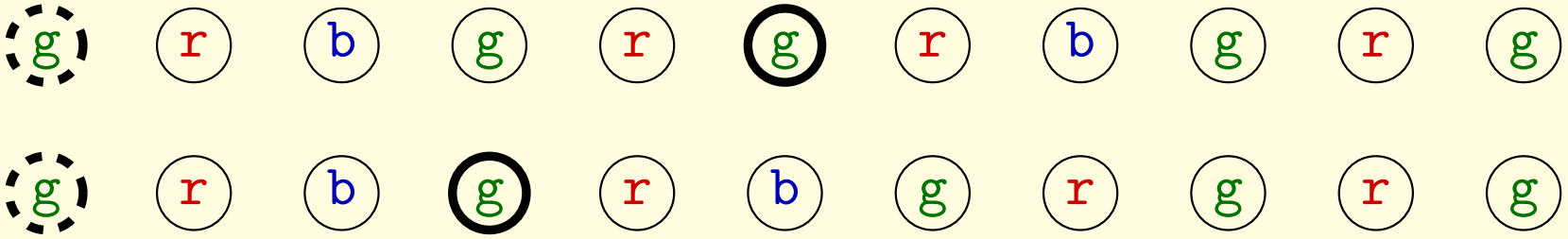
## Example




Player $\mathcal{A}$ plays an **U**-move on the first structure.

# Ehrenfeucht-Fraïssé games

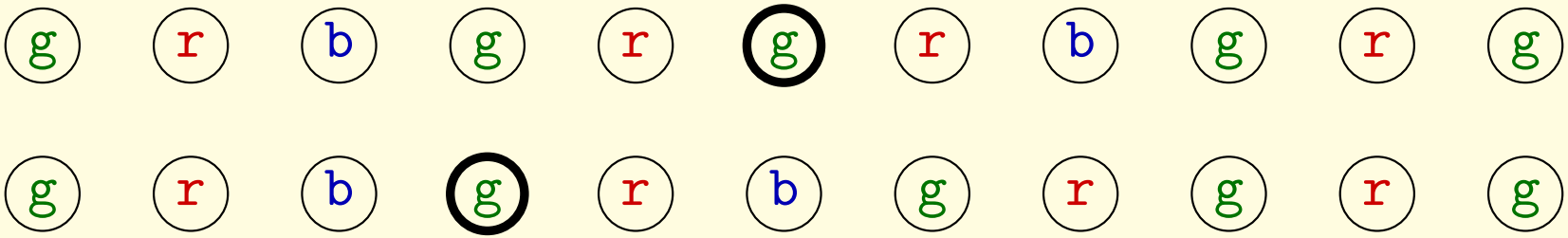
## Example




Player $\mathcal{B}$ responds on the other structure.

# Ehrenfeucht-Fraïssé games

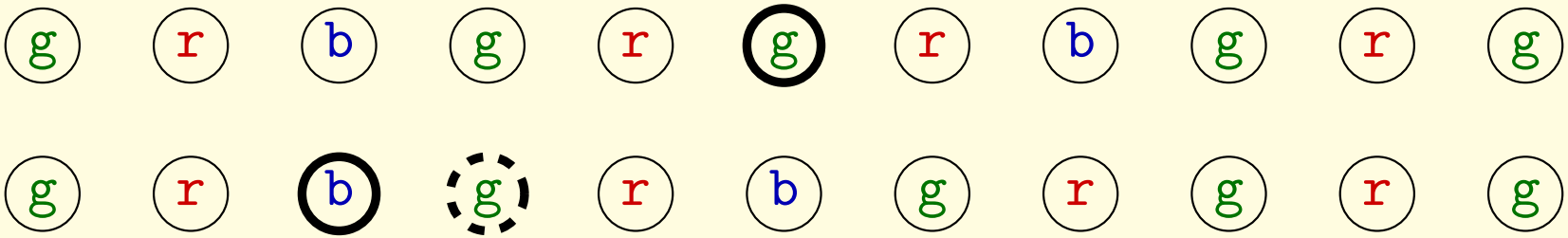
## Example




This is the new configuration.

# Ehrenfeucht-Fraïssé games

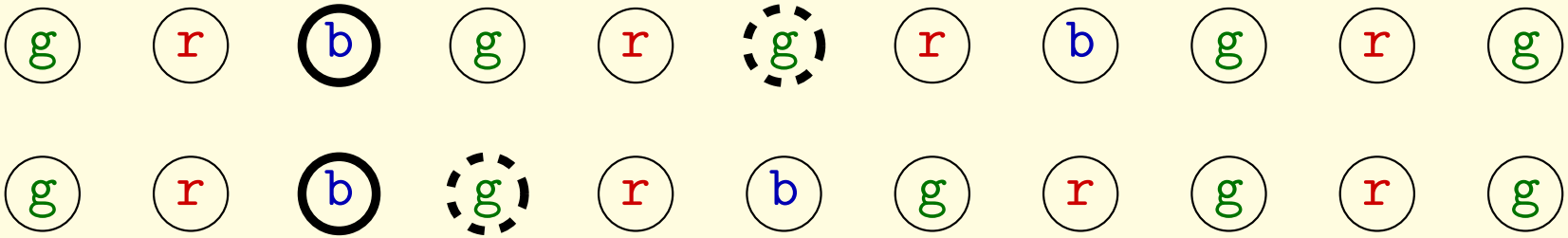
## Example




Player $\mathcal{A}$ plays a **S**-move on the second structure.

# Ehrenfeucht-Fraïssé games

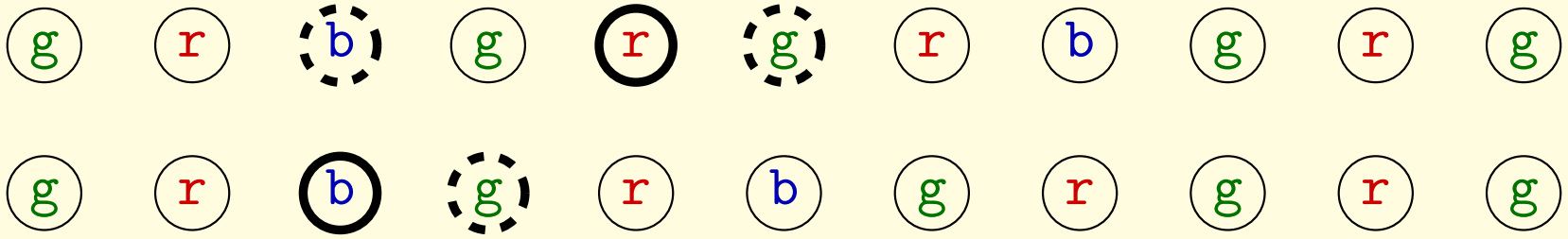
## Example




Player $\mathcal{B}$ responds on the other structure.

# Ehrenfeucht-Fraïssé games

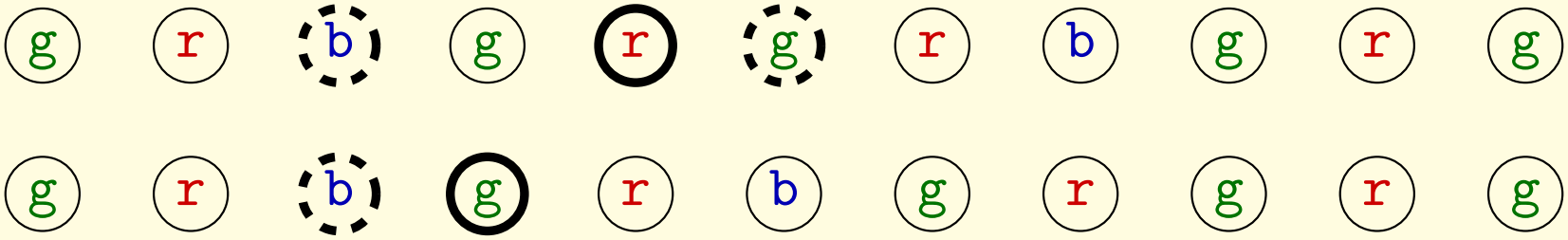
## Example




Player $\mathcal{A}$ picks an intermediary position on the first structure.

# Ehrenfeucht-Fraïssé games

## Example




Player $\mathcal{B}$ responds...

# Ehrenfeucht-Fraïssé games

## Example

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| g | r | b | g | **r** | g | r | b | g | r | g |
| g | r | b | **g** | r | b | g | r | g | r | g |

Player $\mathcal{B}$ responds... and loses.

# EF games and the Until-Since hierarchy

**Definition**

We write $\mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^k)$ for the fragment of $\mathcal{L}(\mathbf{U}, \mathbf{S})$ where the temporal height is bounded by $k$.

# EF games and the Until-Since hierarchy

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**Example**

$$\texttt{green}\ \mathbf{U}\ ((\texttt{green}\ \mathbf{S}\ \texttt{red})\ \mathbf{S}\ (\texttt{red}\ \mathbf{U}\ \texttt{green} \wedge (\texttt{green}\ \mathbf{U}\ \texttt{blue})))$$

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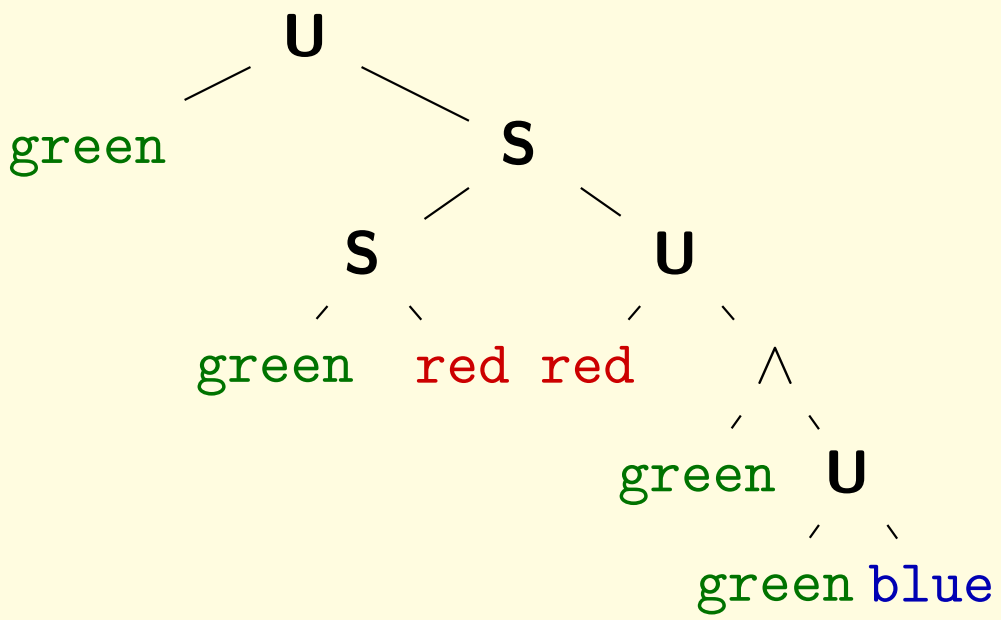

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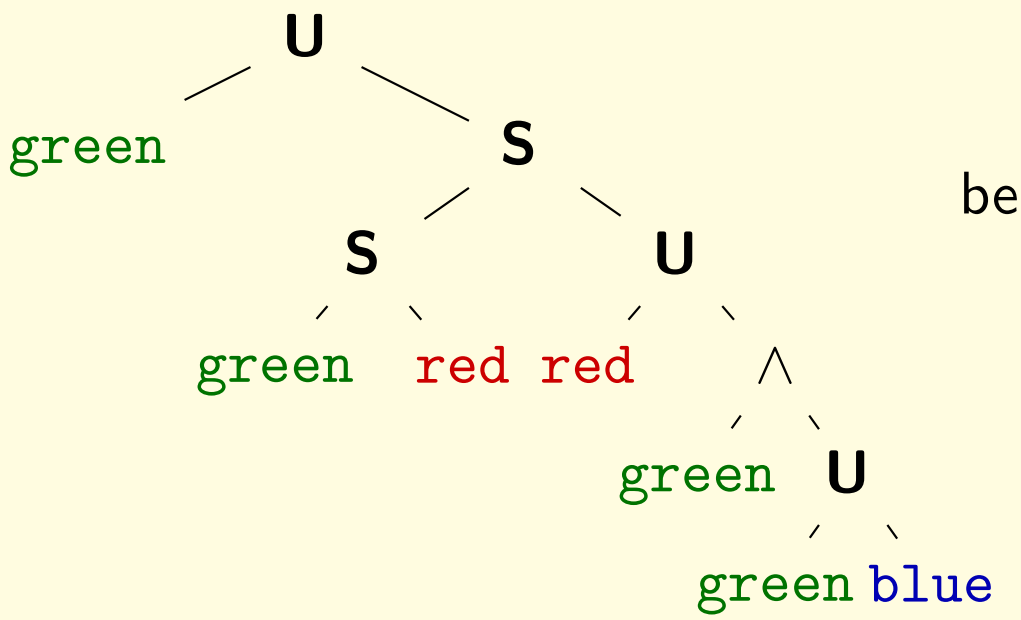


belongs to $\mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^4)$

# EF games and the Until-Since hierarchy

**Definition**

We write $\mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^k)$ for the fragment of $\mathcal{L}(\mathbf{U}, \mathbf{S})$ where the temporal height is bounded by $k$.

**Theorem (Etessami & Wilke, 2000)**

*Player $\mathcal{B}$ has a winning strategy in the $k$-round game from a configuration $\langle t, u \rangle$ iff, for any formula $\varphi \in \mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^k)$, we have*

$$\langle T, t \rangle \models \varphi \;\Leftrightarrow\; \langle U, u \rangle \models \varphi.$$

# EF games and the Until-Since hierarchy

Now, consider the following two structures:

$$T_k = ((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,(\mathtt{r}\,\mathtt{b})^{k} \quad \mathtt{g}\,\mathtt{b}\,((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,\mathtt{y}^{\omega}$$

$$U_k = ((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,(\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b}\,((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,\mathtt{y}^{\omega}$$

# EF games and the Until-Since hierarchy

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For instance:

$$T_3 = \mathtt{rbrbgbrbrbgbrbrbrbgbrbrbgbrbrbgby}^{\omega}$$
$$U_3 = \mathtt{rbrbgbrbrbgbrbrbgbrbrbgbrbrbgby}^{\omega}$$

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$$U_3 = \mathtt{rbrbgbrbrbgbrbrbgbrbrbgbrbrbgby}^{\omega}$$

**Lemma**

*Player $\mathcal{B}$ has a winning strategy in the $k$-round game based on $T_k$ and $U_k$ from configuration $\langle 0, 0\rangle$.*

# EF games and the Until-Since hierarchy

Now, consider the following two structures:

$$T_k = ((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,(\mathtt{r}\,\mathtt{b})^{k} \quad \mathtt{g}\,\mathtt{b}\,((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,\mathtt{y}^{\omega}$$
$$U_k = ((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,(\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b}\,((\mathtt{r}\,\mathtt{b})^{k-1}\,\mathtt{g}\,\mathtt{b})^{k-1}\,\mathtt{y}^{\omega}$$

For instance:

$$T_3 = \mathtt{rbrbgbrbrbgbrbrbrbgbrbrbgbrbrbgby}^{\omega}$$
$$U_3 = \mathtt{rbrbgbrbrbgbrbrbgbrbrbgbrbrbgby}^{\omega}$$

**Lemma**

*There exists a formula $\varphi \in \mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^{k+1})$ that distinguishes between $\langle T_k, 0\rangle$ and $\langle U_k, 0\rangle$.*

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**Lemma**

*There exists a formula $\varphi \in \mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^{k+1})$ that distinguishes between $\langle T_k, 0\rangle$ and $\langle U_k, 0\rangle$.*

For instance:

$$\top\ \mathbf{U}\,(\mathtt{b} \wedge (\mathtt{b}\ \mathbf{S}\,(\mathtt{r} \wedge \mathtt{r}\ \mathbf{S}\ \mathtt{b})) \wedge (\mathtt{b}\ \mathbf{U}\,(\mathtt{r} \wedge \mathtt{r}\ \mathbf{U}\,(\mathtt{b} \wedge \mathtt{b}\ \mathbf{U}\ \mathtt{r}))))$$

# EF games and the Until-Since hierarchy

**Theorem (Etessami & Wilke, 2000)**

*The hierarchy $\mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^k)$ is strict (w.r.t. distinguishing power).*

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Similar techniques can be used to prove that:

**Theorem (Etessami & Wilke, 2000)**

*The hierarchy* $\mathcal{L}(\{\mathbf{U}, \mathbf{S}\}^k, \{\mathbf{X}, \mathbf{F}, \mathbf{X}^{-1}, \mathbf{F}^{-1}\}^r)$ *is strict (w.r.t. distinguishing power).*