completion continued:

We have a metric space (X, d). We considered the space X^1 of Cauchy sequences (x_n). Defined an equivalence relation $(x_n) \sim (y_n)$ if they appear to be converging to the same point; more precisely, if $d(x_n, y_n) \to 0$. X^* is the space of equivalence classes or bags as we called them. Define $d^*([x], [y]) = \lim d(x_n, y_n)$. We showed that this limit exists and does not depend on the sequences you have taken from the two bags.

Defined the map $\varphi : X \to X^*$ as follows. For $p \in X$, $\varphi(p)$ is the bag containing the constant sequence all of whose terms equal p. We showed that this is distance preserving map.

Recall that, for a metric space (Z, ρ), a subset $D \subset Z$ is dense in Z if every non-empty open set contains a point of D. equivalently, given $z \in Z$ and $\epsilon > 0$, there is $p \in D$ such that $\rho(z, p) < \epsilon$..

We now show that range $\varphi(X)$ is dense in X^*. To do this, take $[x] \in X^*$ say the bag containing the Cauchy sequence (x_n). Let $\epsilon > 0$. We shall show $p \in X$ such that $d^*(\varphi(p), [x]) < \epsilon$. Choose N such that $n, m \geq N$ implies $d(x_n, x_m) \leq \epsilon/2$. Take $p = x_N$. We show that this does. Since $\varphi(p)$ is the constant sequence $\{p, p, p \cdots\}$ we see, by calculation,

$$d^*(\varphi(p), [x]) = \lim d(x_N, x_n) \leq \epsilon/2 < \epsilon.$$

We now show that X^* is a complete metric space. But before we do this let us make an observation which will avoid some notational confusion later.

Let (S, d) be a metric space and $T \subset S$ be a dense subset.

Assume the following: (x_n) is a Cauchy sequence; each x_n belongs to T then there is a point of $x \in S$ such that $x_n \to x$. Then the metric space S is complete.

Equivalently, every Cauchy sequence in S converges.

In other words, if you know Cauchy sequences with terms coming from T converge (in S, we are not saying that they converge to points in T) then every Cauchy sequence in S converges.

Proof is simple and as follows. let (x_n) be a Cauchy sequence in S. for each n, using denseness of T pick $y_n \in T$ so that $d(x_n, y_n) < 1/2^n$. Now we see that (y_n) are all points in T. The plan is to show that (y_n) is a Cauchy sequence and get its limit guaranteed by hypothesis, and show that original sequence (x_n) also converges to this limit. Towards showing (y_n) is Cauchy, fix $\epsilon > 0$. Choose N so that $m, n \geq N$ implies $d(x_n, x_m) < \epsilon/2$ and also $\sum\limits_{m \geq N} 2^{-m} < \epsilon/2$. if now, $m, n \geq N$

$$d(y_m, y_n) \leq d(y_m, x_m) + d(x_m, x_n) + d(x_n, y_n) \leq \epsilon/2 + \epsilon/2.$$

So let $y_n \to y$. Towards showing that $x_n \to y$, you only need to note that $d(x_n, y_n) \leq 1/2^n \to 0$. Or, explicitly, given $\epsilon > 0$, choose N so that $|y_n - y| < \epsilon/2$ for $n \geq N$ and also $1/2^N < \epsilon/2$. Then for $n \geq N$,

$$d(x_n, y) \leq d(x_n, y_n) + d(y_n, y) \leq \epsilon/2 + \epsilon/2.$$

This completes the observation. Let us now return to showing that X^* is complete. From the observation, it suffices to show that Cauchy sequences whose points come from the dense set $\varphi(X)$ converge. Accordingly, take a Cauchy sequence $\varphi(p_n)$ where $p_n \in X$ for each n. Since φ is distance preserving map, we conclude that $x = (p_n)$ is a Cauchy sequence in X and is an element of X^1, the space of Cauchy sequences. We show $\varphi(p_n) \to [x]$. Take $\epsilon > 0$. Choose N so that $d(p_n, p_m) < \epsilon/2$ for $n, m \geq N$. We show now $d^*(\varphi(p_i), [x]) < \epsilon$. for $i \geq N$. Keep in mind that $\varphi(p_i)$ is the constant sequence

$$p_i,\ p_i,\ p_i,\ p_i, \cdots$$

and x is the sequence

$$p_1,\ p_2,\ p_3,\ p_4, \cdots.$$

Use definition of d^* to see

$$d^*(\varphi(p_i), [x]) = \lim_n d(p_i, p_n) \leq \epsilon/2.$$

The last inequality is from the fact that $i \geq N$ and as soon as $n \geq N$ we know $d(p_i, p_n) \leq \epsilon/2$.

definition of Completion:

Let (X, d) be a metric space. By completion of (X, d) we mean a metric space (Z, ρ) and a map $\varphi : X \to Z$ such that the following hold.
(i) Z is a complete metric space.
(ii) φ is distance preserving map.

(iii) Range $\varphi(X)$ is dense in Z.

Condition (ii) tells you that $\rho(\varphi(x), \varphi(y)) = d(x, y)$. In particular if $x \neq y$ then $\varphi(x) \neq \varphi(y)$. Thus φ is a bijection from X to $\varphi(X)$. Since the distance is also preserved, when you see $\varphi(X)$ it looks exactly like X. In other words you see a replica of X in this new space.

To put it differently, if you rename the point $\varphi(x)$ as x, then you see X in Z. Thus many times we regard $X \subset Z$. Just like, while constructing real number system starting from rationals, we considered either cuts or Cauchy sequences; but no matter what, we regarded Q as a subset of the R we constructed.

Such a view leads to the nice feeling that the extra points, that is points of $Z - X$, are the new ones needed to show as limits of Cauchy sequences. Thus the new space is not as abstract as it appears, but original set of pints with new things thrown in where ever necessary.

Condition (iii) tells you that you have not added unnecessary points. More precisely, if $z \in Z - \varphi(X)$, then denseness of $\varphi(X)$ tells you that there is a sequence $(x_n) \subset X$ such that $\varphi(x_n) \to z$. But then $\varphi(x_n)$ is a Cauchy sequence (recall any convergent sequence is a Cauchy sequence, in any metric space). But φ being distance preserving, we conclude (x_n) itself is a Cauchy sequence in X. It does not converge in X [if it did, say to x then $\varphi(x_n)$ converges to $\varphi(x)$ a point of $\varphi(X)$ contradicting that it converges to a point of $Z - \varphi(X)$.] Thus the point z is essential to show as limit of the Cauchy sequence (x_n).

Of course condition (i) tells you that your new space is complete. Thus we 'embedded' X in a complete space without bringing in un-necessary points in the process.

Thus what we have shown is that every metric space has a completion. (X^*, d^*) satisfies all the three conditions.

We shall now show that a completion is unique. More precisely, if there are two completions (Z_1, ρ_1, φ_1) and (Z_2, ρ_2, φ_2) then there is distance preserving bijection between them that keeps X fixed. This means there is $f : Z_1 \to Z_2$ which is distance preserving and $f(\varphi_1(x)) = \varphi_2(x)$ for all $x \in X$. Remember $x \in X$ looks like $\varphi_1(x)$ in Z_1 whereas it looks like $\varphi_2(x)$ in Z_2.

Sometimes statement of a claim itself includes how to start its proof.

Thus let there be two completions with notation as above. We define f as follows. For $x \in X$ we define $f(\varphi_1(x)) = \varphi_2(x)$. Now take any $z \in Z_1$. Since $\varphi_1(X)$ is dense in Z_1 take a sequence $\varphi_1(x_n) \to z$ in Z_1. In particular, $\{\varphi_1(x_n)\}$ is a Cauchy sequence in Z_1. Since φ_1 is distance preserving we conclude that (x_n) is Cauchy in X. But now φ_2 is distance preserving tells us that $\{\varphi_2(x_n)\}$ is Cauchy in Z_2. since Z_2 is complete the limit $\lim \varphi_2(x_n)$ exists in Z_2 and this limit is defined as $f(z)$.

This is a good definition because if some one takes a different sequence $\varphi_1(y_n) \to z$ then we see

$$\rho_2(\varphi_2(x_n), \varphi_2(y_n)) = d(x_n, y_n) = \rho_1(\varphi_1(x_n), \varphi_1(y_n)) \to 0$$

so that $\varphi_2(y_n)$ also converges o the same limit as $\varphi_2(x_n)$.

This completes the definition of f on Z_1 to Z_2. It is distance preserving because for two points $z, w \in Z_1$ take $\varphi_1(x_n) \to z$ and $\varphi_1(y_n) \to w$ and see

$$\begin{aligned} \rho_2(f(z), f(w)) &= \lim \rho_2(\varphi_2(x_n), \varphi_2(y_n)) = \lim d(x_n, y_n) \\ &= \lim \rho_1(\varphi_1(x_n), \varphi_1(y_n)) = \rho(z, w). \end{aligned}$$

It follows that f is one-one too. Indeed if $z \neq w$, then

$$\rho_2(f(z), f(w)) = \rho_1(z, w) \neq 0$$

showing that $f(z) \neq f(w)$.

Shall show that f is onto Z_2. Indeed, if $w \in Z_2$, then using $\varphi_2(X)$ is dense, take $\varphi_2(x_n) \to w$. Repeating the earlier argument conclude $\{\varphi_2(x_n)\}$ and hence $\{x_n\}$ and hence $\{\varphi_1(x_n)\}$ are Cauchy (in their respective spaces) Using completeness of Z_1 get limit z of this last sequence and argue $f(z) = w$.

This completes the proof of the fact that every metric space has a completion and it is unique in the sense (loosely speaking) between any two completions there is a distance preserving bijection that is identity on X, which is inside both. Of douse, more precise statement is as mentioned earlier.

This process is due to Cantor. What we have done earlier to construct real numbers starting from rationals is precisely this completion; no more and no less.

This analysis shows some general facts. here is one.

Suppose (Z_1, ρ_1) and (Z_2, ρ_2) are two complete metric spaces. suppose D is a dense subset of Z_1 and f is a distance preserving map defined on D into Z. Then the map can be extended to a distance preserving map on Z_1 into Z_2. Proof is already contained in what we have done above. Take any $z \in Z_1$; use D is dense, get $x_n \in D$ such that $x_n \to z$, conclude (x_n) is Cauchy, conclude $f(x_n)$ is Cauchy, use Z_2 is complete, get its limit and declare that as $f(z)$. One shows that this is well defined; does not depend on the sequence (x_n) you have taken from D.

Also this is distance preserving just as above. Such an extension is unique too, because any distance preserving map has the property that when $x_n \to x$, then $f(x_n) \to f(x)$. This shows that for z, its value $f(z)$ must equal to what we defined.

of course this may not be onto. One can show that it is onto iff $f(D)$ is dense in Z_2.

Here is another fact that comes out of the analysis above. Let X^1 be a set and $d^1 : X \times X \to [0, \infty)$ satisfying the following:
$d^1(x, x) = 0$.
$d^1(x, y) = d^1(y, x)$.
$d^1(x, z) \leq d^1(x, y) + d^1(y, z)$.

Thus d^1 is nearly a metric. It falls short of being metric only because it may not satisfy: $d^1(x, y) = 0$ implies $x = y$. If this condition is also satisfied then this is indeed a metric.

Let us start with (X^1, d^1) as above. then define $x \sim y$ if $d^1(x, y) = 0$. This is an equivalence relation. Indeed, $x \sim x$ because $d^1(x, x) = 0$. If $x \sim y$ then $y \sim x$ by symmetry of d^1. Triangle inequality shows that $x \sim y$ and $y \sim z$ implies $x \sim z$.

Let us consider the space of equivalence classes, denote, X. Define $d([x], [y]) = d^1(x, y)$. This is well defined. Indeed, if $u \in [x]$ and $v \in [y]$, then

$$d^1(u, v) \leq d^1(u, x) + d^1(x, y) + d^1(y, v) = 0 + d^1(x, y) + 0 = d^1(x, y).$$

and

$$d^1(x,y) \leq d^1(x,u) + d^1(u,v) + d^1(v,y) = 0 + d^1(u,v) + 0 = d^1(u,v).$$

Thus $d^1(x,y) = d^1(u,v)$.

It is not difficult to verify that (X,d) is a metric space. Thus d satisfies all the three rules above and the missing rule too.

If you carefully see the completion process this is precisely what we did. The space X^1 is the space of Cauchy sequences and
$d^1((x_n),(y_n)) = \lim d(x_n,y_n)$.
The above equivalence relation is precisely what we used to make bags of Cauchy sequences and the new metric was d^*.

In other words the process of completion due to Cantor not only outlined how to complete a metric space, producing a construction of real numbers; it has also thrown out certain general techniques like the above two. [Of course, historically, matters are different. Cantor did for real line construction. But what he did was so powerful that it applied for metric spaces when they were discovered].

Some times, completion has a concrete representation. For example consider $(0,1)$ with usual $d(x,y) = |x-y|$. The space is not complete. To complete it you should consider space of cauchy sequences. But in this case completion is just $[0,1]$. This is obvious pif you consider the map φ to be the map $\varphi(x) = x$ on $(0,1)$. Here $X = (0,1)$, $Z = [0,1]$. We know Z is complete, φ preserves distance and its range $\varphi(X)$ is dense in Z.

But then what happens if you repeated the construction? Well you see apart from the bags containing constant sequences $\{p,p,p\cdots\}$ for $p \in (0,1)$; you will get only two new bags. They are: bag containing the sequence $\{1/2, 1/3, 1/4, 1/5, \cdots\}$ and bag containing $\{1/2, 2/3, 3/4, 4/5, \cdots\}$. If you name these two bags as zero and one you have $[0,1]$.

category, pseudo metric, isometry:

I have resisted the temptation of mentioning technical words. This is because many times students use such a technical word but unfortunately can not explain what it means. You should know that technical word is only an agreement to use a compact word instead of long expression. The most important thing is to know and understand what the word stands for.

The d^1 we described above which nearly satisfies the axioms of a metric is called a pseudo-metric.

Thus **pseudo metric** on a set X is a function d defined on $X \times X$ which satisfies axioms of metric except possibly: $d(x, y) = 0$ implies $x = y$. Thus a metric is a pseudo-metric, but a pseudo metric need not be metric. But what we described above produces a metric space from a pseudo-metric space.

For example, you can consider R^3 and define

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_3 - y_3)^2}; \quad x = (x_1, x_2, x_3), y = (y_1, y_2, y_3).$$

This is a pseudo metric but not a metric.

Distance preserving maps are called isometries. Thus an **isometry** from a metric space (Z_1, d_1) to a metric space (Z_2, d_2) is a map T satisfying $d_1(x, y) = d_2(Tx, Ty)$.

As mentioned at the beginning of our excursion into metric spaces, the plan is to execute convergence, continuity, and imitate some of our calculus concepts in a more general setting. We discussed sequences and convergence. A function from a metric space X to a metric space Y is said to be **continuous** if the following holds: whenever $x_n \to x$ then $f(x_n) \to f(x)$.

Several routine results from calculus can be imitated. For example a function is continuous iff inverse image of open sets are open. Also, the (ϵ, δ) definition holds. More precisely, define f, as above from X to Y is **continuous at a point** a if the following holds: $x_n \to a$ implies $f(x_n) \to f(a)$.

Then one can show f is continuous at a iff given $\epsilon > 0$, there is a $\delta > 0$ such that the following holds: $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x, a) < \delta$. Also one can show that f is continuous iff it is continuous at every point. Proofs are exactly same as in the usual case discussed in Calculus.

Returning to isometries, you can immediately see that an isometry is a continuous map. It is more than continuous. It satisfies a stronger property, namely, preserves distances.

What we called small sets are called nowhere dense sets. Thus **nowhere dense set** is a set A such that closure of A has no interior point. That is, the only open set contained in $\overline{A}$ is the empty set.

Here is the reason for this word. if you take Q the set of rationals then it is dense in R. It was referred to as everywhere dense. Suppose I take D to be the set of all integers union set of rationals in $(0,1)$. Then it is clear that D is not dense in R, for example, the interval $(14,15)$ does not have any point of D. But you can see that this set D is dense in a part of R, namely, in the open set $(0,1)$ (or $[0,1]$ does not matter).

A small set is not dense in any open set $U \neq \emptyset$. That is why it was called nowhere dense. A set is **first category** if it is union of countably many nowhere dense sets. A set is of **second category** if it is not of first category.

Thus we can state **Baire's theorem** as: Avery complete metric space is of second category.

Try to understand these technical words, then you can follow literature. Nobody uses the word small sets, distance preserving maps etc.

Banach's contraction mapping principle:

We need to discuss compactness and continuous maps. But let us discuss one more important theorem in Complete metric spaces. This is due to Banach.

Let X be a metric space. A map $T : X \to X$ is called a contraction if there is a number c such that $0 \leq c < 1$ and for all points $d(Tx, Ty) \leq cd(x, y)$. Thus application T reduces distances in a strong sense. It is not simply distances are reduced, they are reduced by an assured proportion c.

Of course, contraction is a continuous map. Indeed if $x_n \to x$ then

$$d(Tx_n, Tx) \leq cd(x_n, x) \to 0.$$

Theorem (Banach fixed point theorem, Banach contraction mapping principle).

If T is a contraction of a complete metric space X to itself then there is unique point x^* such $Tx^* = x^*$.

Such a point $Ta = a$ is called a fixed point, because T fixes it. That fixed point is unique is not surprising. If a and b are fixed points then

$$d(a,b) = d(Ta, Tb) \leq cd(a,b); \quad i.e. \quad d(a,b) = 0.$$

Existence o fixed point is also easy, we show that whatever point x you start with, the sequence $x, Tx, T^2x, T^3x, \cdots$ converges to x^*.

This theorem has several applications and several generalizations. We are not concerned with any generalization. There are several ways contraction maps arise and are useful.

suppose that f is a C^1 map of $[a, b]$ to itself. Since derivative of f is continuous and the interval is closed bounded, there is a number c such that $|f'| \leq c$. Suppose that (by chance) $c < 1$. Then the map f is a contraction map. Indeed by mean value theorem, if you take $x \neq y$ then

$$\frac{|f(x) - f(y)|}{|x - y|} = |f'(\theta)| \leq c.$$

In other words, $|f(x) - f(y)| \leq c|x - y|$.

Or equivalently, we have $d(f(x), f(y)) \leq cd(x, y)$.