Understanding R:

If you go close to the real line you see that you can add and multiply numbers. But in a superficial look at the line, you only see what comes after what. We shall now isolate some order-theoretic properties of R, whose presence in a loset helps us to recognize that that loset is none other than R itself.

This understanding will familiarize with structure of R. Also motivates and tells us 'how to proceed' to construct R.

Let $(X, <)$ be a loset. A non-empty subset $S \subset X$ is said to be *bounded above* if there is an $a \in X$ such that for all $x \in S$ we have $x \leq a$. Such an element a is also called an *upper bound* for the set S. An element $s \in X$, if exists, is said to be *supremum* of the set S if it is an upper bound and for any upper bound b we have $a \leq b$. In other words, supremum is the least upper bound. If there is supremum, then it must be unique. Note that neither upper bound nor supremum need belong to the set S.

Similarly, a subset $S \subset X$ is said to be *bounded below* if there is an $b \in X$ such that for all $x \in S$ we have $b \leq x$. Such an element b is also called a *lower bound* for the set S. An element $b \in X$, if exists, is said to be *infimum* of the set S if it is a lower bound and for any lower bound c we have $c \leq b$. In other words, infimum is the greatest lower bound. If there is an infimum, then it must be unique.

A set is said to be *bounded* if it is bounded below as well as above. You should not be frightened with several concepts being thrown in. You should notice that these concepts are not new. You saw them in the context of R. We know what is meant by bounded, supremum, infimum etc for subsets of real line. It just so happens that these are meaningful and useful in the context of losets too.

You should be careful with your intuition and not be swayed by appearance. For example consider the set $X = [0, 1) \cup [2, 3)$ with usual order. This is a loset. Take $S = [0, 1) \subset X$. You might think that this set has no supremum simply because 1 is not in our set. In fact 2 is its supremum as far as the loset X is concerned. In other words, in the loset before your eyes this element

2 is an upper bound for the set and there is no upper bound smaller than this.

In fact, as a loset the above set X is isomorphic to $[0, 2)$. The map $f(x) = x$ for $0 \leq x < 1$ and $f(x) = x - 1$ for $2 \leq x < 3$ sets up an order isomorphism of X with $[0, 2)$. Here is a simple observation regarding existence of infimums and supremums.

Theorem: Let X be a loset. The following are equivalent.
(i) Every non-empty bounded set has supremum.
(ii) Every non-empty set bounded above has supremum.
(iii) Every non-empty bounded set has infimum.
(iv) Every non-empty set bounded below has infimum.

Proof: If (i) holds then (ii) can be shown as follows. Take $A \neq \emptyset$ bounded above. Pick $a \in A$. Consider $B = \{x \in A : a \leq x\}$. This is non-empty because $a \in B$. Also B is bounded above by the bound of A. Moreover, B is bounded below by a. Easy to see that supremum of B, which exists by (i), works as sup of A. In fact, A and B have the same set of upper bounds.

Obviously (ii) implies (i).

Similarly (iii) and (iv) are equivalent.

Assume (ii) holds. We can argue (iii) as follows. Take $A \neq \emptyset$ which is bounded. Take the set B of all lower bounds of A. Since A is bounded, this is non-empty. Also every element of A is an upper bound of B. Use (i) to get $s = \sup B$.

We claim that s is infimum of A. In fact, every element of A being an upper bound of B, we conclude that $s \leq a$ for each $a \in A$. That is, s is a lower bound for A. Further, if x is a lower bound of A, then $x \in B$ by definition of the set B and hence $x \leq s$. Thus s is the greatest lower bound of A, that is, $\inf A$.

Similarly, one shows (iii) implies (i).

Say that a loset is *boundedly complete* if any of the above conditions hold.

Let X be a loset. Sets of the form $\{x \in X : a < x\}$, $\{x \in X : x < b\}$, $\{x \in X; a < x < b\}$ are called *open intervals*. Remember, open intervals in the real line are precisely of the form $(-\infty, a)$ or (a, b) or (b, ∞). This is exactly what we are saying for any loset.

A subset $D \subset X$ is dense, if every non-empty open interval contains a point of D. A loset is *separable* if there is a countable dense set D.

Just as we have a photograph of Q, the set of rationals in the theorem of Cantor, the following is a photograph of the real line.

Theorem (characterization of R): Let $(S, \leq)$ be a loset such that (i) it has no first point, no last point, between any two distinct points there is some other point; (ii) there is a countable set which is dense; (iii) every non-empty bounded subset has supremum. Then S is order isomorphic to R.

Proof: We imitate arguments involved in proving that there is a homeo of R that sends one countable dense set onto another such set. Instead of two countable dense subsets of R, we start with Q of R and countable dense set D of the loset X. Observe that D can not have a first point. In fact if $d \in D$ then the open interval $\{x \in X; x < d\}$ is non-empty (because X has no first point) and hence must contain a point of D. Similarly D has no last point. Thus D has no end points. Also given $a < b$ from D, the open interval $\{x \in X : a < x < b\}$ is non-empty because of hypothesis on X. And D being dense, there is $d \in D$ such that $a < d < b$.

Now we can appeal to Cantor and fix an order preserving isomorphism $f : D \to Q$ and repeat earlier arguments. In other words, take $x \in X$, observe that the set $\{\varphi(d) : d \in D; d < x\}$ is bounded above in R and its sup be denoted by $\varphi(x)$. First justify that when $x \in D$ then this sup indeed same as $\varphi(x)$ so that there is no clash of notation. This is order preserving isomorphism.

This completes proof.

It can be shown (you will do that later) that violation of any one of the features in the above picture will not give you R. In other words, given any one of the above conditions, there are losets that do not satisfy the given condition but satisfy all other conditions; and not order isomorphic to R.

We discuss one more characterization of the reals using its order properties. We need some definitions. Let X be a loset. A *cut* means a partition $X = L \cup U$ such that $a \in L, b \in U \to a < b$. We shall consider only cuts where both L and U are non-empty. There are exactly four possibilities: L has an upper bound in L (hence $= \sup L$) or does not have; U has a lower bound in U (hence $= \inf U$) or does not have. Accordingly we have four possibilities.

Both exist: A cut is said to be a *jump* if L has upper bound in L and U has a lower bound in U.

None exists: A cut is said to be a *gap* if L has no upper bound in L and U has no lower bound in U.

Exactly one happens: A cut is said to be *Dedikind cut* if L has upper bound in L but U has no lower bound in U OR U has a lower bound in U but L has no upper bound in L.

For example let $X = [0,1] \cup [2,3) \cup (3,4]$ with usual order. Then $L = [0,1]$ leads to a jump; $L = [0,1] \cup [2,3)$ leads to a gap; $L = [0, \frac{1}{2})$ or $L = [0, \frac{1}{2}]$ leads to Dedikind cut.

Notice that in describing the cuts above, we have only described L. This is enough because U is necessarily the complement of L. It is not necessary to carry the extra baggage, U.

Here is another photograph of the real line and to recognize that this is a photo of R we compare this with the previous photo.

Theorem: Suppose that $(X, \leq)$ is a loset such that (i) it has no first point and no last point (ii) it has a countable dense set (iii) every cut is a Dedikind cut. Then X is order isomorphic to R.

Proof: Need to verify X satisfies previous theorem. Take $a < b$. If there are no points in between, then $L = \{x : x \leq a\}$ and $U = \{x : b \leq x\}$ is gap.

Let $A \subset X$ be any non-empty set bounded above. Put

$$L = \{x \in X : x \leq a, \text{ for some } a \in A\} = \bigcup_{a \in A} \{x : x \leq a\}.$$

This gives a cut. Indeed, $\emptyset \neq A \subset L$. If b is any upper bound of A, using the fact that X has no last element get t such that $b < t$ to see $t \in L^c$. Thus both L and L^c are non-empty. If $x \in L$, then there is $a \in A$ such that $x \leq a$ and anything smaller than x is also smaller than a and hence is in L. So it is a cut. Hence this must be a Dedikind cut.

Suppose L has a sup $s \in L$. Since $A \subset L$, we see that s is an upper bound of L. As noticed by you, $s \in L$ tells that there is $a \in A$ with $s \leq a$. But s is sup of L tells $s = a$. In other words this upper bound s of A is in A and is hence sup A.

Suppose L^c has infimum $s \in L^c$. Since $A \subset L$ we see that s is an upper bound of A. Let $t < s$. As noted at the beginning of the proof, we can get $t < u < s$. Since s is inf L^c, we conclude that $u \in L$. Hence there is $a \in A$ with $u \leq a$. In particular $t < a$ showing that t can not be upper bound of A. Thus s is the least upper bound of A.

Thus every non-empty set bounded above has a sup.
This verifies conditions of the earlier theorem to complete the proof.

Construction of R:

We shall now proceed to the construction of the real number system. Before we do this, you should be convinced that it is necessary.

For example a number like 13/9 has a clear and concrete existence in our minds. On the other hands, some numbers are difficult to understand, difficult to believe they exist and so on. For example $\sqrt{2}$ does not have as concrete existence as 13/9. Of course, if you take some trouble you can visualise this number. For example (and this is not the only way), you can say, let us draw a concrete right angled triangle with two sides of unit length. Let us measure the diagonal.

If you consider

$$\sqrt{3}^{\sqrt{17}}$$

I am sure none of us have any idea what it is. We would even wonder if there is such a number at all. So it is necessary to convince ourselves that such numbers exist.

You might be deceived by your exposure to calculators. You might say, what is there, if I punch square root instruction and press the x^y function I can get this and show you. But this is illusion. What you get is an approximation. I am sure many of you probably do not even know what your calculator is showing. You just believe 'it *should* show what I asked for'.

You might also momentarily wonder: Did we not meet all numbers already? Did we not show last semester existence of square roots and powers. We did this in several steps, not only that, we developed several laws of indices and so on concerning expressions of the form a^b. Yes, you are right, we actually did all the hard work (thank God, it was over!).

But remember, all that was achieved using properties of the R, real number system, we announced at the beginning. Everything done so far used all the properties (and only those properties). Since we were clear headed and have been careful to list the properties we used, our job now is well focussed. We need to answer the question: is there a system satisfying those properties we listed. This is what we do now.

Thus constructing R simply means exhibiting a system satisfying those properties we listed. We need to exhibit a set R and define operations $+$, $\cdot$ along with elements 0, $1 \neq 0$ and relation $\leq$ such that those axioms hold. We shall recall those axioms only briefly now:

(I) deals with $+$ and 0; says we have a group;

(II) deals with $\cdot$ and $1 \neq 0$; says nonzero things form a group;

(III) deals with $+$ and $\cdot$; says these two things are friendly;

(IV) deals with $\leq$; says we have a loset;

(V) deals with $\leq$ and $+$ and $\cdot$; says they are friendly;

(VI) deals with subsets: says bounded non-empty subset has sup.

R from Q:

So for the next couple of hours we should not use real numbers, we do not have them, we construct them. However we do have rational numbers before us and we use them. It comes as a surprise that real numbers are very simple and nothing profound or complicated. What plays a crucial role is the intuition we gained from our earlier discussion.

Recall that, every real number was earlier described as sup of the set of rational numbers below it. This single sentence (and nothing else) is at the heart of what we do now.

Let us say that a subset $x \subset Q$ is a cut if it satisfies the following:
(i) $\emptyset \neq x \neq Q$;
(ii) $p \in x$, $q \leq p$ imply that $q \in x$;
(iii) $p \in x$ implies there exists r with $p < r$ and $r \in x$.

Thus a cut is a non-empty proper subset of Q such that if you take an element in it then everything smaller than that is also there and somethings larger than that are also there. (We can not say everything larger than that is also there).

The collection of all cuts is denoted by R. This is our real number system.

Notice that sets are usually denoted by A, B, C and so on. But now we are using the symbols x, y, z and so on for cuts, which are some special subsets of Q. We denote elements of Q by p, q, r, s and so on.

We do easy things first. We define an order on R by saying

$$x \leq y \ \leftrightarrow x \subset y.$$

Since $x \subset x$ conclude that $x \leq x$.
If $x \leq y$ and $y \leq z$ then $x \leq z$ as a consequence of set inclusions.
If $x \leq y$ and $y \leq x$ then $x = y$.
Thus we have a poset. Now we show that it is a loset. Take x and y. Suppose $x \not\leq y$. Thus $x \not\subset y$. So pick $p \in x$ and $p \notin y$. The second property of p implies that nothing larger than p is in y (remember y is a cut). Thus everything in y must be smaller than p. The first property of p implies that everything smaller than p is in x (remember x is a cut). Thus everything which is in y must also be in x. Thus $y \subset x$, that is, $y \leq x$.
Did I use too much English? You can rewrite using symbols.

Thus R is a loset. We shall now show that this loset has sup property. Take $\emptyset \neq S \subset R$. Thus S consists of some cuts. Let S be bounded, say $x \leq z$ for all $x \in S$. We exhibit sup for the set S. Let x^* be the union of all the cuts x that belong to S — just keep in mind that cuts are certain subsets of Q.

Since S is not empty, take $x \in S$ and take $p \in x$ to see $p \in x^*$. Thus $x^* \neq \emptyset$. Since $x \subset z$ for each $x \in S$ we see that $x^* \subset z$. Since z is a cut take $q \notin z$. Clearly $q \notin x^*$. Thus $x^* \neq Q$.

Let $p \in x^*$, take $x \in S$ so that $p \in x$. But then everything smaller than p is in this x (remember x is a cut) and hence in x^* too.

Let $p \in x^*$, take $x \in S$ so that $p \in x$. But then a little larger than p is also in this x (remember x is a cut) and hence in x^* too.

These three observations show that x^* is a cut. Since $x \subset x^*$ for each $x \in S$, that is, $x \leq x^*$, we see x^* is an upper bound for S. If z is any upper bound of S, then for each $x \in S$ we have $x \leq z$, that is, $x \subset z$ so that $x^* \subset z$, that is, $x^* \leq z$. That is x^* is the smallest upper bound. Thus S has a sup.

Hope you appreciate how trivial things are, and how we exhibited sup in a painless way. We shall now go on to define addition and multiplication.