CMI (BSc II)/BVRao Analysis, HA-1 04-08-2014

This semester we shall not continue with calculus. So this exercise set does not represent our topics. It is intended to bring you back to the mood and also remind you that there are many interesting matters we did not discuss.

1. Show that there are no real numbers a and b such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$. But there are complex numbers satisfyng it.

2. Let a and b be real numbers. Show

$$\int_0^\infty \frac{1}{1+x^2} \, \frac{x^b - x^a}{(1+x^a)(1+x^b)} \, dx \;=\; 0.$$

Deduce

$$\int_0^\infty \frac{1}{1+x^2} \, \frac{1}{1+x^a} \, dx \;=\; \frac{1}{2} \int_0^\infty \frac{1}{1+x^2} \, dx \;=\; \frac{\pi}{4}.$$

3. Fibonacci numbers are defined by $F_0 = F_1 = 1; F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Show $\sum\limits_0^\infty F_n t^n \;=\; \frac{1}{1-t-t^2}$.

4. Let f and g be functions on R to R which are differentiable as many times as needed below. Show Leibniz's rule:

$$D^p(fg) \;=\; \sum_{k=0}^{p} \binom{p}{k} D^k f \; D^{p-k} g.$$

Calculate the first ten Bernoulli numbers defined by

$$\frac{t}{e^t - 1} \;=\; \sum_0^\infty \frac{B_n}{n!} t^n.$$

Show

$$\sum_{j=0}^{n-1} e^{jt} \;=\; \frac{e^{nt} - 1}{t} \, \frac{t}{e^t - 1} \;=\; f(t)g(t) \quad \text{say.}$$

Show

$$\sum_{j=0}^{n-1} j^p \;=\; \sum_{k=0}^{p} \binom{p}{k} (D^k f)(0) \; (D^{p-k} g)(0).$$

$$= \sum_{k=0}^{p} \binom{p}{k} \frac{n^{k+1}}{k+1} \, B_{p-k} \;=\; \sum_{k=0}^{p} \binom{p+1}{k+1} \frac{n^{k+1}}{p+1} \, B_{p-k}.$$

Deduce

$$1+2+3+\cdots+n \;=\; \frac{1}{2}n^2+\frac{1}{2}n.$$

$$1^2+2^2+3^2+\cdots+n^2 \;=\; \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n.$$

$$1^3+2^3+3^3+\cdots+n^3 \;=\; \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2.$$

$$1^{10}+2^{10}+3^{10}+\cdots+n^{10} \;=$$

$$\frac{1}{11}n^{11}+\frac{1}{2}n^{10}+\frac{5}{6}n^9-n^7+n^5-\frac{1}{2}n^3+\frac{5}{66}n.$$

5. Let f be C^1 on $[1,\infty)$. Show

$$\sum_1^n f(k) \;=\; \int_1^n f(x)\,dx \;+\; \frac{f(1)+f(n)}{2} \;+\; \int_1^n \left(x-[x]-\frac{1}{2}\right) f'(x)\,dx$$

$P_1(x) = x - 1/2$ on $[0,1)$, extended periodically to all of R. Deduce

$$\sum_1^n f(k) \;=\; \int_1^n f(x)\,dx \;+\; \frac{f(1)+f(n)}{2} \;+\; \int_1^n P_1(x)f'(x)\,dx$$

This is known as the first derivative formula (of Euler).

Let us now assume f is C^2. Set $P_2(x) = x(x-1)+1/6$ on $[0,1)$, extended periodically to all of R. Show

$$\sum_1^n f(k) = \int_1^n f(x)\,dx+\frac{f(1)+f(n)}{2}+\frac{f'(n)-f'(1)}{12}-\frac{1}{2}\int_1^n P_2(x)f''(x)\,dx$$

This is the second derivative formula.

These are Euler Summation formulae.

P_2 is continuous and determined by $P_2' = 2P_1$; $\quad \int_0^1 P_2(x)dx = 0.$

Here are special cases. Read Err as Error.

(a) $f(x) = 1/x$ gives $\sum\limits_1^n 1/k = \log n + C + Err(n)$.

$$C = 1 - \int_1^\infty \frac{x - [x]}{x^2}\,dx; \quad Err(n) = \frac{1}{2n} + \int_n^\infty \frac{P_1(x)}{x^2}\,dx.$$

(b) $f(x) = \log x$ gives

$$\log n! = (n + 1/2)\log n - n + C - Err(n).$$

giving Stirling formula $n! \sim e^C e^{-n} n^{n+1/2}$. I hope you remember that $C = \log\sqrt{2\pi}$.

(c) Fix a number $s > 0$ and $s \neq 1$. Then $f(x) = 1/x^s$ gives

$$\sum_{k=1}^n \frac{1}{k^s} = \frac{n^{1-s}}{1-s} + C_s + s\int_n^\infty \frac{t - [t]}{t^{s+1}}\,dt$$

where

$$C_s = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{t - [t]}{t^{s+1}}\,dt.$$

Letting $n \to \infty$, note that C_s for $s > 1$, is nothing but the Riemann zeta function $\sum 1/k^s$. This last summation does not make sense for $0 < s < 1$ where as the above expression for C_s makes sense.

(d) $f(x) = \log x/x$ gives

$$\sum \frac{\log k}{k} = \frac{1}{2}(\log n)^2 + \frac{1}{2}\frac{\log n}{n} + C - Err(n)$$

6. Let C be the set of all pairs (a, b) of real numbers with usual addition $(+)$ and multiplication given by $(a, b) \times (c, d) = (ac - bd, ad + bc)$. Show that $(C, +, \times)$ is a field.

Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a, b are real numbers. With usual addition and multiplication of matrices, show that it forms a field.

Let Q be the set of all polynomials $p(t)$ in one variable t with real coefficients. Consider usual addition and multiplication of polynomials.

Say $p_1 \sim p_2$ if there is a polynomial p such that $p_1 - p_2 = (t^2+1)p$. Let P be the set of equivalence classes. Show that the equivalence relation respects (?) the operations of addition and multiplication. Show that P is a field.

Show that the three fields above are isomorphic in a 'canonical way'.

For even integer $n \geq 1$, show that there are $n \times n$ invertible real matrices A and B such that

$$A^{-1} + B^{-1} = (A+B)^{-1}.$$

7. Let Q be R^4 with points written as $(a,b,c,d) = a + bi + cj + dk$. Use coordinate-wise addition. Multiplication is 'prescribed' by
$i^2 = j^2 = k^2 = ijk = -1$.
Show $ij = k, jk = i, ki = j$ and $ji = -k, kj = -i, ik = -j$.
Show that Q is nearly a field, it misses only commutativity of multiplication. It is called skew-field.

Show M, the set of 4×4 real matrices of the form $\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$
is 'canonically' isomorphic to the above skew field. This skew field is called 'Quaternions.'

8. Let X be any set and $P(X)$ be its power set, that is, collection of all subsets of X. Define for $A, B \subset X$, their symmetric difference, by. $A\Delta B = (A - B) \cup (B - A) = \{x : x \quad \text{is in exactly one of the sets.}\}$.

   Show that $A_1 \Delta A_2 \Delta A_3 \Delta \cdots \Delta A_n$ consists of points that belong to an odd number of these sets; in whichever order you operate.

   Show that $P(X)$ is a group under the operation Δ.

   Let $H \subset P(X)$ be the collection of finite sets. Show that H is a subgroup.

   Let $H \subset P(X)$ be the collection of countable sets. Show that H is a subgroup. If H is the collection of countably infinite sets then is it a subgroup.

9. Let G be a group and X a non-empty set. The collection of functions on X to G, denoted by G^X, is a group with pointwise (?) operations.

   The set $2 = \{0, 1\}$ is a group under addition modulo 2. What is the relation between the group 2^X and the earlier $P(X)$ group.

10. Instead of finite union, you can define union of *any* family of sets. Suppose $\mathcal{C}$ is a collection of sets. Then their union, $\cup\mathcal{C}$, is the set of all objects x such that $x \in C$ for some sets$C \in \mathcal{C}$. Similarly, $\cap\mathcal{C}$ is the collection of all objects x such that $x \in C$ for every $C \in \mathcal{C}$.
    Prove DeMorgan's laws

$$(\cup\mathcal{C})^c = \cap\mathcal{C}^c; \quad (\cap\mathcal{C})^c = \cup\mathcal{C}^c; \quad \text{where} \quad \mathcal{C}^c = \{C^c : C \in \mathcal{C}\}.$$

11. For a sequence of sets (A_n), we define $\limsup A_n$ to be the set of all objects x such that x belongs to A_n for infinitely many values of n. When this happens for an object x then we also say that $x \in A_n$ *frequently*. Similarly, $\liminf A_n$ is the set of all objects x such that there is an n_0 and $x \in A_n$ for every $n \geq n_0$. When this happens for an object x we also say that $x \in A_n$ *eventually*.

Prove De Morgan's laws:

$$(\limsup A_n)^c = \liminf A_n^c; \quad (\liminf A_n)^c = \limsup A_n^c.$$

If X is the universe of discourse (this means all sets we now consider in this exercise are subsets of this X), we define for a set $A \subset X$ its *indicator function* or *characterstic function* to be the following function defined on X: $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.
This function I_A is also denoted by χ_A.

Show that $\limsup I_{A_n} = I_{\limsup A_n}$; $\quad \liminf I_{A_n} = I_{\liminf A_n}$

Let $A_n = [0,1]$ if n is even and $A_n = [2,3]$ if n is odd. Find limsup and liminf.

Let $\{a_n\}$ be a sequence of real numbers and $A_n = (-\infty, a_n)$ and $B_n = (-\infty, a_n]$. Calculate $\liminf A_n$ and $\limsup A_n$ in terms of liminf and limsup of a_n. Do the same for (B_n).

In what follows $N = \{1, 2, \cdots\}$. (Do you understand what exactly is hidden in the dots?).

12. Write a complete proof of the fact: If $m < n$ then there is no bijection between $\{1, 2, \cdots, m\}$ and $\{1, 2, , \cdots, n\}$.

13. If X is a countable set, then show that there is a $1-1$ function on X to (into or onto) N. Conversely, if there is a $1-1$ function on X to N then show that X is countable.

    If X is countable set, show that there is a function on N onto X. Conversely, if there is a function on N onto X, show that X is countable.

    Let X be a countable set and $Y \subset X$. Show that Y is countable.

    Let X_i for $i = 1, 2, \cdots$ be countable sets. Show that $\cup_i X_i$ is a countable set. This is stated as 'countable union of countable sets is countable'.

    (Hint: For each i fix a $1-1$ function $f_i : X_i \to \{1, 2, \cdots\}$. Here is f. Take x in the union. Take the first i such that $x \in X_i$ put

    $$f(x) = 2^i 3^{f_i(x)}.$$

    If X and Y are countable, show that $X \times Y$ is also countable.

    Let seq(N) be the set of all finite sequences of integers. Show that it is countable.

Let Y be the set of all infinite sequences of integers. Show Y is not countable.

14. If X is the set of all 5×5 real matrices and $Y = R^{25}$, exhibit a canonical bijection and conclude $|X| = |Y|$.

    If X is the set of all real symmetric 5×5 matrices and $Y = R^{15}$ show that $|X| = |Y|$. If Z is the set of all upper triangular 5×5 matrices then $|Z| = |X|$.

15. If X is an infinite set, then there is a subset $Y \subset X$ which is countably infinite. Prove this.

    If X is an infinite set then show that there is a proper subset $Y \subset X$ and $Y \neq X$ such that $|X| = |Y|$. Conversely, if X is a set and if there is a proper subset $Y \subset X$ and $Y \neq X$ such that $|X| = |Y|$ then show that X is an infinite set.

16. If $|X| = |Y|$ and $|A| = |B|$ then show that $|X \times A| = |Y \times B|$.

    With the same assumption show that $|X^A| = |Y^B|$.

    Here X^A is the set of all functions from A to the set X.

17. Let Y be the set of all ordered pairs of real numbers, in other words, $Y = R^2$. Show $|Y| = |R|$.

    More generally show that $|R^n| = |R|$ for any integer $n \geq 1$.

    Let seq(R) be set of all finite sequences of real numbers and $X =$ the set of all infinite sequences of real numbers. Show both have same cardinality as R.

18. Verify if the following are equivalence relations on the sets prescribed.

    Let $X = R$. Say $a \sim b$ if $|a - b| \leq 25$.

    Let $X = R$. Say $a \sim b$ if $a - b$ is an integer. Say $a \sim b$ if $a - b$ is rational. Say $a \sim b$ if $a - b$ is irrational.

    Let $X = P(R)$. Say $A \sim B$ if $A \Delta B$ is finite. What if finite is replaced by countable? What if finite is replaced by countably infinite?

    Let X be the set of all functions from R to R. Say $f \sim g$ if $f - g$ is a continuous function. What if 'continuosu function' is replaced by 'polynomial'.

Let $X = R^{29}$. Say $a = (a_1, \cdots, a_{29}) \sim b = (b_1, \cdots, b_{29})$ if there is a permutation π of $\{1, 2, \cdots, 29\}$ such that $a_i = b_{\pi(i)}$ for all i.

Same X as above. Say $a \sim b$ if there is a permutation π of $\{1, 2, \cdots, 25\}$ such that $a_i = b_{\pi(i)}$ for all i with $1 \leq i \leq 25$.

Same X as above, say $a \equiv b$ if $\sum a_i = \sum b_i$.

19. First recall The Cantor set $C \subset [0, 1]$. Let $\mathcal{I}$ be the collection of all deleted open intervals of $(0, 1)$. Or equivalently $[0, 1] - C = (0, 1) - C$ is an open set and hence can be written, in a unique way, as a disjoint union of open intervals and $\mathcal{I}$ is the collection of exactly these open intervals.

    If $I = (a, b)$ and $J = (c, d)$ are in $\mathcal{I}$ then say that $I \leq J$ if either they are the same interval or $b < c$ (that is, the interval I sits to the left of J). Show that the collection $\mathcal{I}$ with this order is another manifestation (?) of Q.

    Here is another Cantor set. Instead of dividing $[0, 1]$ into three parts and removing the middle part, divide into five intervals of equal length and remove second and fourth, keep first, third, fifth. Doing this at each stage get $D \subset [0, 1]$. Let $\mathcal{I}$ be the collection of deleted open intervals. Show again that $\mathcal{I}$ looks like Q (as far as order is concerned).

I hope you appreciate what AC is doing for us. Of course, set theory has many 'axioms' which are self evident (no dispute) and hence we do not talk about them. However about AC doubts exist; for very very good reasons. Here I tell you a story where AC is the main character.

20. Let $N = \{1, 2, \cdots\}$ as usual. Let $\mathcal{U} \subset P(N)$. That is, $\mathcal{U}$ is a collection of subsets of N. It is called an ultrafilter if it satisfies the following conditions:
(i) $N \in \mathcal{U}$; $\emptyset \notin \mathcal{U}$;
(ii) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$
(iii) $A \in \mathcal{U}, A \subset B \Rightarrow B \in \mathcal{U}$
(iv) $A \subset N \Rightarrow A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

Thus it is closed under finite intersections, supersets, not all of $P(N)$, and for every set either that or its complement is in it.

A collection $\mathcal{F}$ satisfying only the first three conditions is called a filter. Here are examples of filters. Verify.

All sets A such that A^c is finite. This is called co-finite filter. The word co-finite stands for 'complement of finite'.

All sets A such that A^c has only finitely many even numbers. For example the set of even integers is one such set.

All supersets of $\{23, 48, 56\}$

None of these is ultrafilter. Verify.

Here is an ultrafilter: All sets that contain the number 103. Verify.

You (and me too) can not think of other ultrafilters! — except replacing 103 by another number. These ultrafilters as above are called fixed ultrafilters.

Say that a filter $\mathcal{F}$ is maximal if $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{G}$ is a filter imply that $\mathcal{F} = \mathcal{G}$. Thus a maximal filter is a filter which is not proper subset of another filter. Show that a filter is maximal iff it is an ultrafilter.

Use Zorn and verify there are ultrafilters which are not fixed. Such ultrafilters are called free ultrafilters.

Show that an ultrafilter is free iff it misses all singleton sets. Show that an ultrafilter is free iff it includes co-finite filter.

21. Last semester we discussed closure, boundary of sets. In that story these concepts were only supporting actors, main actors were integration and partitions. In the present scene 'closure' is the main actor. It is a good idea to recall. We do so only for R. If you understand you can do for R^n but no need to do now.

    Let $S \subset R$. A point $a \in R$ is a limit point of S if every open interval around a has infinitely many points of S. That is, for every $\epsilon > 0$ the set $A \cap (a - \epsilon, a + \epsilon)$ is infinite. We denote

    $$\overline{S} \;=\; S \cup \text{ limit points of } S.$$

    This is called closure of S.

    Show that $\overline{S}$ is a closed set. Recall closed set means a set that includes all its limit points. Thus you are supposed to show that a limit point of $\overline{S}$ is already in $\overline{S}$ (in other words, it is already a limit point of S).

    Show that if C is a closed set and $S \subset C$ then $\overline{S} \subset C$. Thus $\overline{S}$ is the smallest closed set that includes S. That is, closure of a set is the smallest closed set that includes the set.

    Just to get practice, find closures of the following sets.

    $S = Q$, the set of all rational numbers.

    $S = Z$, the set of all integers.

    $S = \{1,\ 1/2,\ 1/3,\ 1/4,\ 1/5, \cdots\}$.

    $S = \{n + (1/n) : n = 1, 2, \cdots\}$.

    Remember The Cantor set? Take one point from each of the deleted intervals. Let S be the set so obtained. Do not be obsessed with AC now. Alright, take the mid point of each deleted interval. What is closure of S.

    Take the 1/10-th point of each deleted interval, what is its closure? 1/10-th point of (a, b) is the point $(9a + b)/10$. Why is it called so?

What if you took one point from each, except some 55 of the deleted intervals?

What if you took 5 points from each deleted interval?

You understand a friend only if you keep interacting with the friend, not by a cursory hello. Same holds with math concepts too.

22. Now we combine ideas of the two previous exercises. Let $\mathcal{U}$ be a free ultrafilter on N. Let $\vec{x} = (x_i : i \in N)$ be a *bounded* sequence of real numbers. let

$$\lim \vec{x} = \bigcap_{A \in \mathcal{U}} \overline{\{x_i : i \in A\}}$$

Show that the right side is non-empty and in fact is a singleton. This number is defined as limit of the sequence $\vec{x}$ *along the ultrafilter* $\mathcal{U}$ and is denoted $\lim\limits_{\mathcal{U}} x_n$.

Show that $\liminf x_n \leq \lim\limits_{\mathcal{U}} x_n \leq \limsup x_n$.

Show $\lim\limits_{\mathcal{U}} x_n = a$ iff given $\epsilon > 0$, there is a set $A \in \mathcal{U}$ such that $\{x_i : i \in A\} \subset (a - \epsilon, a + \epsilon)$.

Show that, $\lim x_n = a$ in the sense of last year Calculus iff given $\epsilon > 0$ there is a set A in the co-finite filter such that $\{x_i : i \in A\} \subset (a - \epsilon, a + \epsilon)$.

Do you see similarity between calculus definition and the present definition?

Limit along the ultrafilter remains same if you change finitely many terms of the sequence. Show this.

Show that $\lim\limits_{\mathcal{U}} x_n$. is a limit point of the sequence (x_n). Remember limit point of a sequence is any number a such that whatever $\epsilon > 0$ you take $x_n \in (a - \epsilon, a + \epsilon)$ for infinitely many n.

Show that if the sequence (x_n) actually converges (in the sense you have learnt last year) then the limit is same as limit along the ultrafilter.

Show that for any bounded sequences,

$$\lim_{\mathcal{U}} (x_n + y_n) = \lim_{\mathcal{U}} x_n + \lim_{\mathcal{U}} y_n; \qquad \lim_{\mathcal{U}} (57 x_n) = 57 \lim_{\mathcal{U}} x_n.$$

$$\lim_{\mathcal{U}} (x_n y_n) = \lim_{\mathcal{U}} x_n \cdot \lim_{\mathcal{U}} y_n. \qquad \forall n \ \ x_n \leq y_n \Rightarrow \lim_{\mathcal{U}} x_n \leq \lim_{\mathcal{U}} y_n.$$

23. Given one specific sequence, and one specific limit point of this sequence, it is possible to choose one ultrafilter so that limit along this ultrafilter for this sequence equals this given number. Have I confused you?

24. Even if your ultrafilter is a fixed ultrafilter, the above prescription of limit works; its conclusion can be explained in simpler terms and the limit could change if one term of the sequence is changed and the concept is useless. I do not want to spoil the suspense, Think.

Limit of a sequence, as learnt in Calculus last year, associates a number — namely its limit — with 'convergent' sequence. But the above procedure, with free ultrafilter, associates a number with *every* bounded sequence. In fact it associates a limit point of the sequence and so does not destroy what you learnt in Calculus. Of course, you might say use AC to pick one of its limit points. But the profound fact is this: the above selection respects addition, multiplication, monotonicity etc.

There is absolutely no easy way to achieve this and this is what you should appreciate.

25. We showed that there are discontinuous functions $f : R \to R$ satisfying $f(x+y) = f(x) + f(y)$ for all x, y. Suppose I want the function to satisfy both the earlier equation and also $f(xy) = f(x)f(y)$ for all x, y then what are the solutions. Answer: $f(x) \equiv 0$ and $f(x) \equiv x$. No more!

    Here is a way to prove. If $f(1) = 0$ show $f \equiv 0$. Let now $f(1) \neq 0$.

    Show (i) $f(x) \geq 0$ if $x \geq 0$; (ii) $f(x) \geq f(y)$ if $x \geq y$; (iii) $f(1) = 1$; (iv) $f(q) = q$ for rational q.

    If for some x we have $f(x) > x$ take q strictly in between and argue $f(q) \geq f(x)$, recognize this contradiction.

    Similarly there is no x with $f(x) < x$.

26. Let $S \subset R$ be open. Show that $|S|$ is either 0 or c.

    Let $S \subset R$ be closed. Its cardinality can be any of the following: $0,\ 1,\ 2,\ 3,\ 4, \cdots,\ \aleph_0,\ c$. Give examples of each.

    Can it be anything else? No. To see this, assume S is not countable.

    First suppose that every point of S is a limit point of S. Show that there are two disjoint closed intervals I_0 and I_1, each having points of S. Show that there are two disjoint closed intervals within each I_k having points of S. Continue and conclude $|S| = c$.

    To understand the general case, say that an open interval J is small if it has rational end points and $S \cap J$ is countable. Let V be the union of small intervals. Show $S_1 = S \cap V^c$ is closed, uncountable, every point of S_1 is a limit point of S_1. Conclude the proof.

27. We consider $V = R^{37}$ as a vector space, as usual, over the field R.

    An ordered basis is a sequence of vectors $\langle v^1, v^2, \cdots, v^{37} \rangle$ where $v^i \in R^{37}$ for all i; which forms a basis for V. Let X be the set of all ordered bases of V. Show $|X| = c$.

    Let G be the set of all 23-dimensional subspaces of V. Show $|G| = c$.

Show that each of the following collections in R^2 has power c: the set of equilateral triangles, set of rectangles with sides parallel to the axes, the set of all rectangles, set of circles, set of regular hexagons.

28. Show that the set of *all* functions from R to R has power strictly larger than c. This means there is a injection from R to this set but no bijection.

29. Let X be the space of infinite sequences of real numbers. That is

$$X = \{(x_1, x_2, \cdots) : x_i \in R \quad \text{for all} \quad i\}.$$

Show that $|X| = c$.

Let $C(R)$ be the collection of all real valued continuous functions on R. We define a map from $C(R)$ to X as follows. Fix once and for all an enumeration of Q, rationals: $q_1, q_2, q_3, \cdots$.

Here is the map. For $f \in C(R)$ associate the sequence

$$s(f) \quad = \quad \{f(q_1), f(q_2), f(q_3), \cdots\}.$$

Show this is an injection and conclude that $|C(R)| = c$.

What is the power of the set of real valued continuous functions on R^5.

What is the power of the set of continuous functions on R^{17} to R^{71}.

The set of real valued continuous functions on R has the same power as R.

Let P_1 be the set of polynomials in one variable x with real coefficients. $|P_1| = c$. Show.

Let P_k be the set of polynomials in k variables $x_1, x_2, \cdots x_k$ with real coefficients. $|P_k| = c$. Show.

What if P_k is the set of polynomials in k variables as above but with rational coefficients.

Let Homeo(R) be the set of homeomorphisms of R onto itself. Show that its power is c.

30. Let M be the set of monotone increasing functions on R to R. That is $f \in M$ if $f : R \to R$ and $x < y$ implies $f(x) \leq f(y)$. The function need not be strictly increasing. Show $|M| = c$.

Just raise your heels and stretch your hands to reach this result.

First: recall from last semester (prove again) that $f \in M$ implies that f has only countably many discontinuities.

Second: Fix a countable set $D = \{d_1, d_2, \cdots\} \subset R$.

For $f \in M$, associate the sequence of numbers

$$s(f) = \{f(q_1), f(d_1), f(q_2), f(d_2), \cdots\}.$$

Denote by M_D those functions in M whose set of discontinuities are contained in D. That is, those functions which are continuous at every $x \in R - D$. Show that the above map is injective on this set and conclude $|M_D| = c$.

How many countable subsets of R are there? What is $|R \times R|$?

Use answers to the above two queries to conclude $|M| = c$.

Show that the set of all monotone functions (increasing or decreasing) from R to R has power c.

31. How large a collection of non-empty subsets of N can you get so that any two are disjoint? That is, I want a family of sets $(A_\alpha : \alpha \in \Delta)$, each $\emptyset \neq A_\alpha \subset N$; such that for distinct indices $A_\alpha \cap A_\beta = \emptyset$. How large can the set Δ be? Show that it must be countable.

What if we need the non-empty sets to be only almost disjoint? That is, I want a family $(A_\alpha; \alpha \in \Delta)$ of non-empty subsets of N so that for distinct indices $A_\alpha \cap A_\beta$ is a *finite set*, could be but need not be empty. How large can Δ be? Answer: $|\Delta| = c$. To see this, First argue $|\Delta| \leq c$.

Fix an enumeration $\{r_1, r_2, \cdots\}$ of rational numbers. You can do this explicitly without using AC (but this is beside the point). For $x \in R$, let us define a set as follows. x_1 is the first rational in $(x-1, x+1)$. x_2 is the first rational that occurs after x_1 which is in the interval $(x - 1/2, x + 1/2)$. In general x_k is the first rational that occurs after x_{k-1} which is in the interval $(x - 1/k, x + 1/k)$. Let

$$A_x = \{x_1, x_2, \cdots\}$$

$A_x \subset Q$; if $x \neq y$ then $A_x \cap A_y$ is finite.

Returning to the problem we started with, show $|\Delta| = c$.

32. Let P be a partially ordered set and $C \subset P$ be a chain.

Understand the difference between the following statements:

(1) $\neg\ \exists(p \in P)\ \forall(x \in C)\ (x < p)$.

(2) $\exists(p \in P)\ \neg\ \forall(x \in C)\ (x < p)$.

(3) $\exists(p \in P)\ \forall\ (x \in C)\ \neg\ (x < p)$.

Want to say: there is no point of P which is larger than every element of C. Which of the above says this? why?

I was too lazy in the class!

Justify why the other two statements do not say that?

33. Consider the set $X = [0,1] \times [0,1] - \{(0,0); (1,1)\}$ with dictionary order.

Show that it has no first point; no last point; between two different points there is some thing strictly in between; every non-empty subset which is bounded above has a supremum; has no countable dense set.

Is this order isomorphic to R?

Suppose you considered all of $[0,1]\times[0,1]$ with dictionary order. Which of the above properties hold?

Suppose you considered $R \times R$ with dictionary order. Which of the above properties hold?

Suppose you consider $Z \times [0,1)$ with dictionary order. Which of the above properties hold?

34. Test which of the following properties hold for the losets given beow:
(a) has first element;
(b) has last element;
(c) has another point in between two different points;
(d) has countable dense set;
(e) every non-empty bounded subset has sup.

$$(i)\ \ S = [0, 1] \qquad (ii)\ \ S = [0, 1) \qquad (iii)\ \ S = (0, 1]$$

$$(iv)\ \ S = (0, 1] \cup [2, 3); \qquad (v)\ \ S = Q$$

$$(vi)\ \ S = [0, 1] \times [0, 1] - \{(0, 0), (1, 1)\} \quad \text{with dictionary order.}$$

35. Verify the details left out in class: We defined multiplication xy for $x > 0$ and $y > 0$ and observed its properties. Extend the definition as follows:
$x < 0$ and $y > 0$: xy is defined as: $-[(-x)y]$.
$x > 0$ and $y < 0$: xy is defined as $-[x(-y)]$.
$x < 0$ and $y < 0$: xy is defined as $(-x)(-y)$.
$x = 0$ or $y = 0$: xy is defined as 0.
This definition satisfies all the properties required of multiplication, Show. Remember, you need not go to cuts. Use known things.

As you noticed probably, the connection of multiplication with order is only that product of two positive numbers is positive; and the rest (whatever) is a consequence of just this.

36. In our definition of R we have defined linear order which is friendly with addition and multiplication.

Instead, sometimes following is taken: There is a subset $P \subset R$ with the following properties.

(i) for all x; exactly one holds: $x = 0$, $x \in P$, $-x \in P$.

(ii) $x \in P, y \in P$ implies $x + y \in P$ and $xy \in P$.

Show that they and us are doing the same thing, that is, (i) starting with P as above define $\leq$ satisfying our conditions; conversely, (ii) starting from our $\leq$ exhibit P. show also that if you start with P, use (i) and then (ii) for that $\leq$; you get back your starting P.

Subsets of R are called 'unary' relations in R. Subsets of $R \times R$ are called binary relations in R — like $\{(x, y) : x \leq y\}$. Subsets of $R \times R \times R$ are called ternary relations in R— like $\{(x, y, z) : x+y = z\}$. In general you use the word, n-ary relation.

Did you realize how a binary operation like addition is also a relation, it is ternary relation.

37. Recall the definition of Cauchy sequence and definition of convergence of a sequence in R.

    Show that (x_n) is Cauchy iff for any given rational $r > 0$, there is an n_0 such that $|x_m - x_n| < r$ for all $m, n > n_0$. Show that (x_n) converges to x iff given rational $r > 0$ there is an n_0 such that $|x_n - x| < r$ for all $n > n_0$.

    Show that given any real number x, there is a Cauchy sequence (q_n) of rational numbers which converges to x.

    These two statements appear to be below our standard, prove them anyway. They are profound for the following reasons.

    (i) we need only rationals to define Cauchyness and convergence for sequences. $(*)$

    (ii) we need only Cauchy sequences of rational numbers to define real numbers. $(**)$

38. Consider the set of real numbers R. Recall the 'least upper bound axiom':

    (VI) Every bounded non-empty subset has supremum.

    Show that this can be replaced by the 'completeness axiom': $(* * *)$

    (VIa) Every Cauchy sequence converges.

    These three simple observations $(*), (**), (* * *)$ form basis for another construction of real number system due to Cantor. This method rescues us many times in Analysis. Shall do soon.

39. Here is an interesting field that builds on ultrafilter. usually algebraists are uninterested in this, they have field of rational functions which are not ordered fields. Analysts do not bother either, because many do not realize importance of this seed. Logicians have refined this idea as well as this example thoroughly (whatever this may mean).

Let $\mathcal{U}$ be a free ultrafilter on $N = \{1, 2, \cdots\}$. Let R_0 be the set of all infinite sequences $s = (s_n)$ of real numbers. Here are some examples of sequences for you to see.

$$s_n \equiv \sqrt{555}; \quad t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}..$$

$$u_n = n; \qquad u_n^* = n^2; \qquad w_n = \frac{1}{n}; \qquad w_n^* = \frac{1}{n^2}.$$

Here $n = 1, 2, 3, \cdots$..

Let us define $s \sim t$ if $\{n : s_n = t_n\} \in \mathcal{U}$.

Show this is an equivalence relation.

Let the space of equivalence classes be denoted by R^*.

Remember we have addition and multiplication of sequences term by term. We can define these operations on R^*. Take x and y. These are equivalence classes. Take one sequence $s \in x$ and one $t \in y$ and define $x + y$ to be the equivalence class containing the sequence $s + t$. Show this is a good definition, that is, it does not depend on s and t as long as you choose from x and y.

Define multiplication in a similar way.

Show that we have a field.

Define order on R^* as follows. Say $x \leq y$ iff $\{n : s_n \leq t_n\} \in \mathcal{U}$ where $s \in x$ and $t \in y$. Show that this is a good(?) definition.

Show that R^* is a loset with this order.

Show we have an ordered field.

Identify usual set of real numbers as constant sequences. In other words, if $a \in R$ define $\varphi(a) \in R^*$ to be the equivalence class containing the constant sequence with each term equal to a.

Show that this is an embedding of $(R, +, \cdot, \leq)$.

Thus this ordered field R^* contains usual R.

This is the identification we use. Thus when we say the number $\sqrt[7]{33} \in R^*$ we mean the equivalence class containing the corresponding constant sequence.

Using the notation of the examples of sequences given at the beginning, let $[s]$ be the equivalence containing the sequence s.

Show $[u] > k$ for each $k \in R$. In some sense $[u]$ is an infinite integer.

For each $\epsilon > 0; \epsilon \in R$, show that $0 < [w] < \epsilon$. In some sense $[w]$ is an 'infinitesimal', strictly positive but smaller than all ϵ's *we know*.

So also is $[w^*]$.

Show the infinitesimal $[w^*]$ is smaller than the infinitesimal $[w]$.

We shall not discuss more about this field.

40. Suppose (X, d) is a metric space. Let us define

$$d_1(x, y) = \min\{d(x, y), 1\}.$$

Show that d_1 is also a metric.

Show that $x_n \to x$ in d iff $x_n \to x$ in d_1.

Show that a sequence is Cauchy in d iff it is Cauchy in d_1.

Show that a set is open in the metric d iff it is open in the metric d_1.

Show that the metric d_1 is bounded, whether d is bounded or not.

Suppose that in taking the minimum above, if I took minimum with 0.0001 how do the answers to above questions change?

Start with a metric space (X, d). Instead of the above, define

$$d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Test whether all the above statements are still valid for d_2.

41. Consider the function $\tan x : (-\pi/2, \ +\pi/2) \to R$. It is strictly increasing, onto and one-one. Let $\tan^{-1}$ be the inverse map. Let us define

$$d_1(x, y) = |\tan^{-1} x - \tan^{-1} y|; \qquad x, y \in R.$$

Show that d_1 is a metric. Show that it is equivalent to d, that is, $x_n \to x$ in d iff $x_n \to x$ in d_1.

Do you think the following is correct: A sequence is cauchy in d iff it is Cauchy in d_1.

42. Given any open set U in $C[0, 1]$ with the sup metric, show that there is a polynomial with rational coefficients which belongs to U.

43. In a metric space show $|d(x, z) - d(y, z)| \leq d(x, y)$. More generally $|d(x, z) - d(y, u)| \leq d(x, y) + d(z, u)$.

44. Let $C[0,1]$ be the collection of real continuous functions on $[0,1]$. define

$$d_1(f,g) = \int_0^1 |f(x) - g(x)| dx.$$

$$d_2(f,g) = \left[\int_0^1 |f(x) - g(x)|^2 dx\right]^{1/2}.$$

Show that these are metrics. More generally, fix $p > 1$ and show that the following is a metric

$$d_p(f,g) = \left[\int_0^1 |f(x) - g(x)|^p dx\right]^{1/p}.$$

45. Let l_2 be the space of all (infinite) sequences $x = (x_n : n \geq 1)$ of real numbers such that $\sum |x_n|^2 < \infty$. Show that this is a linear space. Define

$$d_2(x,y) = \left[\sum |x_n - y_n|^2\right]^{1/2}.$$

Show that this is a metric.

Let l_1 be the space of all sequences $x = (x_n : n \geq 1)$ of real numbers such that $\sum |x_n| < \infty$. Show that this is a linear space. Define

$$d_1(x,y) = \sum |x_n - y_n|.$$

Show that this is a metric.

Give examples of sequences which are in l_2 but not in l_1. Do you think there are sequences which are in l_1 but not in l_2.

What happens if we consider sequences of complex numbers in both the above l_2 and l_1. Show that they are linear spaces and metric spaces.

More generally, consider for a fixed $p > 1$, the space l_p of all complex sequences $z = (z_n : n \geq 1)$ such that $\sum |z_n|^p < \infty$.

Show that this is a linear space and the following is a metric on the space.

$$d_p(z,w) = \left[\sum |z_n - w_n|^p\right]^{1/p}.$$

You can consider the space l_∞ also. It is the space of all bounded complex sequences. That is, the space of all sequences $z = (z_n : n \geq 1)$ with $\sup |z_n| < \infty$. Show that the following is a metric on this space.

$$d_\infty(z, w) = \sup_n |z_n - w_n|.$$

46. Consider

$$X = [0,1]^\infty = \{x = (x_1, x_2, \cdots) : x_i \in [0,1]; \quad i = 1, 2, 3, \cdots .\}$$

$$d(x, y) = \sum \frac{|x_i - y_i|}{2^i}.$$

Show that d is a metric.

Show that $x^n \to x$ in the metric d iff $x_i^n \to x_i$ for each $i \geq 1$. That is, iff coordinate-wise convergence holds.

You can also take

$$X = R^\infty = \{x = (x_1, x_2, \cdots) : x_i \in R; \quad i = 1, 2, 3, \cdots .\}$$

Show that if you define d exactly as above then the series may not converge. Define

$$d_1(x, y) = \sum \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}.$$

If you have done last part of exercise 43, you will not be frightened with this expression.

Show that this is a metric. Show that $x^n \to x$ iff $x_i^n \to x_i$ for each $i \geq 1$. That is, iff coordinate-wise convergence holds.

47. Let $X = Q$, the set of rational numbers.

We define, for $x \in Q$, order of x as follows: If x is an integer, then $O(x)$ is the largest power of 7 that divides x. If $x = a/b$ rational (with a, b integers), then $O(x) = O(a) - O(b)$. Show that this definition does not depend on how you represent the rational as a fraction.

For $x \in Q$, we define $||x||$ as follows:

$$||0|| = 0; \qquad ||x|| = 7^{-O(x)} \quad x \neq 0$$

note the negative sign for exponent.

Show $\quad (i) ||x|| = 0$ iff $x = 0$.

$(ii) ||x + y|| \leq ||x|| + ||y||. \qquad (iii) ||xy|| = ||x|| ||y||.$

Define $d(x, y) = ||x - y||$. Show that this is a distance on Q. Actually this satisfies a better condition than triangle inequality.

Let $x \geq 1, y \geq 1$ be integers. Show $d(x, y) \leq 1/7^n$ iff $x = y \bmod(7^n)$.

Calculate for $n, m \in Z$,

$(i) \quad d(7^n, 7^m) \qquad (ii) \quad d(7^{-n}, 7^{-m})$

Wonder: $7, 7^2, 7^3, 7^4, \cdots \to 0$.

Convince yourself that this metric takes (apart from zero) only the values

$$\cdots, \cdots \frac{1}{7^3}; \ \frac{1}{7^2}; \ \frac{1}{7}; \ 1; \ 7; \ 7^2, \ 7^3, \cdots\cdots.$$

Try to think which fellows are sitting exactly at a distance $1/7$ from zero. Who are sitting at a distance 7 from zero.

48. Calculate interior, closure, closure of interior, interior of closure and so on of the following sets:

 (a) $X = R$.
 $A = C^c$ where C is cantor set in $[0, 1]$.
 $A = (0, 1) \cup (1, 2)$.
 $A = (0, 1) \cup (1, 2) \cup \{6, 7, 8, 9, \cdots .\}$
 $A = (0, 1) \cup (1, 2) \cup \{\text{rationals in } (3, 4)\} \cup \{6, 7, 8, 9, \cdots .\}$

 (b) $X = R^2$ and the sets are $A \times A$; take A to be each of the above sets.

 (c) $X = C[0, 1]$ and A is the set of all polynomials; or A is the set of all continuous functions taking values in $(0, 1)$; or A is the set of all continuous functions taking values in $[0, 1]$.

In what follows (X, d) is a metric space. A metric gives, distance between points, provides one measurement. You can use this to define several other measurements. Naturally, they depend on this metric.

49. For non-empty set $A \subset X$ and for $x \in X$ define

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

In a sense, measures the least distance you need to travel from your location x to reach the town A.

 (a) $X = R$ usual metric $|x - y|$. Calculate $d(x, A)$ when
 $A = [0, 1]$. $x = 25$ or $x = -25$ or $x = 1/2$
 $A = (0, 1)$. x same as above.

 (b) $X = R^2$ Euclidean metric.
 $A = x_1$-axis. $x = (4, 5)$; or $x = (1, 0)$; or $x = (0, 1)$
 $A =$ the line $x_1 + x_2 = 4$. Same points as above.
 $A = B(0, 1/2)$. Same points as above.

(c) $X = C[0,1]$ with sup metric.

A is the set of all functions $z \in X$ with $z(0) = 0$. x is the function $x(t) = t^2$; or $x(t) = \cos(2\pi t)$; or $x(t)$ is a polynomial in t.

A is the set of all functions $z \in X$ with $z(0) = 0 = z(1)$. And x as above.

(d) $X = R$ with $d(x, y) = 0$ or 1 according as $x = y$ or not. This is called discrete metric. Calculate $d(x, A)$ for each $x \in R$ and each non-empty $A \subset R$.

(e) Show that $|d(x, A) - d(y, A)| \leq d(x, y)$.

(f) Show that x is in the closure of A iff $d(x, A) = 0$.

(g) How does $d(x, A)$ change when A is made larger?

(h) If A is compact, then show that the inf is actually minimum. Do you think that if the infimum is attained then A should be compact?

50. Suppose A and B are two non-empty sets. The distance between the sets is defined as

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

In a sense the smallest length of bridge needed to connect the two towns A and B. calculate $d(A, B)$ for the sets given below.

(a) $X = R$.

A is set of rationals in $[0, 1]$ and B is set of irrationals in $(2, 3)$.

$A =$ Cantor set in $[0, 1]$ and
$B = [4/27,\ 5/27] \cup (11/27,\ 16/27) \cup [64/81,\ 71/81]$.

$A = \{1, 2, 3, \cdots\}$ and
$B = \{1 + 1, 2 + (1/2), 3 + (1/3), 4 + (1/4), \cdots\}$

(b) $X = R^2$.

A is unit disc and $B = [2, 3] \times [2, 3]$

A is the x_1-axis and
$B = \{(x_1, x_2) : x_1 > 0, x_2 > 0, x_1 x_2 = 1\}$

(c) $X = R$ with discrete metric. Describe $d(A, B)$ for all pairs of nonempty subsets.

(d) Show that $d(A, B) \neq 0$ implies that $A \cap B = \emptyset$. Do you think the converse is true?

(e) Show $d(A, B) = \inf\{d(a, B) : a \in A\} = \inf\{d(b, A) : b \in B\}$.

(f) Show that distance between two sets is same as the distance between their closures.

(g) Do you think triangle inequality holds:
$d(A, C) \leq d(A, B) + d(B, C)$?

(h) If $A_1 \subset A; B_1 \subset B$, how do you compare $d(A, B)$ and $d(A_1, B_1)$.

(i) Show that if A and B are compact, then the inf is actually minimum. Do you think that if the inf is attained then at least one of the sets should be compact?

51. For a non-empty set A, we define the diameter of A by

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}.$$

In a sense, measures how far are the farthest corners of the town A from each other. Calculate diameters of the sets below.

(a) $X = R$. A is the interval $[0, 1]$; or the Cantor set; or set of rational numbers in $(0, 1)$; or set of irrational numbers in $(0, 1)$; or the interval $(0, \infty)$.

(b) $X = R^2$. A is a line segment; or a triangle; or the open unit disc; or closed unit disc; or ellipse (with semi-major axis a and semi-minor axis b); or the unit square $[0, 1] \times [0, 1]$.

(c) $X = C[0, 1]$. A is the set of all x whose graph lies in the unit square; or set of all x whose graph lies in the unit disc.

(d) $x = R$ with discrete metric. A is the ball of radius $1/2$ around 5; or ball of radius $3/4$ around 5; or ball of radius 2 around 5; or ball of radius 5 around 5.

(e) Show that $\delta(B(a, r)) \leq 2r$. Do you think equality always holds? Show diameter of a set A is same as the diameter of its closure.

(f) Show that if A is compact then sup is actually maximum. Do you think that if the sup is attained then A should be compact?

52. A set $A \subset X$ is said to be bounded if its diameter is finite. Emptyset by convention, is bounded. But anyway, let us consider non-empty sets only.

Sometimes the following definition is used. A set A is bounded relative to a point $x \in X$ if $\sup\{d(x,a) : a \in A\} < \infty$, that is, there is a number M such that $d(x,a) \leq M$ for all $a \in A$.

If A is bounded relative to one point, show it is bounded relative to any other point. Show that this definition is same as saying that the diameter is finite.

Show that any Cauchy sequence is bounded.

Show that union of finitely many bounded sets is bounded.

Do you think every closed bounded set is compact?

Consider R^n with any of the l_p distances $1 \leq p \leq \infty$. Show that this definition of boundedness coincides with the notion of boundedness we adapted last year in R^n. recall $A \subset R^n$ is bounded if $||x|| \leq M$ for all $x \in A$.

53. Here is another metric spacce. Let $X = l_\infty$ be the space of all bounded sequences of real numbers. Define for $x, y \in X$.

$$d(x,y) = \sum_{n \geq 0} |x_n - y_n| e^{-5} 5^n \frac{1}{n!}.$$

This appears complicated, you would not understand whether the number $\exp\{-5\}$ is necessary at all. The essential point is: this is nothing but expectation of $|x - y|$ w.r.t. the Poisson probabilities you have learnt.

Show that this is a metric. do you think the space is complete?

54. We showed last year that $\sqrt{p}$, for prime $p > 1$, is irrational number. Thus $\sqrt{2}, \sqrt{3}, \sqrt{17}, \cdots$ are all irrational numbers. Of course there are many others. We should attend to some fillable gaps in our understanding.

(a) You can use the same reasoning to show that for any integer $k > 1$, either $\sqrt{k}$ is an integer or is irrational. Thus if $\sqrt{k}$ is rational then it must already be integer. Show this.

(b) Here is how you prove e is irrational.
Let if possible $e = a/b$ ratio of two positive integers.
Show then that

$$\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots$$

is an integer. Show $b > 1$ and the above can not be an integer.

(c) Here is how you prove π is irrational. (Proof by Ivan Niven)
Let if possible $\pi = a/b$ ratio of positive integers. Define, for a positive integer n, two functions

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x).$$

Show

$$f(x) = f\left(\frac{a}{b} - x\right).$$

Show that the function $n!f$ has integer coefficients.
Show that f and its derivatives have integer values for $x = 0$ and $x = \pi = a/b$. Show

$$\frac{d}{dx}\{F'(x)\sin x - F(x)\cos x\} = F''(x)\sin x + F(x)\sin x = f(x)\sin x.$$

$$\int_0^\pi f(x)\sin x dx = F(\pi) + F(0) \quad \text{is an integer.}$$

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!}; \quad 0 < x < \pi.$$

Note that f depends on n. Argue that if you choose n large enough the above inequation leads to a contradiction.

(d) Actually the above numbers are transcendental, proof is not easy.

(e) We do not know if γ, Euler constant, is irrational.
When you read such a sentence, you should pause, recapitulate definition of γ and understand what is said; though there is nothing to prove in here.

55. Giving explicit examples of nowhere differentiable continuous functions is not easy. First such example was by Karl Weierstrass. Here is one from Rudin.

Define the function φ on R by $\varphi(x) = |x|$ for $-1 \le x \le 1$ and $\varphi(x+2) = \varphi(x)$ for all $x \in R$. Show $|\varphi(t) - \varphi(s)| \le |t - s|$.

Define
$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x).$$

Show f is continuous.

Fix any $x \in R$. Let
$$\delta_m = \pm\frac{1}{2}\frac{1}{4^m}.$$

Show you can choose the sign for δ_m so that there is no integer between $4^m x$ and $4^m(x + \delta_m)$. We fix this sign now. Having fixed m, set

$$\gamma_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m}.$$

Show that $\gamma_n = 0$ for $n > m$ while $|\gamma_n| \le 4^n$ for $0 \le n \le m$. Show

$$\left|\frac{f(x + \delta_m) - f(x)}{\delta_m}\right| \ge 3^m - \sum_{0}^{m-1} 3^n \ge \frac{1}{2}(3^m + 1).$$

Show f is not differentiable at x.

56. Let (X, d) be a metric space.

(a) Consider a subset $C \subset X$ (always non-empty, in this context). Restrict d to pairs of points in C; still use d for this. then (C, d) is a metric space. Show.

(b) Let (X, d) be complete. If $C \subset X$ is closed then show that (C, d) is a complete metric space.

Conversely, let $C \subset X$ and (C, d) be complete. Then show that C is a closed subset of X.

(c) Let (X, d) be a complete metric space. Let $U \subset X$ be open set. ¿From the above you know that (U, d) is not complete unless U is closed.

However, without changing the notion of convergence, we can change the metric for U as follows. Keep in mind our examples of metrics on $(0, 1]$ and $(0, 1)$ discussed earlier.

First show that $f(x) = d(x, C)$ where $C = U^c$ is a continuous function and is never zero on U. Secondly, show that

$$\rho(x, y) = d(x, y) + \left| \frac{1}{d(x, U^c)} - \frac{1}{d(y, U^c)} \right|; \quad x, y \in U$$

or (not same, though same notation is used),

$$\rho(x, y) = d(x, y) + \min \left\{ \frac{1}{2^9}; \left| \frac{1}{d(x, U^c)} - \frac{1}{d(y, U^c)} \right| \right\}; \quad x, y \in U$$

is a metric on U. Thirdly, Show that $d(x_n, x) \to 0$ iff $\rho(x_n, x) \to 0$, as far as U is concerned. finally show that (U, ρ) is complete.

(d) Thus for example complement of finite sets in the real line can be given a 'complete metric' without changing notion of convergence.

(e) Let (X, d) be a complete metric space. Let $A = \cap U_n \subset X$ where each U_n is open. Thus A is intersection of countably many open sets. Put $d^*(x, y) = \min\{1, d(x, y)\}$. Put

$$\rho(x, y) = d(x, y) + \sum_n \min \left\{ \frac{1}{2^n}, \left| \frac{1}{d^*(x, U_n^c)} - \frac{1}{d^*(y, U_n^c)} \right| \right\}; \quad x, y \in A.$$

Show (A, ρ) is a complete metric space; d-convergence is same as ρ-convergence.

(f) The set of irrational numbers or the set of transcendental numbers in R is a possible candidate for above A.

(g) Read part (e) again. Following says we can not do any better.

Suppose $A \subset X$ and you can give a metric ρ for A such that the following two hold: (i) For points of A, ρ-convergence is same as d-convergence. (ii) (A, ρ) is complete metric space. Then A must indeed be countable intersection of open sets in X.

We shall not prove, involves some work (beautiful though).

57. Let f be any continuous function on $[0, 1]$. Let f_1 be any function whose derivative is f. Let, in general, f_n be any function whose derivative is f_{n-1}. (Of course, given f_{n-1} you have freedom to choose your favourite constant in getting f_n; no more!)

Show the following: if $(\exists n)$ $(\forall x) f_n(x) = 0$ then $f \equiv 0$.

Show the following: if $(\forall x)(\exists n) f_n(x) = 0$ then $f \equiv 0$.

58. Consider $C[0, 1]$. A function is said to be everywhere oscillating if it is not monotone on any (non-degenerate) interval. Pause. Can you picture such a function? Very very difficult. If I did not put continuity, it is trivial to imagine!

There are many many everywhere oscillating functions. More precisely, Complement of the set of everywhere oscillating functions, in $C[0, 1]$, is countable union of small sets.

59. (Principle of uniform boundedness) Suppose that (X, d) is a complete metric space. Let F be a collection of real valued continuous functions on X.

Assume that the collection is point-wise bounded. that is, given $x \in X$, the set $\{f(x) : f \in F\}$ is a bounded subset of R.

Show that there is an (non-empty) open set U on which these functions are uniformly bounded. That is, $\{f(x) : x \in U; f \in F\}$ is a bounded subset of R.

60. Let (X, d) be a metric space.

Recall $\overline{A}$ is closure of A and A^o is interior of A.

Check: $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Is $(A \cup B)^o = A^o \cup B^o$?

Check $(A \cap B)^o = A^o \cap B^0$. Is $\overline{A \cap B} = \overline{A} \cap \overline{B}$?

Show $(\overline{A})^c = (A^c)^0$. and $(A^o)^c = \overline{A^c}$

Recall that closure of a set A consists of points which are in A or which are close to A — recall precise definition. Boundary of a set A is defined to be the set of all points which are close to A as well as A^c. Precise definition of boundary is $\partial A = \overline{A} \cap \overline{A^c}$.

We needed and used this concept in discussing integration in several variables. But we considered only rectangles and regions within a nice closed curve.

Show $\partial A = \overline{A} - A^0$.

Show $\partial(A \cup B) \subset \partial A \cup \partial B$ and $\partial(A \cap B) \subset \partial A \cup \partial B$.

Equality does not hold in general. Give examples.

Do you think $\partial(A \cap B) \subset \partial A \cap \partial B$.

Do you think $A \subset B$ implies $\partial A \subset \partial B$?

What is ∂Q where Q is the set of rationals in R.

What is boundary of Cantor set in R? What is boundary of complement of Cantor set in R? What is boundary of a disc in R^2? boundary of a circle in R^2 boundary of a rectangle in R^2?

Do you think A and $\overline{A}$ and A^0 all have the same boundaries?

61. Consider $X = C[0, 1]$ with the metric $d(x, y) = \int_0^1 |x(t) - y(t)|dt$. Show the space is not complete? Do you think that convergence in this metric is same as convergence in the sup metric? If A is a non-empty subset which is open and closed in this space then it must be either $\emptyset$ or all of X. Show.

62. Many a times we start with a metric space but soon restrict attention to some subset of it. This happened, for example, in the proof that R is not union of infinitely many disjoint non-empty closed sets.
Let (X, d) be a metric space. Let $Y \subset X$. Keep the same metric on Y. Then (Y, d) is called a subspace of X.

For $a \in Y$, the ball $B^Y(a, r)$ is just $B^X(a, r) \cap Y$. Show.

A set $U \subset Y$ is open in Y iff there is a set $V \subset X$ which is open in X and $U = V \cap Y$. Similarly, a set $C \subset Y$ is closed in Y iff there is a set $F \subset X$ closed in X such that $C = F \cap Y$. Show.

Give an example where a proper subset of Y is open in Y but not open in X. Can you give such an example with Y open in X?

Give an example of a proper subset of Y which is closed in Y but not closed in X. Can you give such an example with Y closed in X?

Let $A \subset Y$. Show that the closure of A in Y is nothing but closure of A in X intersected with Y. That is,

$$\overline{A}^Y = \overline{A}^X \cap Y$$

$$\text{Is } \ (A^o)^Y = (A^o)^X \cap Y?$$

Give an example where a subset of Y is small in X but not small in Y. Can you give an example of a set $A \subset Y$ which is small in Y but not small in X?

In the following find $\overline{A}^Y$.
$X = R; \quad Y = Q; \quad A = \left\{ \sum_{k=0}^{n} \frac{1}{k!}; \ \ n \geq 1 \right\}$
$X = R^2; \quad Y = [0, 1) \times (0, 1]; \quad A$ is the set of all points (x, y) with each of x, y being either $1/n$ or $1 - (1/n)$.

63. Recall that a metric space is connected if empty set and whole space are the only sets which are both closed and open. We say that $Y \subset X$ is connected if (Y, d) is connected.

Show that if $Y \subset X$ is connected then $\overline{Y}^X$ is also connected. Do you think the converse is true? That is $Y \subset X$ and $\overline{Y}^X$ is connected then Y is connected.

Consider U an open subset of R^2. Show that it is path connected, that is, given two points a and b in U, there is a continuous function γ on $[0, 1]$ taking values in U such that $\gamma(0) = a$ and $\gamma(1) = b$.

Such a space is called path connected. More precisely, (X, d) is path connected if given any two points a and b in X there is a continuous function γ on $[0,1]$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Show that a path connected space is connected. Now do you see why R^{35} and $C[0,1]$ with sup metric or d_1 or d_2 or d_p metrics are connected? Show that l_2 (infinite square summable sequences) is connected.

Consider $X = R^2$. Consider the following set.

$$Y = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : 0 \leq y \leq 1\}.$$

Show Y is connected.

Show this Y is not path connected. Thus connected open sets are path connected but connected closed sets need not be path connected. This is also an example of connected space which is not path connected.

64. In R every non-empty open set is union, in a unique way, of countably many non-empty open intervals. Of course, in R^2 you can not talk of intervals.

    Show the following: Every non-empty open set in R^2 is union, in a unique way, of countably many non-empty open connected sets.

65. Here is an application of Baire (a Diophantine approximation).

    Suppose that $\{t_n\}$ is a strictly increasing sequence of positive numbers increasing to ∞. This is given to us. Then there is a large set S of real numbers such that if you take $x \in S$, then the set

    $$\{t_n x + m : n \geq 1; m \in Z\}$$

    is dense in R. of course, we can not exactly specify for which numbers x the above set is dense. However, Kronecker tells that when $t_n = n$ then the above set is dense for every irrational number x.

66. Here is another application of Baire, similar to what we did for integrals. Suppose that f is a C^∞ function on $[0, 1]$.

    If $(\exists k)$ $(\forall x)$ $f^{(k)}(x) = 0$ then f is a polynomial. Show.

    If $(\forall x)$ $(\exists k)$ $f^{(k)}(x) = 0$ then f is a polynomial. Show (not easy).

    In particular, there is *one* k such that the k-th derivative vanishes identically.

67. We learnt Cantor intersection theorem; Baire's theorem; Banach fixed point theorem in complete metric spaces. but we have not shown that some of the standard spaces we saw are actually complete. Yes, we know R is complete.

    (a) Show that R^n is complete.

    (b) Recall that R^∞ is the space of all infinite sequences $x = (x_n)$ of real numbers. The metric is (exercise 46)

    $$d_1(x, y) = \sum \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

    according to which the convergence is again coordinate-wise convergence.

Show that this space is complete. Take a Cauchy sequence $\{x^n\}$ where

$$x^n = (x_1^n, x_2^n, x_3^n, \cdots, x_j^n, \cdots\}$$

Show that for every j the sequence of numbers $\{x_j^n : n \geq 1\}$ is a Cauchy sequence. Let its limit be x_j.

Let $x = (x_j)$. Show that $x^n \to x$ in the space R^∞.

(c) Show $C[0,1]$ with sup metric is complete. Take a Cauchy sequence $\{x_n\}$ — remember this is Cauchy in sup metric.

show that for each t the sequence of numbers $\{x_n(t)\}$ is cauchy.

Show that there is a function x such that $x_n(t) \to x(t)$ for each t.

Fix $\epsilon > 0$. Fix N (using hypothesis) so that

$$\sup\{|x_n(t) - x_m(t)|; 0 \leq t \leq 1\} < \epsilon/2; \quad n, m \geq N$$

Show that $|x_n(t) - x(t)| \leq \epsilon/2$ for each $n \geq N$ and each t. Conclude that

$$\sup\{|x_n(t) - x(t)|; 0 \leq t \leq 1\} \leq \epsilon/2; \quad n \geq N$$

Conclude that (x_n) converges to x uniformly. Conclude that x is continuous and hence $x \in C[0,1]$.

Show that $d(x_n, x) \to 0$.

(d) Recall that l_2 is the space of all sequences $x = (x_j)$ of real numbers such that $\sum x_j^2 < \infty$. Recall

$$d(x, y) = \sqrt{\sum (x_j - y_j)^2}.$$

Show that this space is complete. Take Cauchy sequence (x^n) where x^n is the sequence

$$x^n = (x_1^n, x_2^n, x_3^n, \cdots, x_j^n, \cdots).$$

Show that for very j, the sequence of numbers $\{x_j^n : n \geq 1\}$ is a Cauchy sequence. Let x_j be its limit.

Using the fact that Cauchy sequences are bounded (and Minkowski) conclude that there is a number K such that

$$\sum_j (x_j^n)^2 \leq K; \quad n \geq 1$$

Show that $\sum\limits_{j=1}^{J} x_j^2 \le K$ for every $J \ge 1$. Conclude that $x \in l_2$.

Fix $\epsilon > 0$. Fix N so that $d(x^n, x^m) < \epsilon/2$ for $n, m \ge N$. Show that for each $J \ge 1$

$$\sqrt{\sum_{j=1}^{J}(x_j^n - x_j)^2} \le \epsilon/2; \quad n \ge N.$$

Conclude that $d(x^n, x) < \epsilon$ for $n \ge N$; or $x^n \to x$ in l_2.

You should appreciate the ease with which we try an inequality for finite sums (limits can be taken for finite sums) and then conclude the same for infinite sums. pause and understand this sentence.

(e) Test whether you understood the above two calculations by showing that the space l_1 is complete. Recall this is the space of all sequences $x = (x_j)$ with $\sum |x_j| < \infty$ and

$$d(x, y) = \sum_j |x_j - y_j|.$$

It is also true that all l_p spaces $(p > 1)$ are complete but you need not bother about it now (not for exam either). That will be for a later functional analysis course, not now.

68. Let (X, d) be a metric space. Recall that a subset $D \subset X$ is dense if every non-empty open set contains a point of D, or equivalently, every ball contains a point of D. A metric space is separable if there is a countable dense set.

You know that Q is dense in R and that Q is countable.

(a) Show that the set Q^n; the set of n-tuples of ration numbers is dense in R^n.

Show that this set Q^n is countable.

(b) Recall that R^∞ is the space of all infinite sequences $x = (x_n)$ of real numbers. The metric is (exercise 46)

$$d_1(x, y) = \sum \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

according to which the convergence is again coordinate-wise convergence.

Let D be the set of all sequences with finitely many non-zero terms and those terms are rationals. Show that D is dense in R^∞.

Show that D is countable.

(c) Show that the set P of polynomials with rational coefficients is dense in $C[0,1]$.

Show that this set P is countable.

If you are fed up of polynomials, here is another set. For rational numbers $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ and rational numbers $r_0, r_1, \cdots, r_k$ let x be the function whose graph consists of straight lines joining (t_i, r_i) and (t_{i+1}, r_{i+1}) for $i = 0, 1, \cdots, k-1$. Let D be the set of all these function as $k \geq 1$ varies over integers and t_i and r_i vary over rationals as stated. Show that D is dense. Show D is a countable set.

(d) Consider l_2. Let D be the set of all sequences $x = (x_n)$ which have only finitely many non-zero terms and those terms are rationals. Show that this set is dense in l_2.

Show that D is countable.

(e) Show that the same set as above is dense in l_1.

69. For any set A, let $l(A)$ denote its set of limit points. $N = \{1, 2, 3, \cdots.\}$

(a) Let $A = \{(i, 0, 0, 0) : i \in N\} \subset R^4$. Then A is closed and $l(A) = \emptyset$.

(b) Let $B = A \cup \{(i, \frac{1}{j}, 0, 0) : i, j \in N\} \subset R^4$ Then B is closed, $l(B) \neq \emptyset$ but $l(l(B)) = \emptyset$.

(c) Let $C = B \cup \{(i, \frac{1}{j}, \frac{1}{k}, 0) : i, j, k \in N\} \subset R^4$. Then C is closed, $l(C), l(l(C))$ are non-empty but $l(l(l(C)))$ is empty.

(d) Let $D = C \cup \{(i, \frac{1}{j}, \frac{1}{k}, \frac{1}{m}) : i, j, k, m \in N\} \subset R^4$. Guess and show.

(e) It is not difficult to get similar sets in R itself. Try, if you like challenges.

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The most important scientific tool of all is not anything you can buy. It is your own mind. Your thoughts and ideas are the keys that can unlock the mysteries. In the search for understanding, *questions* are perhaps the most powerful *force* of all.

– "The hidden world of forces" by Jack R White

70. I am not sure if you understood the concept of ball in a metric space. This is crucial. I have plotted some of them in class.

(a) Consider R^2 with the Euclidean metric. Plot the unit ball around the origin. Plot ball of radius $1/2$ around the point $(1,3)$. Plot in each case closed ball as well as open ball.

(b) Do the same when the metric is $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$.

(c) Do the same when the metric is $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(d) Do the same when the metric is $d(x,y) = \sqrt[3]{|x_1 - y_1|^3 + |x_2 - y_2|^3}$.

(e) Consider the real line R with the metric $d(x,y) = \min\{|x-y|, 1\}$ Plot ball of radius $r = 1/2$ with centre $x = 13$. Plot ball of radius $r = 1$ same centre. Plot ball of radius $r = 1.001$ with same centre. What if I said plot ball of same radius $r = 1.001$ with centre $x = 31$.

(f) Consider the open unit interval $X = (0,1)$ with metric $d(x,y) = |x-y|$. Is the set $B = \{x : 1/3 < x < 1/2\}$, that is, the interval $(1/3, 1/2)$ a ball? If so what is its centre and radius?

Is the set $(0.99, 1)$ a ball? There are several centres and radii posible. Think and discover.

(g) Let X be a finite set and d be *any* metric on X. Show that every singleton set is a ball. What is the centre and radius?

(h) Consider the ellipse $B = \left\{(x_1, x_2) : \frac{1}{4}x_1^2 + \frac{1}{9}x_2^2 < 1\right\}$. Can you give a metric for R^2 without changing convergence so that the above ellipse is ball of radius one with centre $(0,0)$?

(i) Take a $k \times k$ positive definite symmetric matrix. Show that $d(x,y) = \sqrt{(x-y)^t A (x-y)} = \sqrt{\sum a_{ij}(x_i - y_i)(x_j - y_j)}$. is a metric on R^k and convergence in d is usual one.

71. Just as we have $B(x, r) = \{y : d(x, y) < r\}$ ball around a point we also have ball around a set. For a non-empty set $S \subset X$,

    $B(S, r) = \{y : (\exists\, x \in S)\;\; d(x, y) < r\} = \bigcup_{x \in S} B(x, r)$.

    If K is a compact and U an open set such that $K \subset U$ show that there is an $r > 0$ such that $B(K, r) \subset U$.

72. Let K be a compact and C a closed set in (X, d). If $K \cap C = \emptyset$ then show $d(K, C) > 0$. Recall $d(A, B) = \inf\{d(x, y) : x \in A;\; y \in B\}$.

    Do you think the statement remains true if both sets are closed?

73. S_n is a decreasing sequence of non-empty compact sets in a metric space. Show $\cap S_n$ is non-empty.

    Do you think this remains true if the sets are just closed instead being compact? What if all are closed and one is known to be compact?

74. Let (X, d) be a compact metric space. Show:

    (*) If $\mathcal{C}$ is a family of closed sets such that intersection of any finitely many of them is non-empty then there is a point common to all of them (in other words, intersection of all of them is non-empty) .

    Conversely show: a metric space satisfying (*) is compact.

75. If every real valued continuous function is bounded, show that the space is compact. All this drama is on a metric space, a nice stage.

    If every real valued bounded continuous function attains its supremum then show that such a function attains its infimum too. Show that when this happens the space is compact.

76. X is a compact metric space and f_n is a sequence of real continuous functions decreasing to a continuous function f point wise. Show that the convergence is uniform. This is known as Dini's theorem and we did it last year for $X = [0, 1]$

77. Describe *all* closed additive subgroups of R. Describe all open additive subgroups of R.

E $\exists$ A $\forall$

is it odd how asymmetrical is symmetry
symmetry is asymmetrical how odd it is.

78. Here are some examples of useful metrics on some useful spaces. Of course, others we discussed earlier are useful too!

(a) Consider a (finite) set S and an integer $N \geq 1$. Consider $X = S^N$, sequences (called strings) of length N from the set S. denote $x = (x_1, \cdots, x_N)$ for points in X. Put

$$d(x, y) = |\{i : x_i \neq y_i\}|,$$

that is, the number of places where the two strings differ. Verify this is a metric.

This distance tells you how many places need to be changed to transform one sequence to the other. This is called *Hamming distance* and is useful in coding theory and DNA analysis.

Obviously the larger the distance the better chances of decoding them correctly; they can 'withstand' one or two transmission errors. Or, a larger distance between DNA strings of two animals will hint that they are different species; other interpretations are also possible.

(b) Let $X = R^N$ and Σ be a symmetric positive definite matrix of size $N \times N$. Define

$$d(x, y) = \sqrt{(x - y)^t \Sigma^{-1} (x - y)}.$$

show that this is a metric on R^N.

This is sometimes called *Mahalanobis distance* and is useful in Statistics.

The points x and y are observations (may be vector consisting of nose length, cheek bone size, forehead width; scalp measurement etc of some skull). We have two observations on two skulls we found. Want to know if they belong to the same tribe. Here Σ comes from an assumed probabilistic model; appearance of its inverse seems mysterious but shall not get into details.

(c) Let X be the set of permutations of $S = \{1, 2, 3, \cdots, N\}$. For two permutations π and η define

$$d(\pi, \eta) = \left| \left\{ (i,j) : i < j; \begin{array}{lll} \pi(i) < \pi(j) & \& & \eta(i) > \eta(j) \quad \text{OR} \\ \pi(i) > \pi(j) & \& & \eta(i) < \eta(j) \end{array} \right\} \right|$$

that is; the number of pairs which are compared by π and η differently.

Show that this is a distance. This is called *Kendall's tau* and is useful in statistics. Its largest value is $N(N-1)/2$. When is this value achieved?

Suppose you and I rank (no ties) fifty selected hotels in Chennai. How different are our rankings? Above distance is one such measure.

(d) Let X be the set of vectors of length N consisting of non-negative numbers whose sum equals one.

$$X = \{(p_1, \cdots, p_N) : p_i \geq 0 \ \ \forall\, i; \ \sum p_i = 1\}.$$

that is, all probability vectors. equivalently, all possible probability models for an experiment which has N outcomes. Define

$$d^2(p,q) = \frac{1}{2}\sum_1^N (\sqrt{p_i} - \sqrt{q_i})^2 = 1 - \sum \sqrt{p_i q_i}.$$

Then d is a distance, called *Hellinger or Bhattacharya* distance. This is useful in functional analysis (Hellinger) and in Statistics (Bhattacharya).

How different are two models? Above is a measure.

(e) Consider $S = \{0, 1, 2, 3, \cdots\}$ and $X = S^N$. The l_1 distance on X, namely,

$$d(x,y) = \sum |x_i - y_i|$$

is also called *block distance*. This is because of the following reason. Imagine houses are located at points of X. roads are laid only along lines parallel to the axes. Thus when you travel you can not go diagonally and so on. To go from house x to y this is the distance you need to travel. Think about it.

79. Here are some examples of metric spaces which are important when you study groups. Do not get panicky. For us these are just some routine examples of subsets of euclidean space.

(a) Let X be R^{400} and M_{20} be the space of 20×20 matrices. We think of them as same in the following way: Given a vector in R^{400}, we break it into 20 consecutive segments each of length 20 and these form the rows of a matrix in M_{20}.

Verify that this is an identification of the two sets(?). Bring the metric of R^{400} to M_{20}. Show matrix multiplication is continuous. (From where to where?)

(b) Show that the set of invertible matrices GL_{20} is an open subset of M_{20}. Show that matrix inverse is a continuous map of GL_{20} to itself. In other words, the group operations are friendly with convergence.

Since this is an open subset of a complete space you can treat it as 'complete' space. Is this connected?

(c) Let SL_{20} be the subgroup of GL_{20} which consists of all matrices A with $|det(A)| = 1$, that is, $det(A)$ equals ± 1. show that this is a closed subset of SL_{20}. Since this is a closed subset, you can treat it as a complete metric space. Is this connected?

What if I considered SL_{20}^{+}, matrices with determinant one. Is it closed? Is it connected?

(d) Suppose I consider the set Sym of symmetric matrices. Is this closed subset of M_{20}? Is this connected?

(e) If I considered P^{+} the set of symmetric positive definite matrices. is it open or closed? Is it connected?

(f) What if I considered normal matrices, that is, matrices with $AA^t = A^tA$. What kind of subset is it?

(g) If I considered the space O_{20} of orthogonal matrices, that is, matrices with $AA^t = A^tA = I$, identity matrix. What kind of subsets is it?

Remember: you can think and solve these problems. You should.

80. Let Q be any bounded closed rectangle contained in R^2. Show that the set of polynomials in x, y is dense in $C(Q)$.

    Generalize to R^k.

81. Here is a non-trivial application of the Stone-Weierstrass theorem. Let (X, d) be a compact metric space. We already know that $C(X)$ the space of real continuous functions on X with sup metric is a complete metric space. Goal: to show that $C(X)$ is separable.

    Thus $C(X)$ will be a Polish space.

    (a) Suppose we could exhibit a sequence of functions $\{f_1, f_2, \cdots\}$ in $C(X)$ which separate points.

    Let $f_0 \equiv 1$ and $D_0 = \{f_n : n \geq 0\}$. Let D_1 be the collection of finite products of functions in D_0. Let D be finite rational linear combinations of functions from D_1.

    Show D is countable. Show D is closed under multiplication.

    Let $\overline{D}$ be closure of D in our space $C(X)$. show that $\overline{D}$ is a vector space; is an algebra. equals $C(X)$. Deduce $C(X)$ is separable.

    (b) Let $p \in X$ and $\epsilon > 0$. Put $f(x) = d(x, p)$. Put $g(x) = \min\{f(x), \epsilon\}$. put $h(x) = g/\epsilon$. Show these are continuous functions.

    (c) Given any open ball $B(p, \epsilon)$ show that there is a continuous functions f with $f(p) = 1$ and $0 < f \leq 1$ on B and $f(B^c) = 0$.

    (d) (done in class) show that there is a *sequence of open balls* such that every open set is a union of some of these balls.

    (e) Show that there is a sequence of functions as required in the first step.

82. Let X be a metric space which is not compact. Goal: to show that the space $C_b(X)$ of bounded real continuous functions with sup metric is not separable.

    (a) Show that the space $C_b(X)$ is complete; not needed for discussing separability but good to know such basic facts; at no extra cost.

(b) Let us consider R. Let $N = \{1, 2, 3, 4, \cdots\}$. For any subset $A \subset N$, show that there is a bounded continuous function f_A on R such that $f_A(x) = 1$ for each $x \in A$ and $f_A(x) = 0$ for each $x \in N - A$. show that $C_b(R)$ is not separable.

(c) Back to metric space. Let A and B be two disjoint non-empty closed subsets of the metric space X. Show

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

is a real bounded continuous function on X.

(d) If X is not compact, there is a sequence which has no convergent subsequence. Show we can take the sequence to consist of distinct points. show every subset of the sequence (?) is a closed subset of X.

(e) Show $C(X)$ is not separable.

83. Let $K(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$. Let us define a map on $C[0, 1]$ by

$$Tx(s) = \int_0^1 K(s, t)x(t)dt.$$

Show T takes the space to itself, in other words, if x is continuous then so is Tx (done last semester, recall proof).

Show: if $\{x_n\}$ is a sequence in $C[0, 1]$ which is uniformly bounded, then the sequence $\{Tx_n : n \geq 1\}$ is precompact.

84. Suppose that $A \subset C[0, 1]$ is a collection of differentiable functions. Suppose that there is a number M such that for all $x \in A$ and for all $t \in [0, 1]$

$$|x(t)| + |x'(t)| \leq M$$

Show that A is precompact.

85. Here is an application of Arzela-Ascoli. This is called Peano's theorem.

We are given an open set $\Omega \subset R^2$ and a point $(t_0, x_0) \in \Omega$ and a continuous function $F : \Omega \to R$. Goal is to show that we can exhibit an interval around t_0 and a differentiable function x on that interval such that (i) $x(t_0) = x_0$ and (ii) $x'(t) = F(t, x(t))$. The last condition means that for each t in the interval exhibited, the point $(t, x(t)) \in \Omega$ and the stated equality holds.

get a closed ball B around (t_0, x_0) contained in Ω and let $|F| < M$ for all points in this ball. Take $\delta > 0$ so that

$$[t_0 - \delta, t_0 + \delta] \times [x_0 - M\delta, x_0 + M\delta] \subset B.$$

Argue that there is a sequence of polynomials $P_n(x, y)$ such that

$$d(P_n, f) \to 0; \quad |P_n(x, y)| < M \ \ \forall (x, y) \in B$$

Show that there is a solution $x_n(t)$ for (i), (ii) with data (t_0, x_0, P_n).

Write the integral equation satisfied by x_n.

Show there is a subsequence of $\{x_n\}$ converging to, say, x^*.

show that x^* is a solution of original problem.

86. consider R^2 and $F(t, x) = x^{2/3}$ and $(t_0, x_0) = (0, 0)$. Then $x(t) \equiv 0$ and $x^*(t) = t^3/(27)$ both solve $x' = F(t, x(t))$.

87. Let C be a closed subset of R. Show that we can express

$$C = D \cup P; \quad D \text{ countable}; P \text{ closed}; \quad P = \emptyset \text{ or } l(P) = P$$

Recall that $l(P)$ is the set of limit points of P and so the last phrase means that every point of P is a limit point of P. [take D = union of $(a, b) \cap C$ which are countable].

Let P be a closed set for which every point is a limit point. Then P must have same power as c — if non-empty.

You Prove it just like Cantor intersection theorem. Get two disjoint closed balls B_0 and B_1. within each get two disjoint closed balls B_{00}, B_{01} and B_{10}, B_{11}. Stare at these.

Open your thoughts first, then your pen.

88. The algebra generated by $\{1, x^2\}$ is dense in $C[0,1]$. Is it dense in $C[-1,1]$? Suppose I take an integer $k \geq 1$. For what I is the algebra generated by $\{1, x^k\}$ dense in $C(I)$? Of course, I here is a closed bounded interval.

89. f is real continuous on $[0,1]$ and $\int_0^1 f(x)x^n dx = 0$ for integers $n \geq 0$ show $f(x) \equiv 0$.

90.]it Unit balls

    Remember the unit ball in R^k is the set of all vectors x such that $d(x,0) = ||x|| \leq 1$. We know it is compact.

    Analogously, unit ball in $C[0,1]$ is the set of all functions $x = x(t)$ such that $d(x,0) \leq 1$. This means the set of all x with $\sup |x(t)| \leq 1$. Show that this is not compact.

    Similarly the unit ball in l_2 is the set of all sequences $x = (x_n : n \geq 1)$ such that $d_2(x,0) \leq 1$. This means $\sum x_n^2 \leq 1$ Show this is not compact.

    However show that the following set is compact in l_2. All points $x = (x_n)$ with $|x_n|, \leq 1/n$ for all n.

91. If a metric space is totally bounded then show that every sequence contains a subsequence which is Cauchy.

    Thus, duty of totally boundedness is to provide a Cauchy subsequence; duty of completeness is to make it converge; together give compactness.

92. *compact subsets of* R^∞

    The diagonal argument has great potential.

    Remember we made R^∞ the space of sequences of real numbers into a metric space. We gave a metric to the space. Of course convergence is just coordinatewise convergence. Show that a subset $K \subset R^\infty$ is pre compact iff for each n, there is number M_n such that
    $x = (x_n) \in K \Rightarrow |x_n| \leq M_n \ \ \forall n$.

93. Here is a further generalization of the space $C(X)$ we considered.

Let X be a compact metric space and Y be a Polish space. Let $C[X, Y]$ be the space of all continuous functions on X with values in Y.

Let $X = [0, 1]$ and Y is Cantor set, describe $C[X, Y]$.

Let $X = [0, 1]$ and Y the set consisting of the vertical lines at $x = 2$ and $x = 4$ in the plane. Describe $C[X, Y]$. Are you able to imagine it as two copies of $C[0, 1]$ sitting side by side.

Let $X = [0, 1]$ and Y be the set of complex numbers. Describe $C[x, Y]$ are you able to imagine it as $C[0, 1] \times C[0, 1]$?

Let $X = [0, 1]$ and $Y = C[0, 1]$. Identify $C[X, Y]$ with $C([0, 1] \times [0, 1])$. What does identify mean here?

back to generalities.

Show $f \in C$ implies range of f is a bounded subset of Y.

Show $f \in C$ implies it is uniformly continuous. This means, given $\epsilon > 0$, there is $\delta > 0$ such that

$$d_X(s, t) < \delta \Rightarrow d_Y(f(s), f(t)) < \epsilon.$$

Define for $f, g \in C[X, Y]$;

$$d(f, g) = \sup\{d_Y(f(s), g(s)) : s \in X\}.$$

Show that this is a metric and $C[X, Y]$ is a complete metric space.

94. *Hausdorff metric*

The setup here appears abstract but not the mathematics. I mention to impress upon you the diversity of metric spaces you can think of.

Consider $I = [0, 1]$ and let Γ be the collection of all non-empty compact subsets of I. Given two sets K and L in this space show that there is at least one $\epsilon > 0$ such that $K \subset B(L, \epsilon)$ and $L \subset B(K, \epsilon)$.

Hausdorff defines the distance between two sets as the smallest such ϵ. That is,

$$\rho(K, L) = \inf\{\epsilon > 0 : K \subset B(L, \epsilon);\ L \subset B(K, \epsilon)\}.$$

Suppose K and L are singletons $\{x\}$ and $\{y\}$ guess what should be their distance and verify.

Suppose K is a doubleton and L is a singleton. Then what is their distance? What if both are doubletons?

What is the distance between $[0, 1/3]$ and $[2/3, 1]$. Guess first and proceed.

Show that ρ is actually distance on Γ.

It is not difficult to show that the space Γ is compact, but let us not spend time on it. Remember, in connection with fixed point theorem we came across this space.

95. *Stone-Weierstrass, Complex version*

    Let X be a compact metric space and $C[X, C]$ be the space of complex valued continuous functions on X. Let A be a sub algebra of this which contains constant functions and separates points. Suppose moreover that $f \in A$ implies its conjugate $\overline{f} \in A$. Show that A is dense in the space. (Reduce the problem to real case).

    Remember if $f(x) = f_1(x) + i f_2(x)$ where f_1 and f_2 are real valued then conjugate of f is defined by $\overline{f}(x) = f_1(x) - i f_2(x)$.

96. *identifying points*

    I said that if you tie the two points zero and one together, the unit interval becomes unit circle. This is actually a precise statement. Let us see what it means.

    Let $X = [0, 1]$ usual metric. Let $Y = (0, 1) \cup \{\spadesuit\}$ Thus Y has all points of X except zero and one, instead it has one extra point. This is the bag containing zero and one. Here is the metric $d^*(s, t)$: it is same as $d(s, t)$ when $s, t \in (0, 1)$; if both are the extra point then distance is zero; if $s \in (0, 1)$ then $d^*(s, \spadesuit) = \min\{|s - 0|, |s - 1|\}$.

    Show that this is a metric. Show this space is homeomorphic to the circle via the map $h(t) = (\cos 2\pi t, \sin 2\pi t)$. Need not worry how to interpret the value at $\spadesuit$, take it as zero or one; does not matter.

    later you will see several such constructions.