Cantor Construction of Reals:

So far we have:

• R_0, the set of Cauchy sequences of rational numbers.

• R the set of equivalence classes, where two Cauchy sequences x, y are equivalent if $d(x_n, y_n) = |x_n - y_n| \to 0$.

• $[x] + [y] = [x + y]$; zero element is the equivalence class, still denoted by 0, containing the Cauchy sequence: $\{0, 0, 0, 0 \cdots\}$ — all terms of the sequence are zero. of course this class contains other sequences too, for example

$$(a_n = 1/n); \quad (b_n = -1/n); \quad (c_n = 1/n^{19}); \quad (d_n = 1/n!) \cdots\cdots;$$

but NOT the sequence $(\exp\{-n\})$ or $(1/\sqrt{n})$ and so on simply because these are not sequences of rational numbers.

• $[x] \cdot [y] = [xy]$; unit element is the equivalence class containing the sequence: $\{1, 1, 1, \cdots\}$ — all terms equal to one. This class contains other sequences too, for example, you can add any of the above sequences to this sequence.

• R is a field with above operations.

• $[x] < [y]$ if there is a rational $r > 0$ and N such that $x_n + r \leq y_n$ for all $n > N$. $[x] \leq [y]$ if either $[x] = [y]$ or $[x] < [y]$.

We shall show that $\leq$ is a linear order.

(i) $[x] \leq [x]$.

This is clear because $[x] = [x]$ and see the definition of $\leq$.

(ii) If $[x] \leq [y]$ and $[y] \leq [x]$ then $[x] = [y]$.

In the two hypotheses, if equality holds somewhere then there is nothing to be proved. We now show that strict inequality at both places is not possible, this will complete the proof.

If possible $[x] < [y]$ and $[y] < [x]$. Get one N and $r > 0$ and $s > 0$ such that

$$n > N \Rightarrow x_n + r \leq y_n; \quad y_n + s \leq x_n.$$

This is impossible because if we take any one $n > N$ we should have

$$x_n + r + s \leq y_n + s \leq x_n; \quad r + s > 0.$$

We know enough about rationals.

[irrelevant discussion: In the first para, why did he say 'if equality holds in some hypothesis'; can equality hold in one hypothesis and strict inequality hold in the other hypothesis. Such doubts are natural to arise, if you are thinking. But you need not worry. As far as our proof is concerned this question has no consequence. We have proved our claim. It is all that matters. You can, of course, return to your doubt; yes, you are right.]

(iii) $[x] \leq [y]$ and $[y] \leq [z]$ implies $[x] \leq [z]$.

Repeat above argument to show, with obvious notation, $x_n + r + s \leq z_n$ for $n > N$.

(iv) Either $[x] \leq [y]$ or $[y] \leq [x]$ holds.

Suppose that both $[x] < [y]$, $[y] < [x]$ fail. We show that $[x] = [y]$ holds. Start observing that the assumption implies

$$(\forall r > 0)\ (\forall N)\ (\exists n > N)\ x_n + r > y_n; \quad (\spadesuit)$$

$$(\forall r > 0)\ (\forall N)\ (\exists n > N)\ y_n + r > x_n. \quad (\clubsuit)$$

We need to show that $|x_n - y_n| \to 0$. Fix any $r > 0$. Remembering that x and y are Cauchy sequences, fix N so that

$$n, m \geq N \Rightarrow |x_n - x_m| < r/4; \quad |y_n - y_m| < r/4.$$

To complete proof, we shall show $|x_n - y_n| < r$ for $n > N$. First, for this N and $r/4$, using ($\spadesuit$), ($\clubsuit$) fix $i > N$ and $j > N$ so that

$$x_i + \frac{r}{4} > y_i; \quad y_j + \frac{r}{4} > x_j;$$

The subtle point is that at this stage we do not know if i and j are same in the inequalities above.

Now take any $n \geq N$. Then

$$x_n \leq x_j + \frac{r}{4} \leq y_j + \frac{r}{4} + \frac{r}{4} \leq y_n + \frac{r}{4} + \frac{r}{4} + \frac{r}{4}$$

and

$$y_n \leq y_i + \frac{r}{4} \leq x_i + \frac{r}{4} + \frac{r}{4} \leq x_n + \frac{r}{4} + \frac{r}{4} + \frac{r}{4}$$

as promised.

(v) For every pair exactly one of the following hold: $[x] < [y]$ or $[x] = [y]$ or $[y] < [x]$.

This is already in the above arguments.

(vi) The order is friendly with addition and multiplication. $[x] < [y]$ implies $[x] + [z] < [y] + [z]$. Also $[0] < [x]$, $[0] < [y]$ imply $[0] < [x][y]$. These follow from definition of order.

We shall now identify Q as a subset of R. Define for $q \in Q$, $\varphi(q)$ to be the equivalence class containing the constant sequence $\{q, q, q, q, q, \cdots\}$. This map is one-to-one, respects (?) addition, multiplication and also order. We simply think of Q as a subset of our R. This needs to be said because, after all, elements of R are *not* rational numbers; they are *not even* Cauchy sequences of rational numbers; they are bags where each bag contains a collection of Cauchy sequences.

We make a useful observation. Given $[x] < [y]$ there is a rational q such that $[x] < q < [y]$. Here is the proof. Fix $r > 0$ and N such that

$$n \geq N \Rightarrow x_n + r \leq y_n$$

If necessary by taking a larger N, we can also assume that

$$n, m \geq N \Rightarrow |x_n - x_m| < \frac{r}{4}.$$

Let

$$q = x_N + \frac{r}{2}.$$

We show this will do. Observe that this is a rational (our sequences are sequences of rational numbers). Let $n > N$.

$$q + \frac{r}{4} = x_N + \frac{r}{2} + \frac{r}{4} \leq x_n + \frac{r}{4} + \frac{r}{2} + \frac{r}{4} \leq y_n$$

showing that $q < [y]$. Also

$$x_n + \frac{r}{4} \leq x_N + \frac{r}{4} + \frac{r}{4} = q$$

showing $[x] < q$.

To complete our construction, we show now that every non-empty subset of R which is bounded above has a supremum. We shall repeat the arguments used in the Dedikind construction. To avoid giving any wrong impression let me add, it is not as though Cantor borrowed the Dedikind construction, they both did around the same time.

Thus let S be a non-empty set bounded above. Take any upper bound $[a]$ and get a rational q such that $[a] < q < [a] + 1$. Similarly if $[x] \in S$, take a rational p such that $p < [x]$. Here we are using the observation made above. And also we identify rational with the constant sequence or more precisely, the class containing the constant sequence.

The upshot of what we did above is to get $p_0 = p$ and $q_0 = q$ so that
(i) $p_0 < q_0$ and
(ii) there exists $[x] \in S$, $p_0 \leq [x]$ and $[x] \leq q$ for all $[x] \in S$.
In words, there are points of S at least as large as p but nothing that exceeds q.

We shall now construct rationals p_n, q_n for $n \geq 1$ such that

(i) $p_n < q_n$. Further, one of these numbers is $(p_{n-1} + q_{n-1})/2$ and the other belongs to $\{p_{n-1}, q_{n-1}\}$.
In other words one of the new numbers is same as the earlier one, and the other is average of the two earlier ones.

(ii) $\exists [x] \in S; p_n \leq [x]$ and $\forall [x] \in S; [x] \leq q_n$.

This is easy. For example consider $r_0 = (p_0 + q_0)/2$. if for all $[x] \in S$ we have $[x] \leq r_0$ then declare $p_1 = p_0$ and $q_1 = r_0$. Otherwise declare $p_1 = r_0$ and $q_1 = q_0$. In general consider the midpoint of the previous two points and proceed.

Clearly, the construction shows the next pair of points are in between the existing pair. Thus

$$p_0 \leq p_1 \leq p_2 \cdots\cdots \leq \cdots\cdots \leq q_2 \leq q_1 \leq q_0.$$

$$q_n - p_n = \frac{q_{n-1} - p_{n-1}}{2}.$$

These in turn show that
(p_n) is a Cauchy sequence and (q_n) is a Cauchy sequence.
$q_n - p_n \to 0$, that is $(p_n) \sim (q_n)$.
As a consequence we can define an element of R by

$$s = [(p_n)] = [(q_n)].$$

We now show that

(i)If $s < [x]$ then $[x] \notin S$; that is, $[x] \in S \Rightarrow [x] \leq s$.
This shows that s is an upper bound of S.

(ii) $[a] < s \Rightarrow \exists [x] \in S; [a] < [x] \leq s$.
This shows that nothing smaller than s will serves as upper bound.

This will then complete the proof.

Proof of (i):

Since $s = [(q_n)] < [x]$, fix $r > 0$ such that $q_n + r \leq x_n$, say for $n \geq N_1$. Since (q_n) is cauchy, we have $|q_m - q_n| \leq r/2$ for $n \geq N_2$. Taking N larger than both N_1 and N_2, we see that for $n > N$

$$q_N + \frac{r}{2} \leq q_n + \frac{r}{2} + \frac{r}{2} \leq x_n$$

In other words $q_N < [x]$ But by construction nothing larger than q_N is in S. Thus $[x] \notin S$.

Proof of (ii):

Use similar argument as above to see that there is N such that $[a] < p_N$ and note that there are points of S at least as large as p_N.

[It is tempting to say that $q_n \to s$; so given $s < [x]$ there is N such that $s \leq q_N < x$. But unfortunately at this moment our vocabulary is limited, we do not know convergence that well. This can however be made rigorous and then used.]

This completes Cantor's construction of R.

I am reminded of a poem.

A centipede was happy - quite;
until a toad, in fun, said
"prey, which leg moves after which"
This raised her doubts to such a pitch
She fell exhausted in the ditch
not knowing how to run.

You should not continue thinking (following Cantor): Aha, I now know real numbers; a real number is a Cauchy sequence of rational numbers. This leads to utter confusion when you later think of sequences of real numbers;

because this would then mean a *sequence of Cauchy sequences* of rational numbers! — not a happy thought.

Or you should not think (following Dedikind): Aha, I know real numbers, real numbers are cuts in Q. This leads to unnecessary confusion when you consider subsets of R. This would then mean *a set of subsets* of Q — not a happy thought.

Then why did we do all this. This gives you practice in handling mathematical objects. This leads to certain confidence. This also familiarizes with mathematical arguments and mathematical proofs. There are at least three other reasons why you should appreciate all this.

Firstly, it is very important for us to know whether there is a system at all satisfying the axioms laid down for R. Once we are sure that such a system exists, we just work with a system satisfying these rules — no matter how such a system is arrived at; either by Dedikind or Cantor method or any other third method. Real numbers attain independent existence irrespective of who constructs or how he/she constructs.

It is just like locating a house. It may be to the right of a hotel or to the left of a building or opposite a shop and so on. Once the house is discovered all these other pointers, hotel, building, shop etc are irrelevant. They might even be confusing.

Second reason is vey important. If you know how to construct a house, you can do very well in the construction business. For example, a customer might want a house with some interesting properties; you can think a little bit with your expertise and make it. Here is one concrete example. I had different example in mind, but after discussion with Uma, I realized this may be more appropriate for you.

I have two vector spaces V and W. Suppose we wanted to multiply vectors in W with vectors in V. We also want again a vector space — let us pretend our vector spaces are over R (it makes no difference). Let us not go into the reasons why such a thing is needed. Can you help? You have the expertise now. You only need to think.

We want to multiply: $v \cdot w$ where $v \in V$ and $w \in W$. Consider such 'things'. But if I write this dot, you would ask me what is the dot. So I say consider (v, w), you accept because you know ordered pairs and the set $V \times W$. So I show you (v, w) but at the back of my mind I have $v \cdot w$. But then this set by itself, is not a linear space. We wanted a linear space.

So I say consider finite linear combinations of the above things, like

$$4\, v_1 \cdot w_1 \;+\; \frac{32}{9}\, v_2 \cdot w_2 \;-\; \sqrt{3}\, v_3 \cdot w_3.$$

or equivalently,

$$4\,(v_1, w_1) + \frac{32}{9}\,(v_2, w_2) - \sqrt{3}\,(v_3, w_3). \qquad (\spadesuit)$$

Since 'linear combination of linear combinations' is again a grand linear combination of original things, there is hope that such linear combinations make a linear space. But you would again raise question: what is this plus sign? So I say consider the function f on $V \times W$ defined by

$$f(v_1, w_1) = 4; \quad f(v_2, w_2) = \frac{32}{9}; \quad f(v_3, w_3) = -\sqrt{3}; \quad (\clubsuit)$$

and $f(v, w) = 0$ for other pairs. You do not object to this, you know functions and the above is a legitimate function on $V \times W$. Thus I show you $(\clubsuit)$ but at the back of my mind I have $(\spadesuit)$.

But there is one problem. For example, consider the following. Take $v_1, v_2 \in V$ and $w \in W$. Put $v = v_1 + v_2$. My mind tells me $v \cdot w$ is same as the linear combination $v_1 \cdot w + v_2 \cdot w$. Or to use the 'things I am showing you' I should identify the two functions:
$f(v, w) = 1$ and f is zero for other pairs.
$g(v_1, w) = 1 = g(v_2, w)$ and g is zero for other pairs.
I identify f and g. Thus I consider the collection of functions f on $V \times W$ (which are zero outside a finite set) and define an equivalence relation — well thought out and driven by what we all feel — on this set. This *exactly* meets our demands.

The resulting house goes by the name of tensor product.

The third reason is the importance of understanding symbols without getting confused. We use some symbols for which you would have no objection but while we use the symbol, we have some thing at the back of our mind. Thus, what is at the back of our mind is more important than the symbol we are using. Think about it.

back to open and closed sets:

Before returning to general metric spaces, let us observe something special about open sets in the real line. Let $U \subset R$ be an non-empty open set. For every point $x \in U$ there is an open interval $(x-\epsilon, x+\epsilon) \subset U$. Obviously U is union of all these intervals. However we can be more specific.

Every nonempty open subset of R is a countable disjoint union of non-empty open intervals in a unique way. We shall prove this now.

Let us start clarifying meanings of the terms. Interval means a subset with the following property: if two points are there, everything in between is also there. More precisely, $A \subset R$ is an interval if

$$x < z < y; \quad x, y \in A \Rightarrow z \in A.$$

Suppose A is an interval. Let $a = \inf A$ and $b = \sup A$. if A is not bounded below then we take this inf to be $-\infty$; if A is not bounded above we take the sup to be ∞.

It is easy to show that, for example when the above inf and sup $a, b \in R$ then the interval A must either be $[a, b]$ or $[a, b)$ or $(a, b]$ or (a, b). If it is an open interval, that is, if it is an interval which is an open set then it must be (a, b).

Similarly, when $a \in R$ but $b = \infty$ then the interval A must be either $[a, \infty)$ or (a, ∞). If it is an open interval, then it must be (a, ∞). Similar remark applies when $a = -\infty$ and $b \in R$.

Of course, union of two intervals need not be an interval. We claim that union of two intervals which have a point in common, is again an interval. Indeed let I and J be intervals and z be in both. Let $a < b$ be two points in the union, say $a \in I$ and $b \in J$. If $a < b < z$ then both a and z are in I and hence so is everything in between, in particular, everything in between a and b is in I and hence in $I \cup J$.

If $z < a < b$, then both z and b are in J and hence so is everything in between them. In particular everything in between a and b is in J.

Suppose $a < z < b$. Then everything in between a and z is in I, whereas everything in between z and b is in J. Thus everything in between a and b is in $I \cup J$.

Thus union of two intervals which have a point in common is again an interval.

Returning to our problem, take $x \in U$. By above observation, the union of all intervals I such that $x \in I \subset U$ is itself an interval. Call it I_x. Suppose $a \in R$ and is an end point of I_x, then we claim that $a \notin I_x$. Because if $a \in I_x \subset U$, then there must be an interval $(a - \epsilon, a + \epsilon) \subset U$. But then $I_x \neq I_x \cup (a - \epsilon, a + \epsilon)$ is an interval because a is in both' Also this union is a subset of U because both are so. Also this interval includes x. This contradicts that I_x is the union of all such intervals.

Thus I_x is an open interval.

Actually, I_x is the largest open interval J with the property $x \in J \subset U$. There is nothing to prove here if you look at the definition of I_x.

Thus for every $x \in U$ we have an open interval I_x such that $x \in I_x \subset U$, largest such interval. We now claim that if $x \neq y$ then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. This is clear because if the intersection is non-empty, then their union is an interval contained in U, includes both x and y.

Let $\mathcal{I}$ be the collection of the *distinct intervals* of the family $\{I_x; x \in U\}$. Since any two intervals in $\mathcal{I}$ are disjoint and all are non-empty we conclude that this family $\mathcal{I}$ must be countable. Take one rational from each to see this.

Thus we can enumerate the family $\mathcal{I}$ as a sequence. Thus

$$U \ = \ I_1 \ \cup \ I_2 \ \cup \ I_3 \ \cup \ \cdots.$$

(finite or infinite) union of disjoint non-empty open intervals.

Suppose that

$$U \ = \ J_1 \ \cup \ J_2 \ \cup \ J_3 \ \cup \ \cdots.$$

union of disjoint non-empty open intervals. We show that the intervals are exactly the same, perhaps enumerated in a different order.

Let $x \in U$. Let $J_k = (a, b)$, say, contains the point x. We argue that J_k is indeed I_x. First observe the following. if a is finite then $a \notin U$. Because, if it were in U, then it must be in one of the other J, but then that J must contain an interval around a, but then such an interval intersects J_k. Remember the intervals J_i are disjoint. Similarly if b is finite then $b \notin U$.

Since $x \in J_k \subset U$ and I_x is maximal such interval we conclude that $J_k \subset I_x$. If I_x is strictly larger than J_k, then an end point of J_k must be finite and must be in I_x. But any finite end point is shown to be not in U. This shows $J_k = I_x$.

This completes the proof.

collection of open sets:

Let (X, d) be a metric space. We defined a subset U to be open if $x \in U$ implies there is an $r > 0$ such that

$$B(x, r) \ = \ \{y : d(x, y) < r\} \subset U.$$

The collection of open sets has the following properties:

(i) As already noted, $\emptyset$ and X are open.

(ii) Union of any collection of open sets is open. This is because, if a point x is in the union it is then in one of the sets which itself already contains a $B(x, r)$.

(iii) Finite intersection of open sets is open. Indeed, let U and V be open. Let $x \in U \cap V$. Then $x \in U$ and $x \in V$ so that there is $r > 0$, and $s > 0$ such that $B(x, r) \subset U$ and $B(x, s) \subset V$. If we take $p = \min\{r, s\}$ then easy to see that $B(x, p) \subset U \cap V$. This being true for every point in the intersection we are done.

Accordingly, we see that $\emptyset$ and X are closed; intersection of any collection of closed sets is a closed set; finite union of closed sets is closed.

We claim that the ball $B(x, r)$ defined above is an open set. So we are justified in calling it open ball. To see this let $x_0 \in B(x, r)$, say

$$d(x, x_0) = \alpha < r; \qquad s = r - \alpha > 0.$$

We claim that $B(x_0, s) \subset B(x, r)$. Indeed

$$y \in B(x_0, s) \Rightarrow d(x, y) \leq d(x, x_0) + d(x_0, y) < \alpha + (r - \alpha) = r.$$

$$\Rightarrow y \in B(x, r).$$

Similarly $C = \{y : B(x, y) \leq r\}$ is a closed set and hence we are justified in calling it closed ball. To see that it is closed, we only need to observe that if $y_n \to y$, and $y_n \in C$ for all n, then

$$d(x, y) = \lim d(x, y_n) \leq r.$$

Thus open ball is indeed an open set and closed ball is a closed set.

interior and closure:

If a set is not open how do we relate it to an open set. we can take the largest open set contained in it. This is called interior.

Let (X, d) be a metric space and $A \subset X$. We define A^o, called interior of A, to be the largest open set contained in A. This makes sense because union of open sets is open. Thus A^o is the union of all open sets contained in A. equivalently, it is the union of all open balls contained in A. A point of A^o is called an interior point of A.

Closure $\overline{A}$ is the smallest closed set that contains A. This is just the intersection of all closed sets that contain A.

We can also say that

$$\overline{A} \;=\; A \cup \{\text{set of limit points of } A\}.$$

Denote the set on left by S. We know that a set is closed iff it includes all its limit points. Thus every closed set that contains A includes all limit points of A. Hence $\overline{A} \supset S$. Conversely, to show that $\overline{A} \subset S$ we show that S is a closed set containing A. Of course $S \supset A$. To show S is closed, let $x \notin S$. Since it is not limit point of A there is a ball $B(x, r)$ which has no point of A other than possibly x. But since $x \notin A$ we conclude that $B(x, r) \subset S^c$ showing S^c is open.

Compact sets:

We shall now imitate to define and discuss notion of compact sets that we studied in R^n.

Let (X, d) be a metric space. A subset $K \subset X$ is said to be compact if the following happens: Every sequence $(x_n) \subset K$ has a *subsequence* that *converges to a point of* K.

The importance of this notion comes from the fact that in R^n compact sets are precisely closed bounded sets; more importantly, every real valued continuous function on a compact set is bounded and attains its bounds.

For general metric spaces the above characterization is too much to expect (though the consequence concerning continuous functions is still correct with exactly the same proof, as we shall see). In fact, 'bounded' does not mean anything because we can always change the metric to another bounded metric without changing convergence.

Also one needs to justify the choice of the word 'compact'. This word gives the impression that the set is not too much 'spread out'; or given lot of material to cover the set we can use only a little of that material to cover the set. Yes, this interpretation is right.

We shall discuss this concept after the midsem.

countable intersection of open sets:

We have discussed the concept of interior and closure. We shall now discuss another way of relating a given set to open sets. This leads to a nice and useful story. We start with simple question.

Let $\mathcal{I}$ be the set of irrational numbers. It is not an open set.

Is it union of open sets? This is silly, union of open sets is open, so we have answered this question. Is this intersection of open sets? This is also silly, every set is intersection of open sets, namely, intersection of all sets $\{x\}^c$ with $x \in A^c$.

We ask, is $\mathcal{I}$ intersection of countable many open sets? This question is better. The answer is, Yes, $\mathcal{I}$ is intersection of $\{x\}^c$ with x running over rationals.

Good, let us repeat with Q the set of rationals. It is not open. Is it intersection of countably many open sets? the answer is not so immediate now. We show it is *not* a countable intersection of open sets.

The above question appears purely set theoretic in nature. However, it is worth recalling that we already needed its answer last year while discussing continuous functions. Let us recall. We constructed a function f on R for which the set of continuity points are precisely the set of irrational numbers. In other words, if x is an irrational number then f is continuous at x; if x is a rational number then f is not continuous at x. naturally, we asked: can we cook up a function f which is continuous at x when x is rational while discontinuous at x when x is not rational.

Returning to our claim, let if possible let

$$Q = U_1 \cap U_2 \cap U_3 \cap \cdots .$$

where each U_i is open. Let $r_1, r_2, r_3, \cdots$ be an enumeration of all rational numbers. We shall manufacture a sequence of intervals $[a_n, b_n]$ such that the following hold.

(i) $r_n \notin [a_n, b_n]$.

(ii) $[a_n, b_n] \subset U_n$.

(iii) $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$.

(iv) $0 < b_n - a_n \leq 1/2^n$.

Conditions (iii) and (iv) and Cantor's intersection theorem tell us that there is a point z common to all these intervals $[a_n, b_n]$. Condition (ii) tells that this point is in all the U_n. Condition (i) tells that this point can not be any rational. This contradicts that Q is the intersection of all the U_n.

We shall now construct such intervals by induction. Start observing that each U_n includes all rationals and is an open set.

Take a non-degenerate open interval contained in U_1. If it includes r_1 take a smaller interval that excludes r_1. Take non-degenerate closed subinterval of this. If it is large, cut it down and make its length at most one. This is $[a_1, b_1]$.

Since $a_1 < b_1$ there is a rational in between and this rational belongs to the open set U_2 as well. Thus get a non-degenerate open subinterval inside $[a_1, b_1]$ which is contained in U_2. If this interval includes r_2 take a smaller subinterval to exclude it. Take a non-degenerate closed subinterval of this. If it is large, cut it down and make length at most $1/2$.

Having got intervals for $n = 1, 2, \cdots, k$ we can write down how to get the $(k+1)$-th interval. The reason that such a thing must be written down, instead of saying 'etc etc' or 'do like this' or 'so on' is the following. You need to convince yourself that this procedure can be repeated *for ever*. What would you do if someone says, no you can not continue like this.

I shall write the inductive step just to make sure you know how to do it. Assume we got the first k intervals satisfying the conditions stated above up to k. (Do you realize how important it is to list the conditions, when you make construction inductively, so that they make sense up to k. After all what I needed earlier was $b_n - a_n \to 0$. So I could have written condition (iv) as $b_n - a_n \to 0$. This would be careless because then saying 'condition (iv) holds up to k' does not make any sense. These subtle points you must pay attention to, at least until you clearly grasp what makes sense and what does not! That is why writing proofs is very important.)

Here then is the inductive step. Since $0 < b_k - a_k$ take a rational between a_k and b_k, then it is in $(a_k, b_k) \cap U_{k+1}$. Thus there is a non-degenerate open interval which is contained in $(a_k, b_k) \cap U_{k+1}$. If this includes r_{k+1} take a non-degenerate subinterval which excludes this point. Take a non-degenerate closed subinterval contained in this interval. If its length does not exceed $1/2^{k+1}$ take it; if it is large, cut it down. The resulting interval is $[a_{k+1}, b_{k+1}]$.

This completes the proof.

Let C be the Cantor set. Suppose that we asked: Is $Q \cup C$ a countable intersection of open sets. Then we can not imitate the above proof. Earlier we could show a non-rational point in $\cap U_n$ by avoiding rationals one-by-one. This was possible simply because Q is a countable set.

Let us ask another question which you feel is unrelated to our present discussion. Can you express R as a union of two disjoint non-empty closed sets? That is

$$R = A \cup B; \quad A \neq \emptyset; \; B \neq \emptyset; \quad A, B \text{ closed, disjoint.}$$

This means A is a non-empty proper closed subset of R — connectedness arguments prevent such a thing.

Very good, let us change it to countable union. Can you express R as a countable union of disjoint non-empty closed sets? That is,

$$R = \cup A_n; \quad (\forall n) A_n \neq \emptyset; \quad (\forall n) A_n \text{ closed }; \quad (\forall n \neq m) A_n \cap A_m = \emptyset.$$

The answer is not immediate. There is an interesting story behind this discussion. The beauty unfolds slowly.

But for now, let us just realize that what we proved above is something that actually proves a better thing.

small sets:

A closed subset $C \subset R$ is said to be small if it does not include a (non-empty) open interval.

Thus for example a singleton set is small. Set of integers is small. The Cantor sets is small. However the interval $[0, 1]$ is not small.

We say that $A \subset R$ is small if its closure $\overline{A}$ is small. That is, there is no open interval (non-empty) contained in $\overline{A}$.

For example every subset of the Cantor set is small, because its closure is contained in C. Set Q of rational numbers is not small, because its closure is all of R. However Q is a countable union of small sets, namely, singleton sets. Here then is a nice theorem whose proof is hidden in the earlier argument.

Theorem: R is not union of countably many small sets.

Recall that we already know that R is not countable, that is it is not countable union of singleton sets. We are now saying better. It can not even be union of countably many sets each 'looking like' Cantor set.

The result we proved earlier can be deduced from this more general theorem. Here is how. Suppose, if possible

$$Q = U_1 \cap U_2 \cap U_3 \cap \cdots\cdots.$$

Then

$$R = \bigcup_{r \in Q} \{r\} \quad \cup \quad \bigcup_n U_n^c.$$

This equality is clear because all rationals are in the first collection of singleton sets; the earlier equality about Q tells that every irrationals is outside some U_n and is hence captured by some U_n^c. This is a countable union. Finally, sets in the first collection are singletons and are hence small. Each U_n^c is already closed and does not contain any rational. Hence U_n^c can not include any open interval (non-empty). So it is also small.

In other words, if you can express Q as above, then R is a countable union of small sets.

So how do we prove this theorem. As I said there is no new idea. if possible let $R = \cup C_n$ countable union of small sets. We assume that the sets C_n are closed, if necessary replace the original sets by their closures. Remember, our definition tells that A is small iff its closure is small.

We shall manufacture a sequence of intervals $[a_n, b_n]$ such that the following hold.

(i) $[a_n, b_n] \cap C_n = \emptyset$.

(ii) $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots$.

(iii) $0 < b_n - a_n \leq 1/2^n$.

By (ii) and (iii) and Cantor intersection theorem there is again z common to all these intervals and (i) says that this point is outside all the sets C_n. Thus the union $\cup C_n$ is not all of R. Here is how you construct these intervals.

C_1^c is open. Since C_1 does not contain any interval, you can pick an non-degenerate interval $[a_1, b_1] \subset C_1^c$. If necessary cut it down to satisfy condition of length.

C_2 being small, $(a_1, b_1) \subset C_2$ is false. So $(a_1, b_1) \cap C_2^c \neq \emptyset$ and being open, contains a non-degenerate subinterval. Take closed non-degenerate subinterval of this. Cut down to fulfil length condition to get $[a_2, b_2]$. Next time manufacture $[a_3, b_3] \subset (a_2, b_2) \cap C_3^c$ and proceed.

This completes the proof.

There are several books dealing with metric spaces, for example the book of A. N. Kolmogorov and S. V. Fomin: Introduction to real analysis. You can look at the Hewitt and Stromberg that I mentioned earlier.