Home Assignment:

Given a compact set K contained in an open set U, **show** an $r > 0$ such that
$$\bigcup_{x \in K} B(x, r) \subset U.$$

Solution : Suppose not. then for every n there are points $x_n \in K$ and $y_n \in U^c$ such that $d(x_n, y_n) < 1/n$. Since K is compact there is a subsequence of (x_n) which converges to a point, say x of K. Let us take (x_n) to be the subsequence itself. If you do not like then use (x_{n_k}) below instead of (x_n). But then
$$d(x, y_n) \leq d(x, x_n) + d(x_n . y_n) \to 0.$$
In other words, $y_n \to x$ and U^c being closed we conslude that $x \in U^c$. But $x \in K$ and $K \subset U$. This contradiction proves the result.

OR: For each $x \in K$ pick $r_x > 0$ so that $B(x, 2r_x) \subset U$. Now consider the balls
$$\{B(x, r_x) : x \in K\}$$
covers K and hence finitely many of them cover K. Say the balls around $x_1, x_2, \cdots x_k$. Take
$$r = \min\{r_{x_1}, r_{x_2}, \cdots, r_{x_k}\}$$
We say this will do. To argue our claim, take $x \in K$ and take $y \in B(x, r)$. We need to show $y \in U$. But y is in one of the selected finitely many balls, say, ball around x_i. Thus
$$d(x, x_i) < r_{x_i}$$
But then
$$d(y, x_i) \leq d(y, x) + d(x, x_i) \leq r + r_{x_i} \leq 2r_{x_i}$$
and we know ball of radius $2r_{x_i}$ around x_i is contained in U. Thus $y \in U$.

OR: Consider the function
$$f(x) = d(x, U^c).$$
We know it is a continuous function. Since K is compact it has a minimum on K and the minimum is attained. Note that since $K \subset U$ we see that

$f(x) > 0$ for each point $x \in U$. Thus it is non-zero at points of K. Thus this minimum of f on K must be $r > 0$. This will do.

K is compact and C is closed and $K \cap C = \emptyset$ then **show** $d(K, C) > 0$.

If $U = C^c$ then U is open; hypothesis implies that $K \subset U$ and previous exercises completes the proof. Then $d(K, C) \geq r > 0$ where r is as obtained above.

Is the above statement true if both sets are given to be closed.
Consider the two sets contained in R:

$$A = \{1, \quad 2, \quad 3, \quad 4, \quad 5 \cdots\}$$

$$B = \{2 + \frac{1}{2}, \quad 3 + \frac{1}{3}, \quad 4 + \frac{1}{4}, \quad 5 + \frac{1}{5} \cdots\}$$

Of course, you could think of high school pictures too. You must have drawn hyperbola many times and also learnt the word *asymptote*. Here are subsets of R^2.

$$A = \{(x, y) : x > 0, y > 0, xy = 1.\}$$

$$B = \{(x, y) : y = 0\}$$

C_n is decreasing non-empty closed sets. **show** $\cap C_n$ is non-empty.
Solution: Take a point $x_n \in C_n$. Since all these are in C_1, a compact set it has a convergent subsequence, say, converging to x. Since the sequence is contained in C_1 we see $x \in C_1$. After the first term all terms of the subsequence are in C_2 and hence $x \in C_2$. In general after the k-th term all terms of the sequence and subsequence are in C_k and hence $x \in C_k$.

OR: If the intersection is empty, then every point of C_1 is outside some C_n. In other words

$$C_2^c, \quad C_3^c, \quad C_4^c, \cdots$$

cover C_1 and hence finitely many of them cover. Since C_i are decreasing, the complements are increasing. Thus one of these covers C_1. Say $C_j^c \supset C_1$, or $C_j \subset C_1^c$. But $C_j \subset C_1$. The only possibility is that $C_j = \emptyset$, contradiction.

Is this true if all the sets are closed, non-empty decreasing.
Take $[n, \infty)$ subsets of R.

is this true if all the sets are closed non-empty decreasing but one is compact. Yes, because if C_k is compact, consider the sets only after the k-th stage. Observe that closed subset of a compact set is compact. Thus all sets after the k-th stage are compact.

How do we show that a closed subset C of a compact space K is compact? Several ways. If you are given a collection of open sets which cover C, include C^c and say now K is covered; take a finite sub cover and delete C^c from this (if it is here). The remaining finitely many cover C. Alternatively, take a sequence from C, since it is also a sequence from K take a converging subsequence, but C being closed the limit of the subsequence must be in C already.

In a compact space a family of closed sets is given. Every finitely many of them have a point in common. **show** all the sets have a point in common.

If not, the grand intersection is empty; equivalently their complements (which are open) cover the space; get finitely many of them which cover and get contradiction (?).

If a metric space satisfies the above condition then **show** the space is compact. This is precisely definition of compactness if you take complements. Think.

If every bounded real valued continuous function attains its supremum, then **show** every such function attains its infimum too.

Take bounded real f. Take $g(x) = -f(x)$. then g is also bounded continuous function, use hypothesis about sup for g and observe that gives inf for f.

When every real bounded continuous function attains its bounds then **show** the space is compact.

First we show that the space is complete. Denote the space by X. Take a Cauchy sequence (x_n) Consider the function $f(x) = \lim d(x, x_n)$. The limit exists because the sequence of real numbers $\{d(x, x_n)\}$ is Cauchy. This is because

$$|d(x, x_n) - d(x, x_m)| \le d(x_n, x_m) \to 0 \quad m, n \to \infty.$$

f is continuous because $|f(x) - f(y)| \le d(x, y)$. Unfortunately f may not be bounded. So take $g(x) = \min\{f(x), 1\}$. Thus g is bounded, continuous, non-negative. Its infimum is zero. Given $0 < \epsilon < 1$ (get N so that for $d(x_m, x_n) < \epsilon$ for all $n, m > N$. Observe $f(x_N) = \lim d(x_N, x_n) \le \epsilon$ and hence $g(x) = f(x) \le \epsilon$. Thus g being non-negative we conclude zero is its infimum. This is attained at say x_0. Clearly then $\lim d(x, x_n) = 0$ or $x_n \to x$.

fix $\epsilon > 0$. Shall show we can cover X by finitely many balls of radius ϵ. if not you can get a sequence $\{x_n\}$ such that distance between any two points is at least ϵ. Let

$$A_n = \{x_n, \quad x_{n+1}, \quad x_{n+2}, \quad \cdots\}.$$

$$f_n(x) = d(x, A_n); \quad g_n(x) = \min\{f(x), 1/2^n\}; \quad n = 1, 2, 3, \cdots.$$

Then each f_n and g_n are continuous functions, so is $g(x) = \sum g_n(x)$ (why?). Note $g(x) > 0$ at all points. Since all A_n are closed subsets we see that g_1 itself is positive at all points outside A_1. And $g_k(x_n) > 0$ for each $k > n$. Thus g is positive at points of A_1 too. Since $g_k(x_n) = 0$ for $k = 1, 2, \cdots, n$ we see

$$g(x_n) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^n}.$$

In other words infimum of g is zero but is not attained at any point.

OR: Instead of separately proving two parts as above you can start by saying suppose there is a sequence without any convergent subsequence. repeat the second step above.

If every real continuous function is bounded, **show** that the space is compact.

Repeat same argument above, take $1/g$.

If you carefully observe, we have repeated our old proofs on the real line. You can first complete the space and also argue as I explain din class. Think about it.

back to Arzela-Ascoli:

Consider $C[0, 1]$ with sup metric. Let K be a compact subset. Let us see some of its properties.

(i) K is closed.

We knew it, but let us recall. If a is a limit point of K then every open ball $B(a, 1/n)$ and hence closed ball $\overline{B}(a, 1/n)$ contains points of K. Since $a \notin K$, complements of these closed balls cover K and can not have finitely many of them covering K.

Or, you can also say take any sequence in K, then it contains a subsequence converging to a point of K, so if the sequence itself converges to a then $a \in K$. Thus K is closed.

Since $[0,1]$ is a compact subset of R, every continuous function on it is bounded. That is, given $x \in C[0,1]$ there is a number M such that $|x(t)| \leq M$ for all t. Even if you take finitely many continuous functions then there is one bound for all of them. In general, if you take infinitely many functions then there may not be a common bound for all these functions. But if you have a compact family then there is a common bound.

(ii) There is a number M such that $|x(t)| \leq M$ for all $x \in K$ and all $0 \leq t \leq 1$.

Indeed we know that the function

$$\varphi(x) = d(x, 0) = \max |x(t)|$$

is a continuous function on $C[0,1]$, in particular, on K. Thus K being compact this function must be bounded on K, that is, there is a number M such that $\varphi(x) \leq M$ for all $x \in K$. This is what we wanted.

Since $[0,1]$ is a compact subset of R, every continuous function on it is uniformly continuous. In other words, let $x \in C[0,1]$. Given $\epsilon > 0$, there is a number $\delta > 0$ such that $|x(t) - x(s)| < \epsilon$ for all t, s with $|t - s| < \delta$. Even if you take finitely many continuous functions then there is one $\delta > 0$ which works for all of them. In general, if you take infinitely many functions then there may not be a common $\delta > 0$ which works for all them. But if you have a compact family then there is one $\delta > 0$ that works for all of them..

(iii) Given $\epsilon > 0$, there is a $\delta > 0$ such that $|x(t) - x(s)| < \epsilon$ whenever $x \in K$ and $|s - t| < \delta$. (Of course, through out it is assumed that s and t are in $[0,1]$).

Proof is simple. Fix $\epsilon > 0$. Remember, K being compact, there are finitely many balls of radius $\epsilon/3$ that cover K; say

$$B(x_i, \epsilon/3); \quad 1 \leq i \leq n.$$

Fix $\delta > 0$ such that

$$|s - t| < \delta \Rightarrow (\forall i;\ 1 \leq i \leq n)|x_i(s) - x_i(t)| < \epsilon/3.$$

This does. For, given now any $x \in K$ and any s, t with $|s - t| < \delta$; let us pick i so that $x \in B(x_i, \epsilon/3)$. remember, this means $|x(u) - x_i(u)| < \epsilon/3$ for all u. Thus

$$|x(s) - x(t)| \leq |x(s) - x_i(s)| + |x_i(s) - x_i(t)| + |x_i(t) - x(t)| \leq 3 \times \epsilon/3 = \epsilon.$$

Arzela-Ascoli Theorem Let $K \subset C[0,1]$. Then K is compact iff the followng three conditions hold.

(i) K is closed.

(ii) There is a number M such that $|x(t)| \leq M$ for all $x \in K$ and all $0 \leq t \leq 1$.

(iii) Given $\epsilon > 0$, there is a $\delta > 0$ such that $|x(t) - x(s)| < \epsilon$ whenever $x \in K$ and $|s - t| < \delta$. (Of course, through out it is assumed that s and t are in $[0,1]$).

Moreover, if conditions (ii) and (iii) hold for a set K, then they hold for its closure and hence its closure is compact.

Last sentence is obvious. Indeed if a sequence of functions converges in the space, then they converge point-wise too. Thus the same bound works and the same δ works.

We have already observed that if K is compact, then the three conditions hold. Let us now assume that K satisfies the three conditions and show that it is compact. So take a sequence (x_n) in K. We need to exhibit a subsequence converging to a point of K. Of course if we just exhibit a converging subsequence then it must converge to a point of K because of condition (i).

Since the space $C[0,1]$ is complete, it is enough to exhibit a subsequence which is Cauchy. This is achieved in two steps.

(A) there is a subsequence

$$x_{n_1}, \quad x_{n_2}, \quad x_{n_3}, \cdots\cdots$$

such that for every rational number $r \in [0,1]$ the sequence of numbers

$$x_{n_1}(r), \quad x_{n_2}(r), \quad x_{n_3}(r), \cdots\cdots$$

converges.

(B) the sequence in (A) is Cauchy.

Let us prove (B) first. To avoid ugly notation, let us rename $y_1 = x_{n_1}$, $y_2 = x_{n_2}$, and so on. Thus (y_n) is a sequence of elements in K and for every rational number r, the sequence of numbers $\{y_n(r)\}$ converges. Let us show (y_n) is Cauchy.

Fix $\epsilon > 0$. Use condition (iii) with $\epsilon/3$ and get a $\delta > 0$. Take partition

$$0 = r_0 < r_1 < r_2 < \cdots < r_k = 1$$

where each r_i is rational and $r_{i+1} - r_i < \delta$. Such a partition can easily be got. For example, you can, by taking smaller δ if necessary, assume your δ is rational. Take now multiples of $\delta/2$ and stop when you go out of $[0, 1]$ taking the last point to be one.

Now choose N such that

$$n, m \geq N \Rightarrow (\forall i \; 1 \leq i \leq k) \; |y_n(r_i) - y_m(r_i)| < \epsilon/3.$$

This is possible because each of the sequences $(y_n(r_i) : n \geq 1)$ is convergent and hence Cauchy. We claim this N will do.

To see this, fix any $n, m > N$ and any $t \in [0, 1]$. we need to show that $|y_n(t) - y_m(t)| < \epsilon$. Let us say, $r_i \leq t \leq r_{i+1}$

$$|y_n(t) - y_m(t)| \leq |y_n(t) - y_n(r_i)| + |y_n(r_i) - y_m(r_i)| + |y_m(r_i) - y_m(t)|$$

$$\leq 3 \times \epsilon/3 = \epsilon.$$

The first and last terms are smaller than $\epsilon/3$ by choice of δ and the middle term by choice of N.

This proves (B)

Let us now prove (A). This is routine 'diagonal argument' we have done once long ago. Let us do it again. Again to avoid ugly notation let us reformulate our problem as follows.

(AA) For each $i = 1, 2, 3, \cdots$ we have a bounded sequence of reals, $\{x^i(n) : n \geq 1\}$. The claim is that there is one subsequence of integers $\{n_1 < n_2 < n_3 < \cdots\}$ so that for each i the subsequence $\{x^i(n_1), x^i(n_2), x^i(n_3), \cdots\}$ converges.

How does this show (A)? We are given a sequence of functions (x_n) from K. So for each rational r we can evaluate our functions at this point to obtain a sequence of numbers $\{x_1(r), x_2(r), x_3(r), \cdots\}$. This sequence is bounded by condition (ii) of the theorem. Thus we have countably many sequences as in the above para. the only difference is that in the earlier para I have one sequence for each i, here we have one sequence for each r. But it does not matter because the set of rationals is countable and we can enumerate it as one sequence. Thus (A) is a consequence of (AA)

We shall now proceed to prove (AA). The idea is simple but execution needs vocabulary. Here is the idea. List your sequences as follows.

$$\begin{array}{c} x^1(1),\ x^1(2),\ x^1(3),\ x^1(4),\ \cdots \\ x^2(1),\ x^2(2),\ x^2(3),\ x^2(4),\ \cdots \\ x^3(1),\ x^3(2),\ x^3(3),\ x^3(4),\ \cdots \\ \vdots \end{array}$$

Since the first row is bounded, select a subsequence which converges. In other words, put cross marks on some terms so that if you read the sequence along the cross marks then it converges. Now read the second row only along the cross marks. Since it is bounded, there is a subsequence which converges. In other words if you make some cross marks into double crosses, then the second row read along the double crosses converges. Read the third row only along the double crosses and make some of these into triple crosses so that when you read third row along the triple crosses it converges. Continue. The required subsequence is the first crossed place, second double crossed place, third triple crossed place etc. This does. Too much english!

Here is the execution. Choose

$$n_{11} < n_{12} < n_{13} < \cdots \tag{1}$$

so that the sequence

$$x^1(n_{11}), \quad x^1(n_{12}), \quad x^1(n_{13}), \quad \cdots$$

converges. Possible because we started with a bounded sequence. Now consider the sequence of numbers

$$x^2(n_{11}), \quad x^2(n_{12}), \quad x^2(n_{13}), \quad \cdots$$

and take a subsequence that converges. That is,

$$n_{21} < n_{22} < n_{23} < \cdots \tag{2}$$

from among (1) so that

$$x^2(n_{21}), \quad x^2(n_{22}), \quad x^2(n_{23}), \quad \cdots$$

converges. Now consider

$$x^3(n_{21}), \quad x^3(n_{22}), \quad x^3(n_{23}), \quad \cdots$$

and take a subsequence that converges. That is

$$n_{31} < n_{32} < n_{33} < \cdots \tag{3}$$

from among (2) so that

$$x^3(n_{31}), \quad x^3(n_{32}), \quad x^3(n_{33}), \quad \cdots.$$

In general choose

$$n_{k1} < n_{k2} < n_{k3} < \cdots \tag{k}$$

from among

$$n_{k-1,1} < n_{k-1,2} < n_{k-1,3} < \cdots \tag{$k-1$}$$

so that

$$x^k(n_{k1}), \quad x^k(n_{k2}), \quad x^k(n_{k3}), \cdots$$

converges. Put

$$n_1 = n_{11}, \quad n_2 = n_{22}, \quad n_3 = n_{33}, \quad \cdots \tag{$*$}.$$

We claim that for each i,

$$x^i(n_1), \quad x^i(n_2), \quad x^i(n_3), \quad \cdots$$

converges. We first argue that $n_k > n_{k-1}$. This is because the k-th selection is a subset of $(k-1)$-th selection. In other words, at the worst we might have

$$n_{k1} = n_{k-1,1}, \quad n_{k2} = n_{k-1,2}, \quad \cdots n_{k,k-1} = n_{k-1,k-1}$$

showing $n_k = n_{kk} > n_{k-1}$. Thus we have a strictly increasing sequence of numbers (n_k).

Since (*) is a subsequence of (1) we see that the first sequence converges along this subsequence. Since (*) is a subsequence of (2) except possibly for the first term, we see second sequence converges along the subsequence.

Since (*) is a subsequence of (3) except possibly for the first two terms we see that the third sequence converges along this subsequence. In general (*) is a subsequence of (k) except possibly for the first $(k-1)$ terms so that the k-th sequence converges along this subsequence.

This completes proof of (AA) and thus of (A) and thus of the theorem

There are just two points that need too be mentioned.

Firstly, I used a phrase like 'the first sequence converges along this subsequence'. we have never precisely defined what it means. I hope it is clear, but here it is. suppose we have a sequence $a = (a_n)$ of real numbers and a strictly increasing sequence of integers $S = (m_i)$. We say that the sequence a converges along the subsequence S, if the sequence $\{a_{m_1},\ \ a_{m_2},\ \ a_{m_3},\ \ \cdots\}$ converges.

Second point is the following. we did some construction by induction. I cautioned once that when ever you do such a construction, you should precisely write down what you are going to do and then carry out. Carry out first step and assuming that you constructed up to $(k-1)$-th step, explain how you would do next step. do not say 'like this' 'like that' 'so on' etc. I may also add that subtle point is that when you *before hand* list properties, they should be interpretable inductively. if you are not clear you should refer too an earlier notes where this was discussed.

I carried out an inductive construction without listing before hand what I am proposing to do. That was done in order not to interrupt the thought process set in motion by the motivation. Here is the claim. We construct for each k a strictly increasing sequence

$$S_k = \{n_{k1}, \quad n_{k2}, \quad n_{k3}, \quad \cdots\}$$

of natural numbers so that (i) elements of S_k are among S_{k-1} and (ii) the k-th sequence (x^k) converges along S_k.