Holder, Minkowski:

Let us first show that the d_p example of last time does indeed satisfy the triangle inequality. In what follows we have two numbers $p, q > 0$ such that

$$\frac{1}{p} + \frac{1}{q} = 1. \qquad (\spadesuit)$$

Of course this already implies $p > 1$ and $q > 1$ and given any $p > 1$ there is exactly one q as above.

We start with the simple observation. Let $a > 0$ and $b > 0$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Since ($\spadesuit$) tells us that the right side is convex combination of a^p and b^q we are advised to look for a convex function and use the definition of convexity.

Take numbers x and y such that

$$\exp\{x/p\} = a; \quad \exp\{y/q\} = b$$

so that

$$e^x = a^p; \quad e^y = b^q$$

Now convexity of exponential function completes the proof.

$$ab = \exp\left\{\frac{1}{p}x + \frac{1}{q}y\right\} \leq \frac{1}{p}e^x + \frac{1}{q}e^y = \frac{a^p}{p} + \frac{b^q}{q}.$$

This simple inequality leads **Holder**:

Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers. Then

$$|\sum a_i b_i| \leq \left(\sum |a_i|^p\right)^{1/p} \left(\sum |b_i|^q\right)^{1/q}.$$

If any one of the right side quantities is zero then the corresponding a's or b's are zero and hence left side is also zero and so both sides are zero.

So let us assume that the two quantities on right side are non-zero, say, c and d respectively. Take $\alpha_i = |a_i|/c$ and $\beta_i = |b_i|/d$ to see

$$\sum \frac{|a_i b_i|}{cd} \leq \sum \frac{|a_i|^p}{c^p} + \sum \frac{|b_i|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

gives the stated inequality since $|\sum a_i b_i| \le \sum |a_i b_i|$.

Holder leads to **Minkowski**:

$$(\sum |a_i + b_i|^p)^{1/p} \le (\sum |a_i|^p)^{1/p} + (\sum |b_i|^p)^{1/p}.$$

To prove this, assume that a's and b's are positive.

$$\sum (a_i + b_i)^p = \sum a_i (a_i + b_i)^{p-1} + \sum b_i (a_i + b_i)^{p-1}$$

use Holder note $(p-1)q = p$,

$$\sum (a_i + b_i)^p \le (\sum a_i^p)^{1/p} \left[\sum (a_i + b_i)^p\right]^{1/q}$$

$$+ (\sum b_i^p)^{1/p} \left[\sum (a_i + b_i)^p\right]^{1/q}$$

Bring the common factor of right side to the left side to complete the proof (if that common factor is zero to start with, nothing need to be proved).

If a_i, b_i not necessarily positive replace by their modulus etc.

This leads to the **triangle inequality**:

$$d_p(x, z) \le d_p(x, y) + d_p(y, z).$$

where

$$d_p(\alpha, \beta) = (\sum |\alpha_i - \beta_i|^p)^{1/p}$$

You only need to use Minkowski with $a_i = x_i - y_i$ and $b_i = y_i - z_i$.

limit points:

Next couple of lectures, the agenda is to imitate whatever we did in R and R^n, namely to define convergence, open sets etc. In what follows (X, d) is a metric space. Sequences are sequences in X and subsets are subsets of X.

Definition: A sequence x_n converges to a point x iff $d(x_n, x) \to 0$, equivalently, given $\epsilon > 0$, there is N such that $d(x_n, x) < \epsilon$ for all $n > N$.

A point x is a limit point of the sequence (x_n) if there are terms of the sequence that 'keep coming close' to x. That is, given $\epsilon > 0$ and N, there is $n > N$ such that $d(x_n, x) < \epsilon$. Equivalently, any ball around x contains x_n for infinitely many values of n.

a point x is a limit point of a set if for every $\epsilon > 0$, the set $A \cap B(x, \epsilon)$ is infinite. That is, every ball around x contains infinitely many points of A.

You should be careful not to confuse sequences and sets. For example in R, the sequence

$$1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \cdots\cdots\cdot$$

has limit points 1,2,3. However if you consider the terms of the sequence, it is just $\{1, 2, 3, \}$ and this set has no limit point at all.

We can say x is a limit point of A if every ball around x contains at least one point of A other than x. Of course, if x is a limit point then the definition says that in fact there are infinitely many points of A in any ball around x.

Conversely, given that there is at least one point different from x we can show that there are actually infinitely many points of A as follows. Let $\epsilon > 0$ be given. Pick x_1 in this ball different from x. Note that $x_1 \neq x$ tells $d(x_1, x) > 0$, take the ball of radius $d(x, x_1)/2$ around x and pick one point x_2 in this ball different from x. Pick x_3 different from x in the ball of radius $d(x, x_2)/2$ around x. continue to see infinitely many points of A in the given ball.

We can also say x is a limit point of A iff there is a sequence of distinct points in A converging to x. That is, there is a sequence (x_n), each x_n is in A; $x_n \neq x_m$ for $n \neq m$ and $x_n \to x$. If this happens then the points are distinct tells that each $B(x, \epsilon) \cap A$ is infinite. Conversely, if x is limit point of A, then you can choose inductively for $n \geq 1$ a point $x_n \in B(x, 1/n) \cap A$ different from previous points. This will do.

We can also say x is a limit point of A iff there is a sequence (x_n) of points in A such that $x_n \neq x$ for each n and $x_n \to x$. If x is a limit point, then the previous para gives you a sequence (x_n) of distinct points in A converging to x. Clearly, at most one of them could be x, remove it.

Conversely, if there is a sequence as described above, set $y_1 = x_1$. since all the points of the sequence are different from x, $d(x_1, x) > 0$. Take a point x_{n_1} of the sequence in the ball of radius $d(x, x_1)/2$ around x. Obviously, this is different from x_1, Name it y_2. Now take the ball of radius $d(x, y_2)/2$ and pick x_{n_2} here with $n_2 > n_1$ and continue. This gives a sequence of distinct points from A converging to x.

open sets, closed sets:

A subset U is open if $x \in U$ implies there is $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. that is, whenever there is a point in the set, a ball around that point is contained in U. A subset C is closed if its complement is open.

If you specialize to real line, as already seen earlier, open ball of radius ϵ centred at a is just the interval $(a - \epsilon, a + \epsilon)$. Thus a set is open if: whenever there is a point in the set, a small interval around that point is contained in the set. This is precisely the definition we adapted last year.

Similarly, when $X = R^2$ and $d_2(x, y)$ is the Euclidean distance, then ball of radius $\epsilon > 0$ centered at a point $a = (a_1, a_2)$, is usual geometric disk and thus again this above definition coincides with what we adapted in case of R^2 as well. The above definition coincides with the definition we adapted last year in R^n.

Returning to general metric spaces, we can say C is closed iff it contains all its limit points. To see this let C be closed, need to show no point outside C can be a limit point of C. But C being closed, C^c is open, so if you take a point $x \in C^c$, there is a ball around x contained in C^c and this ball does not contain any point of C so that x can not be a limit point of C.

Conversely, let C contain all it limit points. Let $x \in C^c$. So it is not a limit point of C. So there is a ball around it which does not contain any point of C, except possibly x. But x anyway is not in C. So this ball has no point from C, in other words, every point of C^c has a ball around it contained in C^c. Thus C^c is open and hence C is closed.

Connected spaces:

In a metric space (X, d) the sets X and $\emptyset$ are always open and they are also closed. Are there any other sets which are both open and closed? There may not be.

For example, take $X = R$ and $d(x, y) = |x - y|$, usual metric. Then there is no other set which is both open and closed. Indeed, suppose there is such a set A. Since $A \neq \emptyset$, pick $x \in A$; since $A \neq R$, pick $y \in A^c$. Assume $x < y$. Let $s = \sup\{z \in A : z < y\}$. This sup is sensible because the set is not empty — x is in it; and bounded above — y is a bound. If $s \in A$ then, $y \notin A$ tells $s < y$. Now A is open tells there is a small interval $(s - \epsilon, s + \epsilon) \subset A$ contradicting s is sup of the set. If $s \in A^c$, then A^c open says an interval as earlier is contained in A^c. Now, s being sup of our set nothing above s is in

A and this interval tells us that actually nothing above $s - \epsilon$ is in A again contradicting s is sup of the set.

Similar argument applies if $y < x$.

On the other hand if we take $X = R$ with the metric $d(x, y)$ to be zero or one according as $x = y$ or not, you see that every singleton set is open (ball of radius 1/2 around that point is contained in it!). This shows that every subset is open. As a consequence every subset is both open and closed.

Definition: A metric space (X, d) is connected if the only subsets which are both open and closed are $\emptyset$ and X. Otherwise, the space is said to be disconnected.

The plane R^2 with usual metric is connected. In fact proceed as above, pick x and y and concentrate on the line joining x and y and arrive at a contradiction. Same argument shows that R^n is connected too. We shall return to connected sets later again.

complete spaces:

One concept we want to imitate is that of a Cauchy sequence. A sequence (x_n) in a metric space (X, d) is Cauchy sequence if given any $\epsilon > 0$, there is an N such that $d(x_n, x_m) < \epsilon$ for every $m, n > N$. A metric space is complete if very Cauchy sequence converges.

Thus a Cauchy sequence is a sequence of points which are coming closer and closer. After some stage distance between any two points is at most one; after a later stage distance between any two points is at most 1/2 and so on. Thus completeness means that any sequence of points which are coming closer and closer are *actually* coming closer to a point. Every Cauchy sequence is heading somewhere (not falling out of a hole!).

This property of completeness is very important, it helps us to discover points in the space which may not be visible to the naked eye. For example in the real line we found that the sequence of points

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}; \quad n = 1, 2, 3, 4 \cdots$$

is a Cauchy sequence and hence converges. it was not any point we knew.

So we named it e. similarly the sequence of points

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n; \quad n=1,2,3,4\cdots$$

is a Cauchy sequence, and its limit is a number we did not see earlier and named it γ (Euler's constant).

Now you have a panorama of metric spaces, you can discover points in the spaces, that you are not able to see outright. For example we have the space $C[0,1]$ with sup metric. You can discover that there are elements in this space which are nowhere differentiable; that is, there are nowhere differentiable continuous functions. In fact you will see later that such functions are far more than differentiable functions.

Example: $X=R$, $d(x,y)=|x-y|$ is a complete metric space.
We knew this last year.

Example: $X=(0,1]$, $d(x,y)=|x-y|$. the space is not complete. Indeed $(1/n)$ is a Cauchy sequence that does not converge to any point in X. Let us consider on the same space the following metric

$$d_1(x,y)=|x-y|+\left|\frac{1}{x}-\frac{1}{y}\right|$$

First of all note that, if $x_n \to x$ in this new metric d_1, then in particular, $|x_n-x| \to 0$ so that $x_n \to x$ in the metric d. Conversely, if $x_n \to x$ in the old metric d, then everything being in the space X, we conclude that $(1/x_n) \to (1/x)$ so that $x_n \to x$ in d_1.

Thus the notion of convergence is same under both metrics, this implies (easy to see) that closed sets are same in both and hence open sets are also same in both.

Interestingly enough the space X is complete with metric d_1. In fact if (x_n) is d_1-Cauchy then observe that both

$$|x_n-x_m| \quad \text{and} \quad \left|\frac{1}{x_n}-\frac{1}{x_m}\right|$$

converge to zero as n and m become large. In particular, the first one tells that $x_n \to x$ in R, and the second one tells that this x can not be zero (then $1/x_n$ becomes unbounded). Thus $x \neq 0$ and hence $1/x_n \to 1/x$ and hence

$d_1(x_n, x) \to 0$. In other words every sequence which is cauchy in d_1 does converge in d_1.

Thus the space (X, d_1) is complete where as (X, d) is not; though both have the same notion of convergence and same closed sets and same open sets. What happened is that non-convergent sequences which are Cauchy in d have been destroyed in d_1, they are no longer Cauchy and hence not obliged to converge.

Here is another example.

$$X = (0, 1); \qquad d(x, y) = |x - y|;$$

$$d_1(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{1-x} - \frac{1}{1-y}\right|$$

Both d and d_1 give the same notion of convergence, closed sets etc. But (X, d) is not complete whereas (X, d_1) is complete. Sequences like $(1/n)$ or $\{n/(n+1)\}$ which are Cauchy in d but not converging are no linger Cauchy in d_1 and hence not obliged to converge.

This discussion should tell you that completeness is something that depends heavily on the metric. This is not surprising because, the notion of Cauchy sequence depends on the metric. However, the concept of convergence does not heavily depend on the metric; it depends on the collection of open sets. Of course, at this stage you might say: so what? open sets depend on the metric. In a later course on topology you will see the distinction.

If a metric space is not complete, is there anything that we can do to make it complete. Of course, one thing we can do is to destroy Cauchy sequences which are not converging as happened in the examples above and make the space complete.

Suppose that there are too many Cauchy sequences and we can not destroy all of them. Or imagine that we do not want to change the metric. Can we do anything to make the space complete? In such a case the only possibility is to attach new points to the space and declare them to be limits of Cauchy sequences (which had no limits at present). Remember, the space is not complete due to shortage of points in the set, there are Cauchy sequences but no points to which they converge.

Yes, Cantor discovered a method of completing a metric space. I could give you the general method and then say you can specialise this process to the set of rational numbers to obtain another construction of the real number system. However this will lead to some confusion due to a technical problem. So we shall first execute cantor's idea to construct R.

Cantor's construction of R:

Let us once again pretend that we do not have real numbers. However we do have the set of rationals Q before us. We are constructing R, following Cantor.

Well, why is Q not a model of real number system? It satisfies all the axioms except the least upper bound axiom. There are sets which are bounded above but have no supremum. In Dedikind's construction, we attached points to Q so that such sets have supremum.

Following our earlier observations, we could also have said that Q is not a model of real number system because there are Cauchy sequences which are not converging. We shall now enlarge the space so that such sequences converge. Again remember the general philosophy: if certain phenomena are not taking place, consider all such phenomena.

To make the above statement concrete and bring right perspective, it is worth recalling some of our past actions. If we could not subtract 4 from 3 we considered pair $(3, 4)$ as a point of our space and, in a sense, this pair represented $3 - 4$. Since we felt $3 - 4$ should be same as $7 - 8$ we identified the two pairs $(3, 4)$ and $(7, 8)$. Considered the space of equivalence classes.

If we could not divide 5 by 7 we considered the pair $(5, 7)$ and this represented $5/7$. We felt that this should be same as $10/14$ and so identified the two pairs $(5, 7)$ and $(10, 14)$. We considered the space of equivalence classes.

If the set S of all rational numbers whose square is less than 2 has no supremum, then we considered this set itself as a 'point' and declared this point as supremum of S.

[Let me add a comment in passing which you may ignore. What we actually did was a little different, we considered not just the above set S but actually the set T which consists of all negative rational numbers along with the set of positive rational numbers whose square is less than 2. This set T gives a Dedikind cut but S does not give you a cut. If you want to know why we did this, the answer is simple: Situation was hopeless and we brought some order. If you understood the above three paras, and if you are on imitating spree, then your tendency would be to consider *all* subsets of Q which are bounded above. You will identify two sets if they *appear* to have same sup and consider the collection of equivalence classes. Yes, but luckily enough, it so happens that each equivalence class has *exactly one cut* in it. Thus though I was looking at the equivalence class, I showed you only the cut from that class and *managed* matters. This helped you to consider *one set* rather than *equivalence class of subsets of* Q. Think about it.]

Returning to our present problem, you would have no hesitation considering all Cauchy sequences. Let R_0 be the collection of Cauchy sequences $x = (x_n)$ of rational numbers. We identify

$$x = (x_n) \sim y = (y_n) \leftrightarrow d(x_n, y_n) \to 0.$$

Triangle inequality helps to show that this is an equivalence relation.

Let R be the space of equivalence classes. This is our real number system. equivalence class containing a sequence x is denoted $[x]$.

We define addition by taking $[x]+[y] = [x+y]$. Thus to define sum of two equivalence classes $[x]$ and $[y]$, take one sequence from each class and add (term by term) those two sequences and the class containing this resulting sequence is the sum. This is a good definition. Firstly, the sum of two Cauchy sequences is cauchy. Secondly, it does not depend on which sequence we choose from the equivalence class. Indeed, let $a \in [x]$ and $b \in y]$ then we show $a + b \in [x + y]$ as follows.

$$d(a_n + b_n, x_n + y_n) \leq d(a_n, x_n) + d(b_n, y_n) \to 0.$$

The rules $[x] + [y] = [y] + [x]$ as well as $[x] + ([y] + [z]) = ([x] + [y]) + [z]$ are easy being consequences of similar rules regarding rational numbers.

The zero element is $[\theta]$ where θ is the sequence having zero for all its terms. It is clear $[x] + [\theta] = [x]$. The inverse is also clear, namely, for a given element $[x]$ we take $[-x]$ where $-x$ is the sequence $(-x_n)$.

We shall denote the additive identity by $[0]$.

We define multiplication also in the obvious way. $[x][y] = [xy]$ where $x.y$ is the term wise multiplication of the sequences. To see that this is a good definition, we need to show that xy is a Cauchy sequence and if someone takes different elements $a \in [x]$ and $b \in [y]$ then $ab \in [xy]$.

First recall that a Cauchy sequence is bounded. Indeed, if (x_n) is Cauchy, pick N so that $|x_m - x_n| < 1$ for $m, n \geq N$. In other words for $n \geq N$ we have

$$|x_n| \leq |x_N| + |x_n - x_N| \leq |x_N| + 1.$$

Thus maximum of the finitely many quantities,

$$|x_1|, |x_2|, \cdots |x_{N-1}|, |x_N| + 1$$

will serve asa bound.

To see $(x_n y_n)$ is Cauchy, first pick some bound M for the two Cauchy sequences x and y.

$$|x_n y_n - x_m y_m| \leq |x_n||y_n - y_m| + y_m|x_n - x_m|$$

$$\leq M|y_n - y_m| + M|x_n - x_m|.$$

Given rational $r > 0$ we can choose N such that $|x_m - x_n| < r$ and $|y_m - y_n| < r$ for all $m, n > N$. which can be made small for all large m, n.

Finally, let $(a_n) \sim (x_n)$ and $(b_n) \sim (y_n)$. Shall show $(a_n b_n) \sim (x_n y_n)$. Fix a bound M for all these four Cauchy sequences.

$$|a_n b_n - x_n y_n| \leq |a_n||b_n - y_n| + |y_n||a_n - x_n|$$

$$\leq M|b_n - y_n| + M|a_n - x_n|$$

which can be made small.

The rules $[x][y] = [y][x]$ and $([x][y])[z] = [x]([y][z])$ re easy. The element $[1]$, the equivalence class containing the constant sequence one is the identity element. $[1][x] = [x]$ is clear from definition.

Let $[x] \neq [0]$. We shall show inverse. We start with an observation. since $[x] \neq 0$ we conclude that $|x_n| \not\to 0$. So there is an $r > 0$ such that for infinitely many n we have $|x_n| > r$. since (x_n) is cauchy, there is N such that $|x_n - x_m| < r/2$ for $n, m > N$. If you take any $n > N$ then $|x_n| > r/2$. Indeed, let $n > N$ and pick $m > N$ so that $|x_m| > r$.

$$|x_n| \geq |x_m| - |x_n - x_m| = r - \frac{r}{2} = \frac{r}{2}.$$

Thus for all $n > N$ we have $|x_n| > s$ where $s = r/2 > 0$.

Let (y_n) be the sequence $y_n = 1$ for $n \leq N$ and $y_n = 1/x_n$ for $n > N$. Since $x_n y_n = 1$ for all $n > N$, we see that $[x][y] = [1]$. But we need to convince ourselves that y is a Cauchy sequence. This is immediate because for $m, n > N$

$$|y_m - y_n| = \frac{x_m - x_n|}{|x_m||x_n|} \leq |x_m - x_n|/s^2.$$

which can be made small.

This completes the proof that $[y] \in R$ and is the inverse of $[x]$.

The distributivity $[x]([y] + [z]) = [x][y] + [x][z]$. is immediate.

S o far it is smooth, We now need to define order on R. The simple minded definition $[x] \leq [y]$ if $x_n \leq y_n$ is not meaningful, because this order does not respect the equivalence classes. For example,

$$(2, 1, 1, 1, , 1 \cdots) \quad (0, 1, 1, 1, 1, 1 \cdots)$$

are in the same class. In fact if you change only finitely many terms of a Cauchy sequence then the resulting sequence is equivalent to the original sequence. Thus we should not make the definition depend on all suffixes.

Suppose we modify to say that $[x] < [y]$ if $x_n < y_n$ for all large values of n, then again we run into problems. If $x_n = 1/n$ and $y_n = 2/n$ then $x_n < y_n$ holds for all values of n but they represent the same element $[0]$.

Thus defining order is a little delicate. In stead of defining $\leq$ we start defining $<$. Say $[x] < [y]$ if there is $r > 0$ and N such that $x_n + r \leq y_n$ for all $n > N$. We say $[x] \leq [y]$ iff $[x] = [y]$ or $[x] < [y]$. This definition satisfies our requirements.

Let us start showing that this definition respects the equivalence classes. Let $x \sim a$ and $y \sim b$. Suppose that there is $r > 0$ and N such that $x_n + r \leq y_n$ for all $n > N$. We exhibit $s > 0$ and M such that $a_n + s \leq b_n$ for all $n > M$. Take $s = r/2$. Take $M > N$ so that

$$n > M \Rightarrow |x_n - a_n| < r/4; \quad |y_n - b_n| < r/4.$$

If now $n > M$ then

$$a_n + \frac{r}{2} \leq x_n + \frac{r}{4} + \frac{r}{2} \leq y_n - r + \frac{r}{4} + \frac{r}{2} \leq b_n + \frac{r}{4} - r + \frac{r}{4} + \frac{r}{2}.$$

We need to show that it is a linear order and is friendly with addition and multiplication.