**Cantor intersection theorem:**

In the real line we have shown that a decreasing sequence of closed non-empty intervals with diameter converging to zero have a point in common. We shall now generalize this result to metric spaces. But before doing so, let us understand that the intervals should be as stated in the theorem. A sequence of sets which are decreasing are called nested.

The intervals $\{[n, \infty) : n \geq 1\}$ are closed and nested. But they have no point in common because their diameter is not converging to zero.

The intervals $\{(0,\ 1/n); n \geq 1\}$ have diameter decreasing to zero, they are nested but have no point in common because the sets are not closed.

The intervals $\{[n,\ n+1/n]; n \geq 1\}$ are closed and have diameter decreasing to zero, but they have no point in common because they are not nested.

Theoem: Let $(X, d)$ be a complete metric space, let

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

be nested sequence of closed setts with diameter converging to zero. Then they have exactly one point in common.

The essential thing is that there is a common point.

If there are two different points $x$, $y$ in common then the diameter of each of the sets must be at least $d(x, y) > 0$ and hence can not go to zero.

If we take one point $x_n \in C_n$, then the sequence $\{x_n\}$ is Cauchy. Indeed $\{x_n : n \geq N\} \subset C_N$ so that as soon as the diameter of $C_N$ is smaller than $\epsilon$, we can conclude that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$. By completeness of $X$ this sequence must converge to a point $x$. Since each $C_N$ is closed and $x$ is also limit of the sequence $(x_n : n \geq N)$ which is contained in $C_N$ we see that $x \in C_N$. This is true for every $N$.

Thus the intersection is exactly one point.

Actually the above property characterises completeness. More precisely, let $(X, d)$ be a metric space. Suppose every nested sequence of closed sets with diameter converging to zero have a common point. Then the space is complete. This is seen as follows. Let $(x_n)$ be a Cauchy sequence. Set $C_n = \overline{\{x_i : i \geq n\}}$. Here the overline denotes closure. This is a nested sequence of closed sets. The Cauchy property of the given sequence shows that their diameter is converging to zero. Thus there is a point common to all, say, $x$. We claim $x_n \to x$.

**Baire's theorem:**

Let $(X, d)$ be a metric space. A closed set $C \subset X$ is small if it does not contain any non-empty open set. In other words, it has no interior points. More generally, we say that a set $A \subset X$ is small if its closure does not contain any non-empty open set. Here then is an extremely powerful theorem.

Theorem (Baire Category theorem):
A complete metric space can not be written as a countable union of small sets.

In other words, if $C_1, C_2, \cdots$ is a sequence of closed sets each of which has no interior point, then there is a point of $X$ which is not in any one of these sets. Of course, there will be, not one but plenty of, points outside all the sets $C_n$, as we will see.

Proof of this theorem is very simple, but the theorem is itself very powerful. It will help you to see objects which you can not see with ordinary eyes! Let us see two applications before proving this theorem. You will learn later many many applications.

**nowhere differentiable functions:**

This application is due to Banach. Consider the set $X = C[0, 1]$, space of real valued continuous functions on the interval $[0, 1]$. We show that there are plenty of functions which do not have finite derivative at any point whatso-ever. In fact, the set of such functions is much larger than differentiable functions.

We know that $X$ is a complete space. For integers $n, m > 1$ consider the following set.

$$E_{n,m} =$$

$$\left\{x \in C[0,1] : \left(\exists t; 0 \leq t \leq 1 - \frac{1}{n}\right) \left(\forall h; 0 < h < \frac{1}{n}\right) \left|\frac{x(t+h) - x(t)}{h}\right| \leq m\right\}.$$

Observe that if a function is differentiable at some point $t < 1$, then it is in one of the above sets for some $n, m > 1$. Indeed let $x$ be a function having a finite derivative at a point $t_0 < 1$. Get $n$ so that $t_0 < 1 - 1/n$. Consider the fhe function $\varphi(h) = [x(t_0 + h) - x(t_0)]/h$ on $[0, 1/n]$. Of course at $h = 0$, the value is its limit which is the finite right derivative of $x$ at $t_0$. This is continuous and hence bounded, say, by $m$. Then clearly $x \in E_{n,m}$ for this $n, m$.

The set $E_{n,m}$ is closed. To see this first recall that if $x_n \to x$, then $x_n$ converges uniformly to $x$. If $t_n \to t$ then $x_n(t_n) \to x(t)$. This is because, given $\epsilon > 0$, we can fix $N$ so that $d(x_N, x) < \epsilon/2$ for $n \geq N$. Since $x$ is continuous, we fix $N_1$ so that $|x(t_n) - x(t)| < \epsilon/2$ for $n \geq N_1$. if now $n \geq N \vee N_1$, then

$$|x_n(t_n) - x(t)| \leq |x_n(t_n) - x(t_n)| + |x(t_n) - x(t)| < \epsilon.$$

Now to see that $E_{n,m}$ is closed, take $x_i \in E_{n,m}$ with $x_i \to x$. get $t_i$ for $x_i$. If necessary take subsequence and assume that $t_i \to t^* \in [0, 1/n]$. Now take any $h$ with $0 < h < 1/n$. Then the facts $t_i \to t^*$ and $t_i + h \to t^* + h$ combined with the observation of the para above show that $x$ also satisfies the required inequality for difference quotients.

Finally we show that the set $E_{n,m}$ does not contain any non-empty open set. This will then show that each $E_{n,m}$ is a small set. We start with an observation.

Let $P$ be a polynomial and $\epsilon > 0$, then we claim that there is a function $x$ such that $d(x, P) < \epsilon$ and $x \notin E_{n,m}$.

How does this help us? If $E_{n,m}$ contains a non-empty open set, then by Weierstrass theorem this open set must contain a polynomial and hence a ball around this polynomial is contained in the open set. In other words, if you assume that interior of $E_{n,m}$ is non-empty, then there is a polynomial $P$ and $\epsilon > 0$ such that $B(P, \epsilon) \subset E_{n,m}$. But the observation refutes precisely such a statement, it tells that there is a $x$ in this ball which is not in $E_{n,m}$.

To prove the observation stated above, fix a bound $c$ for the derivative of $P$ on $[0, 1]$. Consider the following function $z$ on $[0, 1]$; it is made up of straight line segments; it starts with $z(0) = 0$; increases with slope $s = c + 2m$ till it reaches $\epsilon/2$; then decreases with slope $-s$ till it reaches $-\epsilon/2$; then increases with slope $s$ till it reaches $\epsilon/2$ and then decreases etc; all this continues till you reach $t = 1$ and then stops. Convince yourself that you do reach $t = 1$ after finite number of these ups and downs. You then see that this defines a continuous function on $[0, 1]$.

Let $x = P + z$. Then clearly $d(P, x) = \sup |z(t)| < \epsilon$. Now take any $t < 1$. Take any $h > 0$ so that $t + h$ is also before the next corner of $z$. Thus you have plenty of $h > 0$ at your disposal. Then

$$\frac{x(t+h) - x(t)}{h} = \frac{P(t+h) - P(t)}{h} + \frac{z(t+h) - z(t)}{h}$$

By mean value theorem the first term on right side is between $-c$ and $+c$. By construction the second term is either $c + 2m$ or $-c - 2m$. Keep in mind that $t, t + h$ are in the same line segment of $z$. As a result the difference quotient for $x$ is at least $2m > m$.

This shows that each $E_{n,m}$ is small. Similarly, the following sets are also small.

$$F_{n,m} =$$

$$\left\{x \in C[0,1] : \left(\exists t; \frac{1}{n} \leq t \leq 1\right) \left(\forall h; -\frac{1}{n} < h < 0\right) \left|\frac{x(t+h) - x(t)}{h}\right| \leq m\right\}.$$

Thus we have countably many small sets $E_{n,m}$ and $F_{n,m}$ for $n, m \geq 1$. If a function has a derivative at any point of $[0, 1]$ then it must be in one of these sets. actually we can be more precise as follows. If a function has a finite right derivative at any point of $[0, 1)$ then it must be in one of the $E_{n,m}$. If a function has a finite left derivative at any point of $(0, 1]$ then it must be in one of the $F_{n,m}$.

Baire's theorem tells that there are functions outside all these sets. If we take such a function then it can not have a finite right derivative or finite left derivative at any point what-so-ever in $[0, 1]$. In particular it is not differentiable at any point.

Just to make you understand the right and left derivatives, let us consider the function $x(t) = |t - 1/2|$. Then this is not differentiable at the point $t_0 = 1/2$. Convince yourself of this. However this has left derivative equal to $-1$; and right derivative equal to $+1$ at the point $t_0 = 1/2$. Thus the functions whose existence we asserted above can not even be like this. Think about it.

Before going to next application of Baire's theorem, let us remember a little history about the hero of this application: Banach.

There was a time when hell descended on earth; in the form of second world war. Poland suffered very heavily. As far as Maths is concerned here is a brief view.

Some could emigrate early on: Alfred Tarski (logician), Antony Zygmund (analyst), Jerzy Neyman (statistician), Samuel Eilenberg (topologist), Stanislaw Ulam (set theory, computation etc) and several others.

Some stayed on and survived the war, either by going underground or showing that they have pure blood, whatever it may mean: W. Sieprinski, K. Kuratowski, H. Steinhaus and others.

Some committed suicide: F. Hausdorff.

Some were put to death in camps: S. Saks, J. Marcinkiewicz, J. Schauder, A. Rajchman, A. Lindenbaum and many many many others.

Some were saved by Director of lab (in Lwow, a city in Poland) that makes Typhus vaccine. Since Germans needed it, they allowed him to choose volunteers. This vaccine needs lice. Growing lice is done by carefully packing them and attaching to the calf or thigh of human so that they suck the blood. These volunteers are called lice-feeders. Our hero Banach was one such. He survived the war but died soon after due to failed health (and lung cancer).

**$R$ as union of closed sets:**

Our second application of Baire's theorem is to show the following which answers a question that we raised earlier.

Theorem: $R$ can not be expressed as union of of infinitely many nonempty disjoint closed sets.

We already knew that there is no subset of $R$ which is open and closed except $\emptyset$ and $R$ itself. Thus we can not express $R$ as union of two disjoint nonempty closed sets. This in turn implies that we can not express $R = \cup_1^k C_i$ where $C_i$ are nonempty disjoint closed sets and $k > 1$. If this could be done then, you can simply take $A = C_1$ and $B = \cup_2^k C_i$ to see $R$ is union of two nonempty disjoint closed sets.

Thus $R$ can not be expressed as a finite union of more than one nonempty disjoint closed sets. The theorem says you can not express $R$ even as countably infinite union of nonempty closed sets.

Let, if possible

$$R = C_1 \cup C_2 \cup C_3 \cup \cdots\cdots .$$

Let the interior of $C_i$ be denoted by $J_i$. Denote

$$U = J_1 \cup J_2 \cup J_3 \cup \cdots\cdots .$$

Since none of the $C_i$ can be open we see each $J_i$ is a proper subset of $C_i$, possibly empty. This $U$ is an open set and its complement, denoted by $H$ is therefore a non-empty closed set. Thus $H$ is a complete metric space in its own right, metric is same $d(x, y) = |x - y|$ for $x, y \in U$.

$$H = (C_1 \cap H) \cup (C_2 \cap H) \cup (C_3 \cap H) \cup \cdots\cdots$$

$$= (C_1 - J_1) \cup (C_2 - J_2) \cup (C_3 - J_3) \cup \cdots\cdots .$$

The plan of the proof now is the following. We shall now forget $R$ for a minute and concentrate on the complete metric space $H$. We show that the sets on the right side above are small in $H$. This contradicts Baire's theorem for the complete metric space $H$.

Let us start with an observation. If an interval $(a, b)$ contains a point, say $x$, of $C_1 \cap H$, then it contains points from other $C_i \cap H$ $(i \neq 1)$ as well. This is easy. If the entire interval $(a, b)$ is contained in $C_1$ then it would have been removed as part of $J_1$, So this interval must have points from, say, $C_5$. Let us say $y \in C_5 \cap (a, b)$. Either $y < x$ or $y > x$ because $C_i$ are disjoint. Suppose $y < x$; similar argument applies in the other case. Consider all points in $[y, x]$ which are in $C_5$ and take its sup, name it $z$. This is sensible because the set we are considering includes $y$ and is hence non-empty; moreover it is bounded by $x$. Also $C_5$ being closed we conclude that $z \in C_5$. In particular $z < x$ and points in between these are not in $C_5$. in other words there is no interval around $z$ contained in $C_5$ which means $z \in C_5 - J_5$ and thus $z \in C_5 \cap H$ and $z \in (a, b)$ as claimed.

To complete executing our plan, first notice that each $C_i \cap H$ is a non-empty closed subset of $H$. That it is non-empty is already noted earlier. It is closed in $H$ because whenever a sequence of points from here converge to a limit then the convergence is 'usual convergence' and hence the limit is in both the closed sets $C_i$ as well as $H$.

finally suppose $C_i \cap H$ contains a non-empty set open in $H$. That is, there is a point $x \in C_i \cap H$ and $\epsilon > 0$ such that $\{y \in H : d(x, y) < \epsilon\} \subset C_i \cap H$. In other words $(x - \epsilon, x + \epsilon) \cap H \subset C_i \cap H$. But this is not possible as observed above. This completes the proof that each $C_i \cap H$ is a small set in the complete metric space $H$.

And completes proof of the theorem.

Just to impress upon you the phrase 'small in $H$', let us consider $R$ and $Z$ the set of integers. Each $\{n\}$ is small in $R$. However none of these singleton sets are small in $Z$. In fact each of these are both closed and open in $Z$.

Thus smallness depends on the background. In the background of $R$, each $\{n\}$ is small. In the background of $Z$, each $\{n\}$ is not small. You should understand this point.

**Proof of Baire:**

This is exactly same as the one for real line, there is no new idea. The execution is made possible by Cantor intersection theorem for complete metric spaces.

Before you get the wrong impression that Baire just imitated the real line proof, let me say the following. Even for the real line it is due to Baire. It is not that the theorem existed for $R$ and he extended it to metric spaces. We discussed real line case first only to understand the argument in a familiar territory, so that the general case would pose no problem.

So let $(X, d)$ be a complete metric space and let $C_1, C_2, C_3, \cdots$ be small closed sets. We exhibit a point of $X$ which is not in any of the $C_i$.

In what follows a ball means ball of positive radius. Every open ball contains a closed ball with same center (say, any strictly smaller radius) and every closed ball contains an open ball with same centre (same radius). In what follows we ask you to choose an open ball inside a closed ball or we ask you to take a closed ball inside an open ball. Then you should select with the same center as mentioned above. This is our agreement.

Take any open ball $B_1$ of your choice. We promise to get open balls

$(B_i : i \geq 1)$, closed balls $(F_i : i \geq 1)$ such that the following holds.

$$B_1 \supset F_1 \supset B_2 \supset F_2 \supset B_3 \supset F_3 \supset \cdots \supset B_{n-1} \supset F_{n-1} \supset \cdots \qquad (*)$$

$$F_i \subset C_i^c; \quad i = 1, 2, 3, 4, \cdots \qquad (**)$$

$$\text{diameter}(F_i) \leq 1/2^i. \qquad (***)$$

Let us see what happens then. Condition $(*)$ says that the sets $F_i$ are nested closed sets; condition $(***)$ with Cantor intersection theorem then gives a point $x$ common to all $F_n$; condition $(**)$ says that this point is not in any of the sets $C_i$. as promised. Actually condition $(*)$ tells that this point is in the open set $B_i$ you gave.

Here is how we construct the sets. Since $C_1$ is small surely $B_1 \cap C_1^c \neq \emptyset$ and is open. So take a closed ball

$$F_1 \subset B_1 \cap C_1^c.$$

If you have possibility of choosing large ball, restrain, choose ball of radius at most $1/2$.

Take open ball $B_2 \subset F_1$ as per our agreement above. Since $C_2$ is small $B_2 \cap C_2^c \neq \emptyset$. So select closed ball

$$F_2 \subset B_2 \cap C_2^c.$$

Again make sure diameter of $F_2$ is at most $1/2^2$. Take open ball $B_3 \subset F_2$ as per our agreement. Then select a closed ball

$$F_3 \subset B_3 \cap C_3^c.$$

Make sure its diameter is at most $1/2^3$.

Here is the inductive step. Suppose we got the balls $(B_i)$ and $(F_i)$ for $i = 1, 2, \cdots, n-1$ as satisfying the three conditions. here is how we construct $B_n$ and $F_n$. Of course $B_n$ is the open ball contained in $F_n$ as per our agreement earlier. Since $C_n$ is small take closed ball

$$F_n \subset B_n \cap C_n^c.$$

If you have a possibility of choosing large $F_n$, cut it down to have diameter smaller than $1/2^n$.

This complete the construction and proof of the theorem.

I would like to stress once again two points which I made earlier. First is this. After getting the first three sets you could say, 'continue like this'. But when you write a proof, you must make sure that such a continuation 'for ever' is indeed possible. So it is important for you to show this.

Secondly, you see, what we want is just that diameter$(F_n) \to 0$ to get a point common to all of them; it does not matter how it converges to zero, it does not need to be smaller than $1/2^n$. Thus in stating my conditions suppose I carelessly stated $(* * *)$ as: diameter $(F_n) \to 0$. You will *not be able to construct* sets by induction simply because it does not make sense to say that we have constructed sets up to $n$ satisfying the three conditions.

Thus whenever you need to make an *unending construction* you must be able to write the conditions in a way that they make sense inductively; an you should be able to explain that having done the construction up to a stage it can be continued to the next stage.

Please pay attention and do think about it.

**completion of a metric space:**

Having seen how important it is to have a complete space, the natural question is the following. If the metric space is not complete, is there anything we can do complete it?

Why is the space not complete. There are Cauchy sequences which are not converging. So either we should make sure such sequences are not Cauchy or provide a point (of convergence) for each such sequence. The first alternative works if we can provide suitable metric without changing the notion of convergence. This is possible if the space is already an open subset of a complete metric space. This is possible even if the space is a set which is countable intersection of open sets in a complete space. It stops there and does not work for any metric space.

Besides, in the procedure described above, there are two problems. firstly, I said if your space is an open subset of a complete space you can change the metric. But how do you recognize that there is indeed a bigger space which is indeed complete and our space is indeed an open subset of it? Second point is that one may not like to change the distance. After all, if a particular metric is natural then changing it, just to achieve some other desired property, makes the new metric artificial and devoid of meaning. such a thing should be avoided.

So we look for the second alternative. This is what Cantor did. We mentioned this issue when we described Cantor's construction of real numbers. Exactly the same procedure works, not only for the set of rational numbers, but for any space.

This is what we do now.

So let $(X, d)$ be a metric space. Consider the space $X^1$ of all cauchy sequences in $X$. Thus each element of $X^1$ is a Cauchy sequence $(x_n)$.

Define $(x_n) \sim (y_n)$ if $d(x_n, y_n) \to 0$. This is an equivalence relation on $X^1$. Denote the space of equivalence classes by $X^*$. Thus elements of $X^*$ are bags. Each bag contains Cauchy sequences which are equivalent. If there is one Cauchy sequence $(x_n)$ in a bag then every Cauchy sequence $(y_n)$ equivalent to $(x_n)$ is also in that bag — and nothing else is there. Elments of $X^*$ are denoted $[x]$.

Let us observe that if $(x_n)$ and $(y_n)$ are Cauchy sequences in $X$ then the limit $\lim d(x_n, y_n)$ exists. Indeed, to show this, it is enough to show that the sequence of real numbers $\{d(x_n, y_n)\}$ is a Cauchy sequence of real numbers. But this is easy

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m).$$

and hence can be made as small as we please for all large values of $m$ and $n$. You only need to see that both terms on the right side can be made small.

Further if $(a_n) \sim (x_n)$ and $(b_n) \sim (y_n)$ then $\lim d(a_n, b_n) = \lim d(x_n, y_n)$. This is again because

$$|d(a_n, b_n) - d(x_n, y_n)| \leq d(a_n, x_n) + d(b_n, y_n) \to 0.$$

Let us define for $[x]$ and $[y]$ in $X^*$

$$d^*([x], [y]) = \lim d(x_n, y_n).$$

The above analysis shows that this limit exists and is well-defined on $X^*$.

We show that $d^*$ is a metric on $X^*$. Clearly $d^*([x], [x]) = 0$. Also $d^*([x], [y]) = 0$ implies, by definition of the equivalence relation, that $[x] = [y]$. Symmetry, $d^*([x], [y]) = d^*([y], [x])$ is also clear. the triangle inequality of $d$ leads to the triangle inequality for $d^*$.

We shall now identify $X$ as a subset of $X^*$. For $p \in X$, let $\varphi(p)$ be the constant sequence $x_n = p$ for all $n$, more precisely, $\varphi(p)$ is the bag containing this sequence. Then this map is one-one. This preserves distance too. That is $d(p,q) = d^*([p],[q])$.

We shall show that $X$ is a dense subset of $X^*$; $(X^*, d^*)$ is a complete metric space. Thus $X$ is *enlarged* to a set $X^*$ and the metric is also *extended* to $d^*$ to make the space complete.

By showing that $X$ is dense, we are saying that for every point of $z \in X^*$ there is a Cauchy sequence $(x_n) \subset X$ which converges to it. In other words, in this process of completion we have not added unnecessary points, every new point $z$ we aded, being limit of a Cauchy sequence $(x_n)$ in $X$.

This leads to the feeling that the completion is unique. Yes, this is so. We have not yet defined what is completion. We shall do and then prove that completion is indeed unique.

There are some books that I have consulted from time to time for nice problems apart from the books I already mentioned. I am giving below. But you should feel free (and also make it a habit) to consult any book from the library. It is not enough to be able to understand what I say; it is very important to be able to understand others too.

You should develop the habit of reading material and internalizing it. That is, *do not* classify it as easy or difficult (you are not here to judge, though you can do it); do not reproduce it word-to-word (this is not a memorizing contest, though you can do it); *but* understand and think about it till you are able to explain to others in your own words.


An introduction to complex analysis and geometry
John P D'Angelo.

$p$-adic analysis compared with real
Katok Svetlana.

A primer of real functions
Ralph P Boas.