where are we:

So far we learnt some basic properties of real numbers,

convergence of sequences (helped to 'approximate' unknown numbers with known fractions; discovered new numbers like Euler's constant, convergence helped us to make precise statement like 'how does this number look like for large values of n' as in Stirling formula for $n!$ or Walli's product for $\pi/2$),

convergence of series (helped us to add infinitely many numbers which in turn helped us to make sense of power series — natural extension of the high school concept of polynomial — gives us a way to discover functions),

continuity and differentiation of functions from R to R (a class of nice functions whose graph can hopefully be drawn, functions which can be understood better, which can be used in our descriptions of physical phenomena and so on),

integration of functions (helps in calculating areas; later will also be used to discover functions by knowing certain equations obeyed by their derivatives; helps also to discuss convergence of series; allows us to define new functions and so on),

and then we learnt functions of several variables.

This last topic will be further continued in Calculus III to develop various useful stories. In our course we shall discuss Analysis.

Basically we discuss: sets; numbers; sets of numbers. At the end, if we have time, we shall discuss some Fourier series. Actually, questions concerning Fourier series lead to the (discovery and) development of set theory by Cantor.

But then, did we not discuss already sets of numbers — compact sets, Cantor set and so on? Yes. But there was a gap in our understanding. We did not discuss what sets are and their properties.

We did not know if there are real numbers at all — we only postulated that there is a set with certain operations (addition and multiplication); worked with such a set after making a mental picture of it as a line. Even if there is such a system, we have not proved that there is only one such system. All these issues need to be settled.

Things to think about:

Before we get into the subject, a word of advise is in order.The first thing you should develop is respect for the words you use. Many a times I see that you are using technical words and I get the impression that you do not know its meaning. Probably you know meanings, but are careless. Or probably lack discipline. It is for you to think.

You should try to figure out negation of sentences. For example, not many could write what is meant by f is not continuous. There is no maths in this, really speaking. For example consider,

Every student in this class has a red pen.

You seem to think its negation is the following.

Every student in the class has a green pen.

Every student has no red pen.

Student in other class have red pen.

and so on.

You are under the correct impression that matters are simple, but under the wrong impression that you need not think.

Consider the sentence: $x = 5$. Its negation is simple. $x < 5$ or $x > 5$. This sentence has no 'quantifier'. When you have phrase 'every student' or 'there exists a student' you are using quantifier. Negation of statements involving quantifiers are tricky unless you practice. If there are two quantifiers you need to think a little more. Try negating: for every student there is a problem in this set which is difficult

Many times you seem to think that explanation can be taken as proof. For instance consider the following statement.

BVR does not know Tamil because he is from Andhra Pradesh.

This is of the form: 'S because T', where S and T are statements. You should realize that T is only a plausible explanation for the statement S, but not its proof. In daily use we do accept such explanations as proof, but not in maths. After all, there are persons who satisfy statement T and know Tamil too. Thus T does not allow you to conclude S. In other words, knowing T is true does not mean S is true.

It is possible to define precisely what is meant by proof. Basically, if you make a sequence of sentences and tell me it is proof then (i) each sentence in that sequence must follow from earlier sentences or must be a hypothesis and (ii) the last sentence is the one you are proving. Think about it. I do not want to convert this into Logic class. Moreover knowing such a definition will not help you. The only thing that helps is practice. So practice. Do not try to convert exam into practice session.

Sets:

A set is any collection of objects. If S is a set and x is an object in this set then we write $x \in S$. If x is not an object in this set we write $x \notin S$.

This definition is good enough for us. The reason for making this statement is that if you literally take the above definition you will end up in problems.

You must have heard about the barber paradox. Consider only males. A barber in a town declared that he shaved those and only those who have not shaved themselves. Decide whether the barber shaved himself or not. Thus if you consider the set S of all persons in the town whom the barber shaved, you are unable to decide whether the barber himself is a member of this set or not. Thus even though S is defined as a collection of objects, we are unable to decide if a specific object (namely, the barber) is in this set or not.

So sometimes you see a definition like: a set is a well defined collection of objects. But then, what is meant by well defined? You have to start from somewhere. Thus for us a set is a collection of objects.

There are other such paradoxes too. Let k be the least natural number that can not be described in less than hundred letters. Does this make sense?

Or, Consider the collection S of all sets. This is clearly a collection of objects. Is this a set? If so it should belong to itself. Do you believe it?

Set operations:

If S and T are two sets, then they are same and we write $S = T$; if they have the same objects. In other words $x \in S$ implies $x \in T$ and $x \in T$ implies $\in S$.

We say S is a subset of T and write $S \subset T$; if $x \in S$ implies $x \in T$. Thus every object which is in S is also in T. We also say that T is a superset of S and write $T \supset S$.

If S and T are two sets then their union $S \cup T$ is the collection of those objects which are either in S or in T. Similarly their intersection $S \cap T$ consists of all objects which are in both S and T.

$$S \cup T = \{x : x \in S \text{ or } x \in T\}; \quad S \cap T = \{x : x \in S \text{ and } x \in T\}.$$

We define $S - T = \{x : x \in S; \quad x \notin T\}$, that is, it consists of all objects which belong to S but which do not belong to T.

In case we have a grand set Ω under consideration and if we are only considering objects that belong to Ω, then for $\Omega - S$ we write S^c and is called complement of S. Since objects which are in Ω are under discourse, it is called universe. Sometimes it is also called universal set, but there is nothing universal about it; the word only means that it is the universe now and all objects we are talking about are in this set (whether we say so or not).

Thus, unlike union and intersection, complement must be with reference to something. It may be specified or understood from the context.

Theorem (DeMorgan's laws):

$$(A \cup B)^c = A^c \cap B^c; \quad (A \cap B)^c = A^c \cup B^c.$$

Proof:

$x \in (A \cup B)^c$
$\Rightarrow x \notin (A \cup B)$ (def. complement)
$\Rightarrow x \notin A$; and $x \notin B$ (def union)
$\Rightarrow x \in A^c$ and $x \in B^c$ (def complement)
$\Rightarrow x \in A^c \cap B^c$ (def intersection).
These arrows can be reversed.
The other equality can be proved similarly.

Many times we use the word 'similar'. You should not blindly copy it like a faithful elementary school student. This phrase is an abbreviation for the following. 'I have done the proof once, it is similar to the earlier part and so I omit writing it (probably to spare the reader from getting bored).' As a result you should verify and earn the eligibility to use this phrase.

A set which has no objects is called empty set (that such a thing exists is an axiom). It is denoted by $\emptyset$.

We use numerals: $0, 1, 2, 3, \cdots$ We can precisely define them as follows, you need not keep this abstract way in mind. You can think of them as you have been doing all along.

$$0 = \emptyset; \quad 1 = \{\emptyset\}; \quad 2 = \{\emptyset, \{\emptyset\}\}; 3 = \{\emptyset, \ \{\emptyset\}, \ \{\emptyset, \{\emptyset\}\}\}$$

In general k is the set consisting of all the previous sets. Thus everything is a set. There is nothing before zero. So zero is empty set. Before one there is zero. So one consists of just the empty set.

The purpose of all this is just to convince that everything can be defined using only one symbol: empty set. (Every thing grew from empty set!)

N stands for the set $\{1, 2, 3, \cdots\}$.

Let A and B be two sets. Their Cartesian Cartesian product $A \times B$ is defined as the set of all ordered pairs, with first element of the pair being an object from A and the second element being an object from B.

$$A \times B = \{(x, y) : x \in A; y \in B\}.$$

Of course I have used the word ordered pair (x, y) and you might wonder what the hell it is. If you are wondering, here it is

$$(x, y) = \{x, \{x, y\}\}$$

It is a set consisting of two objects. First object is x. Second object is the set consisting of x and y. Of course you can keep your mental picture in tact and ignore this definition. This is definition of ordered pair is only to reinforce the feeling that ordered pair can be defined precisely. (everything is a set!)

functions and cardinality:

Let X and Y be sets. A function on X to Y is a rule that associates with every element of X one element of Y.

You can keep this in mind and there is nothing wrong with this definition. However you may suddenly get a doubt: what is this 'rule' and 'association' business? What is rule and what is not rule? What is association and what is not association?

If you get such a doubt, there is no need to worry. Here is something that helps. A function is a subset $R \subset X \times Y$ such that

$$\forall x \exists ! y \ \ (x, y) \in R.$$

Here $\exists !$ is an abbreviation for 'there is a unique'. Thus for each $x \in X$ there is exactly one y such that $(x, y) \in R$. If you want to denote your function by f then for a given $x \in X$, this unique point y is denoted by $f(x)$, value of the function f at the point x. Thus R associates with every x one y.

If you go back in time, you realize that a function f is a rule as said earlier and 'graph' of the function f is the set of all points (x, y) such that $y = f(x)$. Now if you want to be rigorous and not use unnecessary words like rule/association etc, then you turn things around. *By definition* the graph is regarded as function. Think about it.

But let me assure you that you should keep in mind the same good old idea of function. What all I said is an explanation, in case you feel you are cheated.

I should say two things. Firstly, as already said, the symbol $\exists ! y$ is an abbreviation for there exists unique y. Thus

$$\forall x \exists ! y \ \ (x, y) \in R$$

means

$$\forall x [\{\exists y \ \ (x, y) \in R\} \quad \wedge \quad \{(x, y_1) \in R, (x, y_2) \in R \Rightarrow y_1 = y_2\}].$$

Second point is the following. Sometimes you see a different (not equivalent to ours) definition of function. It is a subset $R \subset X \times Y$ such that

$$(x, y_1) \in R, (x, y_2) \in R \Rightarrow y_1 = y_2.$$

Thus for any point x there can not be two y. However it is quite possible there is no y at all. Thus for each x there is at most one y such that $(x, y) \in R$. The set of x for which there is such y is called the 'domain' of the function and for x in the domain the unique y is thought of as $f(x)$. In a sense these are 'partial' functions. What we defined are 'total' functions. We stick to our definition.

f is one-to-one, or $1-1$ or injective if two different points have different images. That is

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

f is onto or surjective if image is all of Y. That is

$$\forall y \exists x \ (x, y) \in R.$$

f is bijective if it is both one-one and onto.

Since you have seen several examples of functions, there is no need for me to give examples here in a text book style.

Cardinality:

The number of elements or cardinality of empty set is 0; $|\emptyset| = 0$ If X is a non-empty set and there is a bijection from X to $\{1, 2, \cdots, n\}$ then we say that X has n elements; $|X| = n$

For this definition to make sense you should first convince yourself that the following can not happen: X has bijection with $\{1, 2, \cdots, 25\}$ and also X has a bijection with $\{1, 2, \cdots, 33\}$. Because if this happens then we would be saying $|X| = 25$ and also $|X| = 33$! Yes, such a thing as above can not happen. (Why?)

A set which is either empty set or is in bijection with $\{1, 2, \cdots, n\}$ for some $n \in N$ is said to be a finite set. A set which is not finite is said to be infinite.

If X is in bijection with N, we say that X is countably infinite. If X is either finite or countably infinite, we say that X is countable. If X is not countable then we say that X is uncountable.

Several properties of countable sets were discussed last year. We showed that the set of real numbers R is uncountable. You should recall those.

We have only associated 'numbers' with finite sets. There are numbers associated with infinite sets too. If X is countably infinite, we say $|X| = \aleph_0$ (aleph zero). We say $|R| = c$ (c stands for 'continuum').

How do you compare sets? The good old primitive way. Associate with each student the chair he/she is sitting. If some 'unassociated chairs' are leftover after associating chair to each student, then we say there are more chairs than students. If 'unassociated students' are leftover after associating a student to each chair, then we say there are more students than chairs.

Of course, you might think we have made so much progress, we can as well 'count' the number of students and number of chairs and then compare these numbers. But the point is you are using too many unnecessary words in this counting. Moreover this (so called) progress stops with finite sets and is useless while comparing infinite sets.

Let X and Y be any sets, finite or countable or uncountable anything. We say Y has more elements than X (or X has less elements than Y); write $|X| \leq |Y|$; if there is an injection (one-one function) on X into Y. It is a peculiar situation. We have not defined 'number of elements' in X or Y. But we defined what it means to say X has less elements than Y.

Common sense suggests

$$|X| \leq |X|$$

$$|X| \leq |Y|; \quad |Y| \leq |Z| \Rightarrow |X| \leq |Z|$$

$$|X| \leq |Y| \quad |Y| \leq |X| \Rightarrow |X| = |Y|.$$

The first two statements above are easy to prove using composition of functions. The last statement is true but not trivial. It is known as Cantor-Bernstein-Schroder Theorem. We shall prove it soon.

But are there larger and larger sets? Yes, given any set X we can cook up a set Y such that $|X| < |Y$. This means $|X| \leq |Y|$ and $|X| \neq |Y|$. Here is how.

Let X be any set. Then $P(X)$ denotes the collection of all subsets of X.

There is an injection on X into $P(X)$, namely, associate with $x \in X$ the singleton set $\{x\}$ in $P(X)$. Thus $|X| \leq |P(X)|$. However there is no bijection. Indeed, if there is a bijection, say $f : X \to P(X)$ then let us define the following subset of X

$$S = \{x \in X : x \notin f(x).\}$$

Since S is clearly a subset of X, there must be a $s \in X$ such that $f(s) = S$.

Assume $s \in S$. The definition of the set S tells you $s \notin f(s)$. But $f(s) = S$. So $s \notin S$. contradiction to the assumption.

Assume $s \notin S$. The definition of S tells you $s \in f(s)$. But $f(s) = S$. Thus $s \in S$. contradiction to the assumption.

But s must be either in S or in S^c. The above argument says none of these happen. This completes the proof that there is no bijection of X with $P(X)$.

Theorem: $|X| < |P(X)|$.

The above theorem is due to Cantor and the argument is known as Cantor's diagonal argument. If you use a symbol $\aleph$ for $|X|$ then it is customary to denote $|P(X)| = 2^{\aleph}$.

Of course, we should make sure that such notations do not contradict existing notations. For example suppose X is finite and $|X| = 5$. Then the above notation tells us $|P(X)| = 2^5$. But we already have a meaning for 2^5 and it is 32. So we must convince ourselves that indeed $|P(X)| = 32$. Yes, you can do it.

Generally empty set needs special assumption. In the above proof what happens if X is empty set? You can repeat. The only function from X to anything is $R = \emptyset$, simply because whatever be Y, we have $X \times Y = \emptyset$.

Recall $P(X) = \{\emptyset\}$. In other words $P(X)$ has one element (namely, empty set). Obviously, if you take this element as y, there is no x such that $(x, y) \in R$. Handling empty set is a little troublesome. You are encouraged to think (and free to ask) but not encouraged to worry about empty set.

inverse and forward images:

Let $f : X \to Y$. For $B \subset Y$, we define inverse image of B under f as follows.

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

(If you think of function as $R \subset X \times Y$,then this simply means $\{x \in X : \exists y(y \in B \& (x, y) \in R)\}$. Of course, there is no need for you to keep thinking function as subset of product space).

Simalarly, for $A \subset X$ we define its forward image under f by

$$f(A) = \{f(x) : x \in A\}.$$

Here is a useful property of inverse images.

Theorem:

$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
$f^{-1}(B^c) = [f^{-1}(B)]^c$.
$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
In general, $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.

You should be able to prove this.

posets and losets:

Let X be a set (not empty). A relation on X is simply a subset of $X \times X$. These are actually called binary relations, but we need not bother. We call them relations.

We say a relation is a partial order if

$$\forall x \quad (x, x) \in R$$

$$\forall x, y, z \quad (x, y) \in R \;\; (y, z) \in R \Rightarrow (x, z) \in R$$

$$\forall x, y, \quad (x, y) \in R \;\; (y, x) \in R \Rightarrow x = y.$$

First statement is called reflexivity; second is called transitivity; third is calle anti-symmetric. But you need not remember these technical words at this moment. If xRy holds, we say x and y are comparable and write $x \leq y$. Thus with this new notation we have (i) $x \leq x$; (ii) $x \leq y$ and $y \leq z$ implies $x \leq z$ and finally (iii) $x \leq y$ and $y \leq x$ implies $x = y$.

If the following condition also holds then the partial order is called a linear order.

$$\forall x, y \quad x \leq y \;\; \vee \;\; y \leq x.$$

Thus any two elements are comparable.

A set with a partial order is called a partially ordered set or simply a poset. A set with a linear order is called a linearly ordered set or simply a loset.

The set of all subsets of the real line with $A \leq B$ if $A \subset B$ is a poset but not a loset.

The following sets are losets. I give only sets, the order is the usual order coming from real numbers.

(i) $X = R$
(ii) $X = [0, 1]$

(iii) $X = [0, 1] \quad \cup \quad [2, 3]$
(iv) $X = [0, 1) \quad \cup (2, 3]$.
(v) $X = [0, 1) \quad \cup [2, 3]$
(vi) $X = Q$ the set of rationals .
You might wonder why I am giving these examples rather than saying: take anythingsubset of R. Each set has some peculiar property.

For example, this first set has neither a first element (something smaller than every other thing) nor a last element (some thing which is larger than every other thing). The second set has both of these.

In the first two sets given any two different elements, there is one strictly in between The third set does nto have this property. For example between 1 and 2 which are in this set, there is none in between.
In all these three sets every non-empty bounded set has a supremum. But in the fourth set the subset $[0, 1)$ is bounded but has no sup. The fifth set looks similar and it appears as if the subset $[0, 1)$ has no sup, but actually it has sup.

The sixth set is countable and also has interesting properties which we shall discuss next.