R **from** Q:

We are now in the process of constructing real number system using Q, the set of rational numbers. A cut x is a non-empty proper subset of Q such that for any of its elements, everything below is also there and somethings above are also there. R is the collection of all cuts.

We defined an order $x \le y$ if $x \subset y$ and showed that with this definition R is a loset having least upper bound property.

We now define addition on R. Let $x \in R$ and $y \in R$. Then

$$x + y \;=\; \{p + q : p \in x; q \in y\}.$$

This set on right side is non-empty because, x and y are so and hence any $p + q$ with $p \in x$ and $q \in y$ would do. There are also points which are not here. Indeed if $p_1 \notin x$ and $q_1 \notin y$ then for any $p \in x$ we have $p < p_1$ and for any $q \in y$ we have $q < q_1$. Thus $p+q < p_1+q_1$ showing that $p_1+q_1 \notin x+y$.

Finally, if $p \in x$ and $q \in y$ then x being a cut things r a little larger than p are also in x so that things like $r + s > p + q$ are also in $x + y$. Thus $x + y$ as defined above is indeed a cut.

The equality $x + y = y + x$ follows from the fact that the set of $p + q$ is same as the set of $q + p$. Similarly $(x + y) + z = x + (y + z)$ follows simply because the set of numbers $(p+q)+r$ is same as the set of numbers $p+(q+r)$. See how we are using the corresponding rule for Q.

Let

$$0^* = \{p \in Q : p < 0\}.$$

This is clearly a cut. Note that zero itself is not in the set.

We claim this is zero element. Take any $x \in R$. We show $x + 0^* = x$. If we have $p \in x$ and $q \in 0^*$ then $q < 0$ so that $p + q < p \in x$. Since x is a cut we conclude $p + q \in x$. Thus $x + 0^* \subset x$. Conversely, let $p \in x$. Let $r > p$ and $r \in x$. Then $p = r + (p - r) \in x + 0^*$. Thus $x \subset x + 0^*$.

We shall show inverse. Let $x \in R$. Define

$$y = \{-q : q \in x^c; q \text{ not least element of } x^c\}.$$

Here x^c, as usual, is the complement of x, that is, all rational numbers q such that $q \notin x$.

If $p \in x$ and $-q \in y$ then $q \in x^c$ tells $p < q$ so that $p + (-q) < 0$. Thus $x + y \subset 0^*$. Conversely, if we show that every negative rational is in $x + y$ it follows that $x + y = 0^*$ and completes the proof that y is inverse of x. For this it is enough — because $x+y$ is a cut — to exhibit a sequence of elements $p_n - q_n$ in $x+y$ such that every negative rational is smaller than one of these numbers.

We first make a construction. Take any $p \in x$ and $q \in x^c$. Set $p_0 = p$ and $q_0 = q$. Take mid-point $(p+q)/2$ which is a rational number. If this is in x then this is p_1 and $q_1 = q$. If this mid-point is in x^c then it is q_1 and $p_1 = p$. Now take mid-point of p_1 and q_1. continue and convince yourself you can write the inductive step to construct the sequence (p_n, q_n) so that each p_n is in x and each q_n is in y and $q_{n+1} - p_{n+1} = (q_n - p_n)/2$. Thus, by induction, we have

$$q_n - p_n = \frac{q-p}{2^n}.$$

Observe that given any rational number $r > 0$, there is n such that $q_n - p_n < r$. This is because of the following reason. By binomial theorem (for integers) $2^n > n$ so that by Archimedian property there is an n such that

$$\frac{q-p}{r} < n < 2^n; \qquad \text{or } \; q_n - p_n < r.$$

Every positive rational is larger than one of the $q_n - p_n$.

Let s be any rational number $s < 0$. The above argument shows that s is smaller than one of the $p_n - q_n$. If q_n is not the least element of x^c, this already implies that $p_n - q_n \in x + y$ and hence $s \in x + y$. if q_n is the least element of x^c then we can not say that $p_n - q_n \in x + y$. However, we can get n such that $s/2 < p_n - q_n$, by the above argument. Take a rational q_n^* such that $q_n < q_n^* < q_n - s/2$ (remember $-s > 0$). Then, of course $q_n^* \in x^c$ and is not its least element.

$$s = \frac{s}{2} + \frac{s}{2} < (p_n - q_n) + (q_n - q_n^*) = p_n - q_n^* \in x + y$$

completing the proof that $s \in x + y$. Thus $x + y = 0^*$.

We proved that R with addition so defined is a group (abelian). Let us see its relation with order. Let $x < y$ and $z \in R$. Wish to show $x + z < y + z$.

Of course $x \subset y$ tells that if $p \in x$ then $p \in y$ so that for any $q \in z$ we have $p + q \in y + z$ showing that $x + z \subset y + z$, that is, $x + z \leq y + z$. The only question is whether equality can hold here. If so, we would have $x + z = y + z$; add $-z$ to both sides, use associativity (already proved), replace $z + (-z)$ by 0^*, use $x + 0^* = x$ etc to see $x = y$. But this is false.

We need to define multiplication. This is extremely tricky. You can not blindly put, just like addition, $x \cdot y$ to consist of all pq with $p \in x$ and $q \in y$. Note we are using dot for the multiplication we plan to define, but for product of rational numbers we do not use dot. This will never be a cut, simply because all negative numbers below some stage will be in both x and y showing that all sufficiently large positive numbers are in this suggested set.

The root cause of the problem is that multiplication of two negative numbers is positive number. This suggests us to define product of positive numbers first. This is what we do now.

let $x > 0^*$ and $y > 0^*$. We define

$$x \cdot y \quad = \quad \{q \in Q : q \leq 0 \quad \text{or } q = p_1 p_2,\ 0 < p_1 \in x,\ 0 < p_2 \in y\}.$$

Thus all negative rationals are put in this set right away. This is a cut. In fact, this is clearly non-empty. If $r \notin x$ and $s \notin y$ then you can easily argue that $rs \notin x \cdot y$ so that the suggested set is not all of Q. Let $r \in x \cdot y$ and $s < r$ be rational. We need to show $s \in x \cdot y$.

If $s \leq 0$ then by definition $s \in x \cdot y$. So let $s > 0$. Then $r > 0$ too so that there is $p_1 \in x$, $p_2 \in y$ with $0 < p_1$ and $0 < p_2$ and $r = p_1 p_2$. Thus $0 < s/p_2 < r/p_2 = p_1$ and hence $s/p_2 \in x$. Thus

$$s = \frac{s}{p_2} p_2 \in x \cdot y.$$

Finally let $r \in x \cdot y$. Need to show some thing larger than that in $x \cdot y$. If this $r \leq 0$ then any $p_1 p_2$ will do. So let $r > 0$. Then $r = p_1 p_2$. Take a little larger than p_1 in x (remember p_2 is positive) etc to complete the proof.

Thus the suggested set is a cut and hence it is in R. Need to show $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. These follow as in the case of addition using the corresponding properties in Q. Shall now exhibit multiplicative unit.

$$1^* = \{q \in Q.q < 1\}.$$

This is clearly a cut — need to use between any two rationals there is another one etc to show things a little larger.

Take any $x > 0^*$. We show $x \cdot 1^* = x$. All rationals $r \leq 0$ are in both: left side by definition of product and right side by definition of the element 1^*. Need to show both sides have the same positive elements. If you take a positive element $p_1 p_2$ on left side then $0 < p_2 < 1$ implies that this is smaller than p_1 and hence is in x. conversely, if we take $p \in x$, then a little larger, say, $r > p$ is also in x and hence

$$p = r\,\frac{p}{r} \in x \cdot 1^*.$$

We now show inverse. Let $x > 0^*$. Let

$$y = \{q \in Q : q \leq 0\} \bigcup \left\{\frac{1}{q} : q \in x^c, q \text{ not least element of } x^c\right\}$$

All negative rationals are in y. If $q \in x^c$ then anything larger than that is in x^c so that any positive smaller than $1/q$ is also y. If $1/q \in y$ then q being not the least element of x^c take something smaller than that and one divided by that etc; to see some thing larger than $1/q$ is also in y. Thus y is a cut and $y > 0$.

Need to show now that $x \cdot y = 1^*$. All rationals $r \leq 0$ are in both sides. So let us show they have the same positive things too. Let $r \in x \cdot y$. Thus $r = p_1 p_2$ with $0 < p_1 \in x$ and $0 < p_2 \in y$. But definition of y tells $p_2 = 1/q$ with $q \in x^c$. But then $q > p_2$ showing that $p_1 p_2 < 1$ and hence is in 1^*.

Finally we need to show that $0 < r < 1$ implies $r \in x \cdot y$. Repeat the argument given for addition. Argue, using the same construction, that p_n/q_n goes closer and closer to one, that is, anything smaller than one is smaller than one of these. This is easily achieved by observing

$$0 \ < \ 1 - \frac{p_n}{q_n} \ = \ \frac{q_n - p_n}{q_n} \ \leq \ \frac{q_n - p_n}{p}.$$

and you can make $q_n - p_n$ as small as you please, see earlier calculation.

This completes the proof that y is multiplicative inverse of x.

Let us see the relation of order. We only have the following rule to be verified: $x > 0$ and $y > 0$ implies $x \cdot y > 0$. Hence we can proceed to verify this without completing our definition of multiplication. Just remember, we have not defined multiplication if one of x and y is not positive. Shall do so soon.

But, of course, verification of the above property is trivial because, $x > 0$ tells we have $0 < p_1 \in x$ and similarly $0 < p_2 \in y$ and hence $0 < p_1p_2 \in x \cdot y$ to complete the proof.

Let us complete the definition of multiplication.

$$x \cdot y = \begin{cases} - & [(-x) \cdot y] & \text{if} & x < 0, & y > 0 \\ - & [(x) \cdot (-y)] & \text{if} & x > 0, & y < 0 \\ & [(-x) \cdot (-y)] & \text{if} & x < 0, & y > 0 \\ & 0 & \text{if} & x = 0, & \text{or} \quad y = 0 \end{cases}$$

This may look complicated, but first think of real numbers and observe these are correct. That was the reason for this definition.

This completes the definition of multiplication. That this satisfies all requirements is routine patient verification. You will do in exercises. This verification does *not* need using cuts. You need simply properties already proved and some 'algebra' — I do not mean you have to use theorems from algebra!, I mean *you do* some algebra.

This completes construction of R. This method is due to Dedikind. Later you will se another method due to Cantor.

enter Peano:

Thus if we are granted Q, we could construct

$$(R, +, \cdot, \leq)$$

satisfying all the needed rules. I have been a little loose; I freely used whatever I needed about Q without enunciating beforehand what I expect Q to obey. But do not worry; if some one challenges, you can also enunciate, just read the above argument and see what is used and state those. That is all. This can be easily done. But let us not be too legalistic and spend time on doing this. (This is important, but does not help in our understanding. So we are not spending time.)

But who gives us Q? We shall construct ourselves. You can not get something out of nothing. So what is the basic ingredient needed. Here it is. We shall be brief in the details.

Peano Axioms:

There is a set N_0 and a function $s : N_0 \to N_0$ satisfying the following three rules:

(i) $x = y$ if and only if $s(x) = s(y)$.
(ii) there is a unique element θ which is not in the range of s.
This θ, being unique, shall be denoted by 0.

(iii) If $A \subset N_0$, and $0 \in A$, and $x \in A$ implies $s(x) \in A$; then $A = N_0$.

The function s is called successor function.

If you want you can restate axiom (i) as

$$x = y \leftrightarrow s(x) = s(y).$$

There is no need to put 'iff'. The definition of function already tells you that when $x = y$, then you must have $s(x) = s(y)$.

Second axiom means there is no x such that $s(x) = \theta$. if you want you can restate this as

$$(\exists!\theta)(\forall x) \;\; \neg(s(x) = \theta).$$

The last axiom is called induction axiom, because it allows to use mathematical induction. This is also called well-ordering axiom, because it makes N_0 as well-ordered set; with a natural order to be defined shortly.

It is also possible not to demand uniqueness of θ in (ii). Take any one such θ and name it zero and proceed to axiom (iii). Then it follows that there can not be another such θ. Suppose that $\theta' \neq \theta$, also satisfies (ii) then the set $N_0 - \{\theta'\}$ satisfies conditions of (iii) and hence must equal N_0.

Let us not bother with minor details of no interesting consequence. The above are our axioms. These axioms are called Peano axioms.

You should note something very very interesting about axiom (iii). Usually axioms about a system tell about rules to be satisfied by elements of the system. For example axioms of groups tell you the rules to be satisfied by the elements of the group. Similarly about rings or fields. None of these tell you 'rules that subsets should obey'.

axiom (iii) tells you a rule to be obeyed by subsets of N_0, not by elements of N_0. This is a fine distinction, difficult to appreciate for you, but you should not ignore. The same remark applies to the 'least upper bound axiom' for R. all other axioms tell you rules to be obeyed by real numbers, or existence of special real numbers like zero and one. But the lub axiom is a rule that imposes a rule to be satisfied by subsets of R.

Let us accept that there is such a system (N_0, s). We see later.

Peano arithmetic: from (N_0, s) to $(N_0, +, \cdot, \leq)$:

We shall use the successor map to define addition and multiplication on N_0 as follows. These are defined using induction.

$$x + 0 = x; \qquad x + s(y) = s(x + y).$$

In other words, we first define $x + 0 = x$ for every x. Then we define

$$x + 1 = x + s(0) = s(x + 0) = s(x) = x + 1.$$

and then define $x + 2$ etc. Did we define $x + y$ of every y? Yes. the set of y for which we defined $x + y$ includes 0 and if it includes y then it includes $s(y)$ and hence this is all of N_0.

One can show again several rules like:
(commutativity) $x + y = y + x$ for all x and y;
(associativity) $(x + y) + z = x + (y + z)$ for all x, y, z;
(cancellation law) $x + z = y + z$ implies that $x = y$

We define multiplication

$$x \cdot 0 = 0; \qquad x \cdot s(y) = x \cdot y + x.$$

This satisfies analogues of the above three rules: commutativity; associativity and cancellation law.

Now we can define order:

$$x \leq y \leftrightarrow (\exists z) \;\; x + z = y.$$

You might think this is not right because for every x and y there is always such an z, namely $y-x$. But this is a wrong thought because at this moment we do not know this minus sign; for example there is nothing like $3-4$. At this moment the existential quantifier refers to existence of z in N_0. Since we have nothing else I did not have to say $\exists(z \in N_0)$. We have only N_0 at this stage before us, no more.

Order satisfies several rules:
It is a in linear order;
it has a first element;
it has no last element.

Very interesting, the proof of transitivity uses associativity of multiplication: suppose $x \le y$ and $y \le w$ then we have z and u such that $x+z=y$ and $y+u=w$. Now

$$x+(z+u)=(x+z)+u=y+u=w.$$

As I said we are being brief, but you can verify all the rules.

N_0 to Z:

As already noticed, given x and y there may not always exist u such that $y+u=x$; in other words I may not be able to talk about $x-y$ for some pairs. Of course, the cancellation law tells us that *if there is* one such u then there is *only* one u.

Since we are unable to do subtraction, for some pairs we cook up some ideal elements and put them in our set to make this possible. Basic philosophy is this: The very phenomenon that is not possible is put in as an element and this element represents that phenomenon. But then we should be careful, $3-4$ and $8-9$ should be same.

This is achieved by considering *all possible phenomena*; identify two phenomena if there are reasons for you to believe that they are same. Then hopefully this set would make all phenomena possible. All this is abstract sermon. Let us actually do it.

Let

$$Z_0 = \{(a,b) : a, b \in N_0\}$$

Thus when I say (a,b) I am thinking of $a-b$, but I can not say so because you will question me: what is this minus sign? Till I introduce this sign I should be careful not to go beyond my vocabulary.

Some people think of $b-a$ when they write (a,b). It makes no difference as long as you think one way or other (and not both) throughout.

We identify some pairs. Say

$$(a,b) \sim (m,n) \leftrightarrow a+m=b+n.$$

Actually I wanted to identify when $a-b=m-n$, I said the same thing without using minus sign, that is, using only the means I have now. It is a routine matter to verify that this is an equivalence relation.

Let the space of equivalence classes be denoted by Z. We have enlarged the set N_0. We can identify N_0 as a subset of Z. Take $n \in N_0$ and let $\varphi(n)$ be the equivalence class containing $(n,0)$ — remember, I have $a-b$ at the back of my mind when I wrote (a,b).

Actually, I can make a nice selection of 'one thing' from each equivalence class and use them rather than thinking of the huge equivalence class all the time. Each equivalence class contains exactly one pair where one of the coordinates is zero.

if $a=b$ then easy to see $(a,b) \sim (0,0)$.

if $a<b$, then there is c such that $a+c=b$, then easy to see $(a,b) \sim (0,c)$.

if $b<a$, then there is c such that $b+c=a$, then easy to see $(a,b) \sim (c,0)$.

These are the only possibilities because we have linear order.

Clearly a pair $(a,0) \sim (0,b)$ implies $a+b=0+0$ so that $a=b=0$.

If $(a,0)=(b,0)$ then $a=b$. If $(0,a)=(0,b)$ then $a=b$.

Thus there is exactly one pair (a,b) in each equivalence class with one coordinate zero. Thus we could think of this element instead of the equivalence class. In other words we could have defined

$$Z = \{(a,b) : a=0 \quad \text{or} \quad b=0\}$$

And we could have though of $(a,0)$ as $a-0=a$ and $(0,a)$ as $0-a=-a$.

Yes you are right, we did complicate life. We could have just said 'add a symbol minus n" for every n, then these symbols along with N_0 is precisely our Z. Such a prescription will serve the present purpose. However it misses the general philosophy, it looks artificial, serves only immediate purpose. The philosophy 'if you can not achieve certain phenomenon, consider the set of phenomena' gives a unity for mathematical constructions.

But you can keep both pictures in mind. you will see first picture looks complicated but 'smooth' to operate; second picture looks smooth but complicated to operate.

On this set Z we define addition multiplication and order:

$$(a, b) + (c, d) = (a + c, b + d).$$

$$(a, b) \cdot (c, d) = (ac + bd, ad + bc)$$

Under addition it becomes a group and multiplication obeys all the three rules we had earlier. We define order as follows.

$$(a, b) \leq (c, d) \leftrightarrow ad + bc \leq ac + bd.$$

This definition makes Z a loset and the order is friendly with addition and multiplication.

In the second picture, these definitions take the following shape. Of course, you can repeat the same earlier formulae.

$$(n, 0) + (m, 0) = (n + m, 0); \quad (0, n) + (0, m) = (0, n + m)$$

To define $(n, o) + (0, m)$ we need to consider two cases. If $n > m$, say, $n = k + m$ then the sum equals $(k, 0)$. But if $m > n$ and say $m = k + n$ then the sum equals $(0, k)$. Think about it. Multiplication is as follows.

$$(m, 0) \cdot (n, 0) = (mn, 0); \quad (0, m) \cdot (0, n) = (mn, 0)$$

$$(m, 0) \cdot (0, n) = (0, mn).$$

Definitely messy. However order takes simpler shape (as it should). Each $(0, n)$ is smaller than each $(m, 0)$. Among $(m, 0)$ kind it becomes larger as m becomes larger, whereas among $(0, m)$ kind it becomes as m becomes larger.

Of course all of it just provides you the picture you already have in mind

$$\cdot\cdot\,, -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \cdots\cdots.$$

This is how we think of Z from now on. Once constructed, no matter how, Z has existence independent of how it is constructed and by whom.

Z to Q:

The trouble with Z is that while we can add and subtract, we can multiply but not divide. In other words we have additive inverses but not multiplicative inverses. By the way, using known rules, we understand that anything multiplied by zero has to be zero, so when we say multiplicative inverse, we mean only for non-zero things. We know that we can never have multiplicative inverse for zero.

If I could not subtract earlier $3-4$ then we added $(3,4)$ and when we see this 'symbol' we think of $3-4$. In the same way, if I can not divide $3/4$, I add the symbol $(3,4)$ and when I see this I think of $3/4$.

Of course there is a confusing point. When you see $(3,4)$ do you have at your back of the mind $3-4$ or $3/4$? You can not have both. That is why I said earlier how we think of Z. We do not think of $3-4$ now, it is represented by the symbol -1, that is all. There is no pairs. elements of Z are as described at the end of the discussion above. First understand this point.

So since division between some pairs of points of Z is not possible, let us repeat the earlier process. Let

$$Q_0 = \{(a,b) : a, b \in Z, b > 0\}.$$

But as you see I can not think of $3/4$ and $6/8$ as two separate quantities. Define in Q_0 an equivalence relation

$$(a,b) \sim (c,d) \leftrightarrow ad = bc.$$

I wanted to identify if $a/b = c/d$ but I can not say this because I do not have meaning for these quantities. But you see i said the same thing using the permitted vocabulary.

Let Q be the set of equivalence classes. This is our set of rationals. Define

$$(a,b) + (c,d) = (ad + bc, bd), \quad (a,b) \cdot (c,d) = (ac, bd).$$

First observe that bd is nonzero and hence the definition makes sense, the resulting pairs are in our collection. Also observe that these definitions respect the equivalence relation. That is

$$(a,b) \sim (a_1,b_1), (c,d) \sim (c_1,d_1) \Rightarrow (ad + bc, bd) \sim (a_1 d - 1 + b_1 c_1, b_1 d_1).$$

$$(a,b) \sim (a_1,b_1), (c,d) \sim (c_1,d_1) \Rightarrow (ac,bd) \sim (a_1c_1,b_1d_1).$$

This observation, that the operations respect the equivalence relation, help us to define the operations on the space of equivalence classes, that is, on Q.

These operations make Q into a field.

We define order by

$$(a,b) \leq (c,d) \leftrightarrow ad \leq bc$$

Again, this definition respects the equivalence relation and hence defines an order on Q. This makes Q into a loset and the order is compatible with addition and multiplication.

These operations and order extend the operations and order we already have on Z. Of course, for this sentence to make sense you should first be able to see Z inside Q. The equivalence class containing the element $(a,1)$ is identified with $a \in Z$. If you want to be formal, you can define map φ on Z into Q and say that this map preserves the 'structure'.

In other words Q is an ordered field. It has the following property, called Archimedian property.

$$x \in Q \Rightarrow (\exists\ \ k \in N_0)\ \ x < k.$$

Every number is smaller than some integer.

Peano again:

The final thing we need to do is to convince ourselves that there is a set N_0 and $s : N_0 \to N_0$ satisfying the axioms of Peano. This comes from set theory and we only indicate without going into the details. We define

$0 = \emptyset$ empty set, that is the set which has nothing.

$s0 = \{\emptyset\}$ set with only one element, namely, the empty set.

$ss0 = \{\emptyset, \{\emptyset\}\}$ set with two elements, namely, the empty set and the set which contains the empty set. Equivalently, it consists of the elements of the previous set (which is only one) and the previous set itself.

$sss0 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ set consisting three elements all the elements of the previous set (which are two) and the previous set itself.

In general $s(*)$ consists of all elements of the previous set $*$ along with one more element, namely, the previous set $*$ itself. In symbols you can think of ($*$ may be confusing, think of k and think of $k+1$ instead of sk)

$$k + 1 = k \cup \{k\}$$

before you get confused, see that right side is union of two sets. The first set is k and hence everything which is in k is here. The second set has exactly one thing in it, namely, k itself. Thus this new set has only one extra thing namely the previous set itself.

The set N_0 consists of all these objects we defined and the map is

$$s(k) = k \cup \{k\}$$

This set satisfies all our requirements.

There are two important points in what I said above that make you suspect something is wrong somewhere. Yes, you are justified in not accepting what I said above. Did I already use $0, 1, 2, 3 \cdots$? Can I use them? I finally said N_0 consists of *all these*. What is meant by *all these*? How long should I construct the sets which I was describing above?

This is where set theory enters the picture. Here is a truth.

There is a smallest infinite set ω_0 which has the property

$$x \in \omega_0 \Rightarrow x \subset \omega_0.$$

That is, there is an infinite set ω_0 which satisfies the above condition and if ω is another set satisfying that condition then $\omega_0 \subset \omega$.

We can show that such a set contains all the things listed earlier by us. In other words

$$\emptyset \in \omega_0$$

$$\{\emptyset\} \in \omega_0$$

$$\{\emptyset, \{\emptyset\}\} \in \omega_0$$

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \in \omega_0.$$

and so on. I needed some symbols to show you my listing and thus k etc entered in my earlier description. I could have listed my sets one after the other without using the symbol k. But how long can I do? If I used 'etc' how long should you do?

Actually our N_0 is exactly this set ω_0. See how we described it in one sentence without using symbols like k, $k+1$ and without using words like 'so on'. Axioms of set theory allow you to show that there is such a set (and it is unique).

This is called first infinite ordinal. This definition of ordinal numbers is due to von Neumann. Cardinal and ordinal numbers were invented by Cantor. Cardinal numbers tell you counting without any idea how things are placed: one, two, three, etc.

Ordinal numbers tell you counting when things are standing in a line; first, second, third, etc. You can do this for any well ordered set. In any well ordered set there is a least element. That is the first element. Consider its complement, it has a least element, that is the second element of the set and so on. If you have counted first, second, third etc then remove all these. If the resulting set is non-empty take its first element it is ∞-th element of the set. Remove this also. Least element of the remaining set is $(\infty + 1)$-th and so on. What I called ∞ here is precisely ω_0.

You have learnt a little about cardinal numbers, you have learnt a little about well-ordered sets. It is worth while to learn a little about ordinal numbers too, perhaps later sometime if there is some time.

This completes our discussion of construction of R. That is, we have constructed a set and shown operations and order that satisfy all the properties listed last year.

Metric spaces:

We shall proceed to the next topic of our discussion, namely, metric spaces. These are nothing but sets where there is a notion of distance between pairs of points. You can say how far is one point from other.

Once you have this notion, you can understand the concept of closeness, after all two points are close if the distance between them is not much. Once you can feel closeness of points, you can feel if a sequence of points are getting closer and closer to a given point. In other words, convergence of sequences makes sense. Once you can do this, you can talk about continuous functions.

After all continuous functions are just those that preserve convergence. So you see you can build an excellent story, imitate on your set what you have done on R. Of course, you can not add or multiply unless these are possible in your set.

This is not just for the sake of generalisation, it has applications far beyond expectations.

Let X be a nonempty set. For each pair of points x, y in the set we want to have the concept of distance between the points $d(x, y)$. One thing you feel immediately is that $d(x, x) = 0$. Also if x and y are different points then naturally they are separate and must be at a distance no matter how small; thus $d(x, y) > 0$.

One thing experience tells us is that distance is symmetric. the distance from x to y must be same as the distance from y to x. That is, $d(x, y) = d(y, x)$. Of course, I depended on your experience.

It is also quite possible to think that this ned not be true. For example, suppose you try to measure distance by the energy you need to spend from going from one place to other. For points far apart you need to spend more energy in travelling whereas if the points are close you need not spend too much energy. Such a method appears very reasonable. But imagine x is the point at the base of a hill and y is the point on the top of the hill. You will surely agree that it takes more energy to go from x to y than from y to x. Thus it is also conceivable that distance is not symmetric.

However We adapt symmetry.

Finally again from practice, we see that distance is in a sense 'direct distance' whatever it may mean. Thus going from x to z directly should be not more than going from x to z via y. That is $d(x, z) \leq d(x, y) + d(y, z)$. This is called triangle inequality. This is true for any three points you take in the plane. This we knew from elementary geometry.

We shall adapt just these three as the things to be satisfied by distance and see what can be done and what is its use.

Definition: Let X be a (non-empty) set and $d : X \times X \to [0, \infty)$ satisfying the three conditions.

(i) $d(x, y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$ for any two points x, y.

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for any three points x, y, z.

Then we say that d is a distance function on X and we also say (X, d) is a metric space.

If we have a metric space (X, d) and $a \in X$ and $r > 0$ then open ball of radius r with centre a is defined to be the following set: $\{x \in X : d(x,a) < r\}$.

Example 1: X is real number system R and $d(x,y) = |x - y|$.

In this case $B(a,r)$ is the interval $(a - r, a + r)$. Thus open balls are just open intervals.

Example 2: X is the plane R^2 and for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we put

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

As explained earlier this is just Pythagoras theorem.

In this case $B(a,r)$ where $a = (a_1, a_2)$ is the set

$$\{(x_1, x_2) : (x_1 - a_1)^2 + (x_2 - a_2)^2 < r\}$$

which is the usual disc.

Example 3: X is R^n and

$$d(x,y) = \sqrt{\sum_1^n (x_i - y_i)^2}.$$

Example 4: X is again R,

$d(x,y)$ is zero if $x = y$ and one if $x \neq y$.

Here $B(a,r)$ consists of just the point $\{a\}$ as long as $r \leq 1$, but it equals all of R as soon as $r > 1$.

Example 5: X is R^2.

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}; \quad x = (x_1, x_2) \;\; y = (y_1, y_2).$$

Here ball of radius around $(0,0)$ consists of the square with vertices $(\pm 1, \pm 1)$.

Example 6: X is R^2.

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|, \quad x = (x_1, x_2) \;\; y = (y_1, y_2).$$

Here ball of radius around $(0,0)$ consists of the region bounded by the four lines $(\pm x_1) + (\pm x_2) = 1$.

More generally,
Example 7: Fix $p \geq 1$. X is again R^2.

$$d(x,y) = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{1/p}, \quad x = (x_1, x_2) \;\; y = (y_1, y_2).$$

Triangle inequality here is not obvious. We shall prove.

Example 8: X is $C[0,1]$ the space of real valued continuous functions on the unit interval. For f, g

$$d(f,g) = \sup\{|f(x) - g(x)| : 0 \leq x \leq 1\}$$

Since continuous functions on $[0,1]$ are bounded this makes sense. Since for any three functions f, g, h and any point x we have

$$|f(x) - h(x)| \leq |f(x) - g(x)||g(x) - h(x)| \leq d(f,g) + d(g,h).$$

This is true for every x nd so by taking sup over x on left side we get the triangle inequality.

Here given f and $r > 0$ the ball $B(f,r)$ consists of all continuous functions whose graph lies in the band: graph of $f(x) - r$ and graph of $f(x) + r$.