Continuous functions:

Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a function and $a \in X$. We say that f is continuous at a if the following is true: $x_n \to a$ in X implies $f(x_n) \to f(a)$ in Y. We say f is continuous function if it is continuous at every point $a \in X$.

All the results that we did in calculus concerning functions on R are all true with nearly the same proofs.

Theorem 1: f is continuous at a iff the following holds: for every $\epsilon > 0$ there is a $\delta > 0$ such that $\rho(f(x), f(a)) < \epsilon$ whenever $d(x, a) < \delta$.

Suppose the condition holds. To show f is continuous at a take any sequence $x_n \to a$. We need to show that $f(x_n) \to f(a)$. So fix $\epsilon > 0$. We shall show a stage after which $\rho(f(x_n), f(a)) < \epsilon$. With this $\epsilon > 0$ find $\delta > 0$ as assured. Since $x_n \to a$ we have $d(x_n, a) < \delta$ after some stage and clearly then $\rho(f(x_n), f(a)) < \epsilon$.

To prove the converse, assume that the condition fails. Fix an $\epsilon > 0$ for which we can not find $\delta > 0$ satisfying the condition. Since $\delta = 1/n$ does not satisfy the requirement we can pick x_n so that $d(x_n, a) < 1/n$ and yet $d(f(x_n), f(a)) \geq \epsilon$. Clearly $x_n \to a$ but $\rho(f(x_n), f(a)) \geq \epsilon$ for all n. So f is not continuous at a.

Theorem 2: f is continuous at a iff the following holds: For any set $V \subset Y$ which includes a ball around $f(a)$ the inverse image $f^{-1}(V)$ includes a ball around a.

suppose the condition holds. Then we verify the statement of the previous theorem. So let $\epsilon > 0$ be given for which we need to find $\delta > 0$. Take V to be the ball of radius ϵ around $f(a)$ and use condition to get a ball $B(a, \delta) \subset f^{-1}(V)$. This δ will do.

To prove the converse, suppose that the statement of the previous theorem holds. We show that condition of the present theorem holds. Take V as stated, suppose it includes ball of radius $\epsilon > 0$. get a $\delta > 0$ as in the previous theorem. Observe that $f^{-1}(V)$ includes $B(a, \delta)$.

Theorem 3: f is continuous iff for every open set $V \subset Y$, the set $f^{-1}(V)$

is open in X.

This is immediate from previous theorem. Indeed, suppose f is continuous. Let V be open in Y. To show $f^{-1}(V)$ is open in X take any point $a \in f^{-1}(V)$. Then $f(a) \in V$ and V being open, contains a ball around $f(a)$ so by previous theorem $f^{-1}(V)$ includes a ball around a. Thus $f^{-1}(V)$ includes a ball around each of its points. that is, it is open.

to prove the converse, suppose that condition of the theorem holds. To show f is continuous at say $a \in X$, take $x_n \to a$. Need to show $f(x_n) \to f(a)$. Take open ball of radius $\epsilon > 0$ with centre $f(a)$. Inverse image of this ball is open and also contains a and so by from hypothesis, includes a ball around a, say, $B(a, \delta)$. Since $x_n \to a$ we conclude that $x_n \in B(a, \delta)$ after some stage. But then after that stage $f(x_n)$ is in the ϵ ball around $f(a)$. Thus $f(x_n) \to f(a)$.

Theorem 4: f is continuous iff for every closed set $C \subset Y$, the set $f^{-1}(C)$ is closed in X.

f is continuous and C^c is open implies $f^{-1}(C^c)$ is open and hence $f^{-1}(C)$ is closed. Conversely, Given condition implies that for any open set $f^{-1}(U^c)$ is closed and hence $f^{-1}(U)$ is open.

Theorem 5: (i) If f and g are real valued continuous functions on X, then so are $f + g$, fg and $32f$.

(ii) Fix a point $z \in X$. The function $f(x) = d(z, x)$ is a continuous function. More generally, if $A \subset X$ is a non-empty set then $f(x) = d(x, A)$ is a continuous function. Recall $d(x, A) = \inf\{d(x, z) : z \in A\}$.

(iii) In particular, given a closed set C, there is a non-negative continuous function which takes the value exactly for points in the closed set. Equivalently, given an open set U there is a non-negative continuous function which takes strictly positive values exactly on U.

First part follows because convergence in R respects addition, multiplication etc. Actually the same is true for functions taking values in R^n.

Second part has already been noted earlier as a consequence of triangle inequality.

Third part follows because $d(x, C) = 0$ iff $x \in C$; when C is a closed set. If $C = \emptyset$ take the constant function 1.

Theorem 6: If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

If you have a collection $\mathcal{U}$ of open sets in Y covering $f(X)$ then the family of open sets $\{f^{-1}(U) : U \in \mathcal{U}\}$ covers X and take a finite family that covers X, say, $\{f^{-1}(U_i) : 1 \leq i \leq k\}$ then the finite family $\{U_i : 1 \leq i \leq k\}$ covers $f(X)$. (You should verify this and not simply reproduce this sentence).

In particular, since we know that compact sets are precisely closed and bounded sets we conclude as a special case of the above theorem: if f is a continuous function defined on a closed bounded subset of R, then the range of the function is again closed and bounded. In particular, it attains its bounds too, as they are part of the range.

Theorem 7: If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Observe that $f : X \to f(X)$ is a continuous function. If $f(X)$ is not connected, then there is a non-empty proper subset $A \subset f(X)$ which is both closed and open. Clearly its inverse image is both closed and open and non-empty and proper subset of X.

In particular, since we know that the only connected subsets of R are intervals we conclude the following: Range of a real valued continuous function defined on an a connected metric space is again an interval. In particular range of a continuous function defined on an interval in R is again an interval. as a result the function has intermediate value property, that is, if $f(x) = a$ and $f(y) = b$ and $a < c < b$ then there is a point z such that $f(z) = c$ and $x < z < y$.

Homeomorphism:

A function $f : X \to Y$ is called a homeomorphism if it is one-one, onto, continuous and inverse map is also contious from Y to X.

Remember, homeomorphism need not preserve the metric. for example the function $f(x) = 1/x$ is a homeomorphism of $(0, 1)$ onto $(1, \infty)$ that does not preserve the distance.

Then why is it called homeomorphism? Generally when you say isomorphism or homomorphism etc they preserve the structure that we have. Homomorphisms preserve 'forward' and isomorphisms preserve both ways (for this to make sense isomorphism has to be bijections and etc).

Metric on a space has provided us collection of open sets, which is not visible in the notation. This structure is preserved by a homeomorphism. That is, a homeomorphism is a bijection f with the property: $U \subset X$ is open in X iff $f(U) \subset Y$ is open in Y. Equivalently, the collection of closed sets is preserved. Thus homeomorphism is a bijection f such that the following holds: $C \subset X$ is closed in X iff $f(C)$ is closed in Y. Equivalently convergence is preserved. Thus homeomorphism is a bijection f with the property: $x_n \to x$ in X iff $f(x_n) \to f(x)$ in Y.

Thus, if two spaces are homeomorphic then by renaming, one looks like the other. Renaming x as $f(x)$ the space X looks like Y. But please do remember, this appearance is only for convergence; not for actual distances.

Homeomoprhism is an equivalence relation among metric spaces. Aha, the collection of metric spaces is NOT a set! (Do not waste time on this point.) What we mean is the following. If you take any specific collection of metric spaces, then among the collection homeomorphism is an equivalence relation. We can state the same property without talking about collections at all as follows.

X is homeomorphic to X (identity map).

If X is homeomorphic to Y then Y is homeomorphic to X (take inverse map).

If X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic yo Z (take composition of maps).

Here are some examples:

The space $[0,1]$ is not homeomorphic to any of the spaces $[0,1)$; $(0,1]$; $(0,1)$. Because the former is compact and the later are not compact.

The space $(0,1)$ is not homeomorphic to $[0,1)$; $(0,1]$. There is one end point in the later spaces so that even after removing that point the space is still connected. But in the space $(0,1)$ removal of any one point makes it disconnected. More precisely, if $f : [0,1) \to (0,1)$ is homeo, then f restricted to $(0,1)$ in the domain is a homeomorphism onto $(0,1) - \{f(0)\}$. The domain is connected but the range is not.

Of course the spaces $[0,1)$ and $(0,1]$ are homeomorphic, for example $f(x) = 1 - x$ is one such. This is not the only map which exhibits homeomorphism between these two spaces.

Are any of the above spaces homeomorphic to the unit circle
$S^1 = \{(x, y) : x^2 + y^2 = 1\}$.

No. This is not homeomorphic to $[0, 1]$ because there are two points in the later, namely the end points, so that the space remains connected even after removing those. However removal of any two distinct points makes S^1 disconnected. Of course S^1 is compact but $(0, 1)$ and $[0, 1)$ are not compact.

Of course given bounded interval (a, b) the map $f(x) = a + x(b - a)$ sets up a homeomorphism from the interval $(0, 1)$ to (a, b). The interval $(0, 1)$ is homeomorphic to $(1, \infty)$ by the map $g(x) = 1/x$. The interval $(0, \infty)$ is homeomorphic to $(-\infty, 0)$ via the map $h(x) = -x$. Also $(0, \infty)$ is homeomorphic to (a, ∞) using $\varphi(x) = a + x$. Now by using compositions we can easily deduce the following: Any two (non-empty) open intervals in R are homeomorphic, bounded or unbounded.

We can not say the same about closed intervals. Any two closed bounded intervals (non-degenerate) are homeomorphic. Obviously, a closed bounded interval, which is compact, can not be homeomorphic to unbounded closed interval. Also even among unbounded closed intervals, you can show that $[1, \infty)$ is not homeomorphic to $(-\infty, \infty) = R$. you can do this by arguing that removal of a point still keeps the first set connected where as removal of any one point makes R disconnected. You can also argue as follows. First showing that a homeo must be either strictly increasing or strictly decreasing and thus image of any homeo can only go in one direction from the image of the end point.

Consider the set

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$$

This is not homeomorphic to any of the above spaces. This is a compact space. This is also connected. The earlier ones are all path-connected but this is not. We shall understand this now.

S being a closed bounded subset of R^2 it is compact. To show that S is connected, you can, for example, say the interval $(0, 1]$ is ~~compact~~ and its continuous image $x \mapsto (x, \sin(1/x))$ is ~~compact~~ and hence its closure, namely S, is connected.

Here we used the following fact. Suppose that we have a subset $Y \subset X$. Suppose that Y is connected. Let Y^* be the closure of Y in the space X.

Then Y^* is also connected. In fact, if $A^* \subset Y^*$ is a proper non-empty set which is both closed and open in Y^*, then $A = A^* \cap Y$ is both open and closed in Y. But then Y being connected we should have either $A = \emptyset$ or $A = Y$. If $A = Y$ then the $A^* \supset Y$. But A^* is closed in Y^* so it must be all of Y^*. Similarly, if $A = \emptyset$ then $A^* = \emptyset$

Of course, you can show S is connected with bare hands as follows. Let $A \subset S$ be non-empty subset which is both closed and open. Suppose it has a point $(a, \sin(1/a))$ then the entire curve should be in A. In fact, if some $(b, \sin(1/b))$ is not in A then consider all t between a and b such that $(t, \sin(1/t)) \in A$ and arrive at a usual (?) contradiction by considering the sup or inf of this set of points t. If the entire curve is in A then A being closed, it must include the other points of S from the Y-axis too. Note that each point $(0, y)$ in S is limit of a sequence of points on the curve.

However S is not path connected. For example there is no path joining $P = (1, \sin 1)$ to $Q = (0, 0)$. Prove this statement, if needed read earlier para again. Remember path in S means continuous function φ on $[0, 1]$ with values in S. It is path joining two points P and Q means that $\varphi(0) = P$ and $\varphi(1) = Q$.

You can ask whether R and R^2 are homeomorphic. You can use the argument involving connectedness and deleting point to argue that the answer is no. The same holds for R and any R^n for $n > 1$. It is also true that R^m and R^n are not homeomorphic if $m \neq n$, but the proof is not so simple.

Homeomorphism is an important concept in understanding spaces. For example if you have understood a group, then you have understood all groups isomorphic to it. If you have understood a vector space, then you have understood all vector spaces that are isomorphic to it. similarly, if you have understood a metric space, then you have understood all spaces homeomorphic to it; in some sense. Remember homeomorphism may not preserve distance. Thus understanding here means convergence and continuous functions.

separable spaces:

Recall that a metric space is separable if there is a countable set D such that every non-empty open sets contains a point of D. Such a set D is called dense set.

Theorem: A compact metric space is separable.

Let (X, d) be a compact metric space. For each n finitely many balls of radius $1/n$ cover the space, take such finitely many balls (your choice) and let F_n be their centres. Note that given any point $x \in X$, there is a point $p \in F_n$ such that $d(x, p) < 1/n$.

Let $D = \cup F_n$. Each F_n being finite, we conclude that D is countable. Given any point $x \in X$, there is a point $p_n \in F_n$ such that $d(x.p_n) < 1/n$. In other words $p_n \to x$. thus every ball around x contains points from D. This wing true of every point x we see that every non-empty open ball contains points from D. Thus D is dense, showing that a compact metric space is separable.

Consider R, real line and the collection $\mathcal{B}$ of open intervals with rational end points. Then we have the following properties: (i) the family $\mathcal{B}$ is countable and (ii) every open set is union of some of these intervals from $\mathcal{B}$.

Indeed the fact that there are only countable many pairs of rational tells us that $\mathcal{B}$ is countable. Further given any open set V and a point $x \in V$, there is an interval $(x - \epsilon, x + \epsilon) \subset V$; taking rationals a and b with $x - \epsilon < a < x$ and $x < b < x + \epsilon$ and setting $J = (a, b)$ we see $J \in \mathcal{U}$ and $x \in J \subset V$. In other words V is union of all intervals in $\mathcal{B}$ which are contained in V.

We now show that similar result is true in metric spaces.

Theorem: Let (X, d) be a separable metric space and D be any countable dense set. Let

$$\mathcal{B} = \{B(p, 1/n) : p \in D; n \geq 1.\}$$

then the family $\mathcal{B}$ is countable and every open set is union of some sets from $\mathcal{B}$.

In other words, using open balls of radius $1/n$ $(n \geq 1)$ around points of D we can describe all open sets.

The fact that D is countable tells that $\mathcal{B}$ is a countable collection. Take any ball $B(x, r)$. Pick n such that $(2/n) < r$. D being dense pick $p \in D$ with $d(x, p) < 1/n$. Observe that if $y \in B(p, 1/n)$ then

$$d(x, y) \leq d(x, p) + d(p, y) \leq 2/n < r.$$

In other words $x \in B(p, 1/n) \subset B(x, r)$. as a result, if you take any open set V and any $x \in V$ then use the fact that $B(x, r) \subset V$ for some $r > 0$ to conclude that there is an $B(p, 1/n)$ such that

$$x \in B(p, 1/n) \subset V.$$

In roher words V is union of all balls from $\mathcal{B}$ which are contained in V.

compactness revisited:

Before discussing metric spaces, one motivation was that, if we have concept of convergence then we can discover more and more points in the space. Let us fully recall what it means.

In R we are comfortable with rational numbers because they are described simply as ratio of integers. We somehow knew that the sequence

$$1,\ 1+1,\ 1+1+\frac{1}{2!},\ 1+1+\frac{1}{2!}+\frac{1}{3!},\ \cdots$$

converges and we named its limit as e and discovered that it was not something known to us earlier. Similarly, the sequence

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}; \quad n = 1, 2, 3, \cdots$$

converges. We also know that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n; \quad n = 1, 2, 3, \cdots$$

also converges. Both the sequences described above have the same limit. We named it γ. We do not yet know whether we have discovered a new (non-rational) number.

In the same way in a metric space we know some points and using the notion of convergence we can discover many other points. If the space is compact, then manufacturing a sequence will lead to a subsequence which converges and hence a point is discovered.

For example, in the space $C[0, 1]$ we know that polynomials are dense (remember, unless stated otherwise, the metric is sup metric). Suppose I some how produce a convergent sequence of polynomials, then I can talk about limit of this sequence, thus obtaining a legitimate element of $C[0, 1]$. Hopefully, I can discover properties of this new function by looking at the known sequence of polynomials which are converging to it.

In fact, exponential function, sine and cosine functions are obtained as limits of polynomials. Remember our discussion on power series.

But how do I know whether some sequence is converging or not? One possibility is that the space is complete and your sequence is Cauchy sequence. But to show something is Cauchy involves estimating distance between elements of the sequence. Another easier way is to show that the sequence lies in a compact set. Of course, this does not imply that the sequence converges, in general it does not. But there are convergent subsequences.

So it is important to know what sets are compact and what are not. For example in R or R^n compact sets are precisely closed and bounded sets. The characterization of compact metric spaces is too general and we would now specialise to one particular example.

Arzela-Ascoli:

Let us consider the space $C[0, 1]$. How do you know if a given subset of this space is compact? Answer to this question is provided by a theorem that was proved (independently) by two Italian mathematicians. Arzela and Ascoli. This is what we discuss now.

But first observe the following. For each $n = 2, 3, 4, \cdots$ define a continuous function x_n on $[0, 1]$ as follows. $x_n(t)$ starts at zero when $t = 0$ and increases with slope $2n$ till it reaches 1 at $t = 1/2n$ and then decreases the same way reaching zero at $t = 1/n$ and it stays at that value from then on. Let C be the set consisting these functions $\{x_n; n \geq 2\}$.

All these functions are non-negative and are bounded by one. Thus this is a bounded set. Remember a set is bounded if there is a number M such that distance between any two points of the set si at most M. Also this is a closed set. Indeed, if there is any limit point, equivalently, if there is a subsequence that converges to some x then it must also converge pointwise too and hence $x \equiv 0$. But all these functions are far from zero, indeed, $d(x_n, 0) = 1$ for each n. So there is no limit point.

Thus this set is closed and bounded but not compact. We need more than 'bounded'. This is what we discover now.