what next:

We completed our discussion of metric spaces. We have done some basic results and applications. There are two possibilities for the remaining period — we can discuss either power series with complex coefficients or Fourier series.

We discussed real power series $\sum a_n x^n$ and discussed radius or interval of convergence in our first course. we found a number $R \geq 0$ such that the power series converges (actually, converges absolutely) for every real number x with $|x| < R$; does not converge for any real number x with $|x| > R$; and for $x = \pm R$, it may or may not converge depending on the particular series.

The same analysis can be done even if you have power series $\sum a_n z^n$ with complex coefficients. There is a real number $R \geq 0$ such that whenever you take a complex number z with $|z| < R$ the series converges (converges absolutely). Whenever you take a complex number z with $z| > R$ the series does not converge. Exactly the same proof that we did for real series goes through.

However we shall not continue this line of thought. Mainly because, introducing the theory with the statement that the same proof as the real case goes through, is perhaps the worst way of introducing the splendour of complex analysis. You can tell your complex analysis teacher to carry out the details concerning power series.

Of course, we did discuss complex valued functions of complex variable and their derivatives. This was when we were discussing functions from R^2 to R^2. We derived the Cauchy-Riemann equations; some of its interesting consequences; related the complex derivative to the derivative matrix when you regard the function as a map of R^2 to R^2 instead of from C to C. We shall discuss some relevant aspects of complex analysis as we go along.

We shall now discuss Fourier series. We would first review what we know about C^n, the complex n dimensional space.

complex n-dimensional vector space:

You studied the n-dimensional complex vector space and inner product. Let us recall. C^n is the set of all n tuples $z = (z_1, z_2, \cdots, z_n)$ of complex numbers. We define inner product

$$\langle z, w \rangle = \sum_k z_k \overline{w_k}; \quad \langle z, z \rangle = ||z||^2 = \sum |z_k|^2.$$

$$d(z, w) = \sqrt{\langle z - w, z - w \rangle} = \sqrt{\sum |z_k - w_k|^2}.$$

Here $\overline{w}$ is the complex conjugate of w. Recall if $w = a + ib$ where a, b are real then its conjugate is $a - ib$.

This space has an orthonormal basis consisting of n vectors. Thus there are n vectors $e_1, e_2, \cdots, e_n$ such that

$$\langle e_k, e_l \rangle = 1 \text{ if } k = l; \quad \text{and} \quad = 0 \text{ if } k \neq l.$$

Any vector v can be uniquely written as

$$v = \sum_1^n \langle v, e_k \rangle e_k = \sum \hat{v}_k e_k.$$

It follows from properties of inner product that

$$||v||^2 = \sum_1^n |\hat{v}_k|^2.$$

interesting point, trivial but should be noted, is that this is true whatever orthonormal basis you take Let us take an integer $1 \leq m < n$ and consider only the partial sum

$$v^* = \sum_1^m \hat{v}_k e_k.$$

Then clearly v^* is in the subspace spanned by the first m-basis vectors, that is,

$$v^* \in \text{ span } \{e_1, e_2, \cdots, e_m\} = S; \quad \text{say}.$$

The vector $v - v^*$ is orthogonal to every vector in S. This is because

$$\langle v - v^*; \sum_1^m a_k e_k \rangle = \langle v; \sum_1^m a_k e_k \rangle - \langle v^*; \sum_1^m a_k e_k \rangle$$

$$= \sum_1^m \overline{a_k} \langle v, e_k \rangle - \sum_1^m \overline{a_k} \langle v^*, e_k \rangle = 0;$$

because, $\langle \sum_{l=1}^{m} \hat{v}_l e_l, e_k \rangle = \hat{v}_k = \langle v, e_k \rangle$.

Since $v = (v - v^*) + v^*$ and $v - v^* \perp v^*$ we see

$$||v||^2 = ||v - v^*||^2 + ||v^*||^2.$$

Thus $||v^*||^2 \leq ||v||^2$. We could have got it by direct computation because

$$||v^*||^2 = \sum_1^m |\hat{v}_k|^2 \leq \sum_1^n |\hat{v}_k|^2 = ||v||^2.$$

There is a subtle point in this so called 'direct computation'. It is that, we assume that v itself is a linear combination of the basis vectors. You might think, what else can it be; it has to be a linear combination; every vector is and so on. However you will realize the subtlety later.

In a sense v^* captures the entire part of v in S (whatever it may mean) so that what remains is orthogonal to S. Not only this, v^* is the vector closest to v in the subspace S. That is

$$d(v, v^*) \leq d(v, w) \quad \forall \;\; w \in S.$$

This is seen as follows.

$$v - w = (v - v^*) + (v^* - w)$$

and the two terms on the right are orthogonal we see

$$||v - w||^2 = ||v - v^*||^2 + ||v^* - w||^2 \geq ||v - v^*|^2$$

Fourier series:

The plan is to understand the statement of Fourier:

Every wave is a superposition of sine and cosine waves.

Let us consider the interval $[0, 1]$. Let f be a function on this space with $f(0) = f(1)$. We can extend it as a periodic function of period one on the real line, in a unique way, namely, define $f(x+1) = f(x)$ for all $x \in R$. More precisely it is defined by $f^*(x)$ is the given $f(x)$ on $[0, 1)$; $f^*(x) = f(x - 1)$ on $[1, 2]$ and $f^*(x) = f(x - 2)$ on $[2, 3]$ etc. Similarly on the negative side. It is much easier for you to think of its graph. First imagine on $[0, 1]$ and then extend the curve to all of R.

Conversely given any function f^* on R which is periodic of period one its restriction to $[0,1]$ gives us a function f on $[0,1]$ with the property $f(0) = f(1)$. Moreover in this process, f is continuous iff f^* is continuous.

Such functions on R have a wavy graph they are called waves. Actually you can define what are waves using certain differential equations but let us not go too far now. The simplest waves we know from high school are $s(x) = \sin(2\pi x)$ and $c(x) = \cos(2\pi x)$. Of course any multiple of these functions is also a wave. The functions $\sin(4\pi x)$, $\cos(4\pi x)$ are also such functions. More generally $\sin(2k\pi x)$ and $(\cos(2k\pi x)$ are such functions.

By superposition we mean sum. Thus the function $f + g$ is superposition of f and g. Generally, you do not use this term. This originates in music, where a sound is superposition of some basic sounds; you may call 'notes' and so on. This phrase originates again in differential equations where if you have two solutions f and g of an equation like $f'' + f = 0$, then their sum is also a solution — just as if you have two vectors which are solutions of a homogeneous matrix equation $Av = 0$, then so is their sum. For the differential equation written above you see 'basic' solutions are $\sin x$ and $\cos x$ and other solutions are just linear combinations of these two. This is irrelevant for us now. I am only trying to explain the word 'superposition'.

Thus the statement of Fourier amounts to saying that any periodic function of period one is a sum of linear combinations of the functions: $\sin 2\pi kx$ and $\cos 2\pi kx$. Thus mathematically, the statement amounts to saying that given periodic function of period one, there are numbers (a_n) and (b_n) such that

$$f(x) = a_0 + (a_1 \sin 2\pi x + b_1 \cos 2\pi x) + (a_2 \sin 4\pi x + b_2 \cos 4\pi x) + \cdots$$

$$+(a_k \sin 2k\pi x + b_k \cos 2k\pi x) + \cdots.$$

Such a series is called Fourier series.

I am carrying the baggage $2\pi k$ instead of k because I am wanting period one. If you want period 2π then you just look at $\sin kx$ and $\cos kx$. It is a matter of standardization. If you want to take the interval $[0, 37]$ then we consider the functions $\sin(2\pi kx/37)$ and $\cos(2\pi kx/37)$. Of course $k = 0, 1, 2, 3, \cdots$..

There are several problems. What is the meaning of the series? How should we understand it? In what sense is the equality to be understood? Is such a thing true? If not every function; which functions are like this? So on.

Prior to Fourier problems concerning series of sines and cosines did arise in understanding waves and sounds. Fourier was studying, in a systematic manner, conduction of heat and arrived at this problem. You should recall that last semester we did discuss heat equation, found out that the 'normal density'

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

is fundamental solution, in a precise sense. If you remember, the problem was to describe at every time $t > 0$, the distribution of heat in an (two sided) infinite rod when you know the initial distribution of heat on the rod. The problems for a finite rod are more difficult.

Before making systematic study of the above type of series, let me add most of mathematics originated here. In fact understanding certain problems in Fourier series was the genesis of Set Theory by Cantor. Understanding these issues are behind even in matters like convergence of series and so on. Of course, series were used earlier too.

Exponentials:

The whole theory (at least for us now) can be regarded as an extension of what we learned about C^n earlier. Let us start considering $C[0,1]$. For the time being forget about periodic etc. So it is not necessary to have $f(0) = f(1)$ at this moment —later when theorems appear, we need. But the change of attitude is that *we consider complex valued functions.*

Let us quickly recall. $f : [0,1] \to C$ then for each x, $f(x)$ is a complex number and hence is $f_1(x) + if_2(x)$ where $f_1(x)$ and $f_2(x)$ are real numbers. Thus given a complex valued function, there are two real valued functions f_1 and f_2 so that $f = f_1 + if_2$. The function f_1 is called the real part of f and f_2 is called the imaginary part of f. You must note that these functions are real valued.

Conversely, given two real valued functions, the above equality provides you a complex function. Unless stated to the contrary, when we write $f = f_1 + if_2$ we mean that these are the real and imaginary parts. Also, bringing in the notion of convergence in C, you see that f is continuous iff its real part and imaginary part are continuous.

For later use, let us also make an observation. Suppose you have a sequence ($f^n = f_1^n + if_2^n$) of complex valued functions and a function $f = f_1 + if_2$. Then f^n converges to f uniformly iff the real parts and imaginary parts converge uniformly. This follows from a simple observation. For a complex number $z = a + ib$

$$\max\{|a|, |b|\} \quad \leq \quad |z| \quad \leq \quad |a| + |b|.$$

Thus if $f^n \to f$ uniformly, then $||f_1^n - f_1|| \leq ||f^n - f||$ and hence $f_1^n \to f^1$ uniformly. Similarly $f_2^n \to f_2$ uniformly.

conversely, if the real and imaginary parts converge uniformly, then

$$||f^n - f|| \leq ||f_1^n - f_1|| + ||f_2^n - f_2||$$

showing that $f^n \to f$ uniformly.

We define for any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Then by induction and using the sine and cosine formulae, we can show

$$e^{in\theta} = \cos n\theta + i\sin n\theta.$$

This is called De Moivre's formula and you must have seen it in high school.

Actually, one defines for every complex number z, the series

$$\exp\{z\} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

This series converges. Indeed take the disc $\{|z| \leq M\}$. Let $S_n(z)$ be the n-th partial sum of the above series. Then the sequence $\{S_n(z)\}$ is uniformly Cauchy in the disc because for $n \geq m$

$$|S_n(z) - S_m(z)| \leq \sum_{m+1}^{n} \frac{M^k}{k!} \to 0 \ \text{ as } \ m, n \to \infty.$$

If you take θ to be real number, then the above definition along with the fact that $i^2 = -1$ tells us

$$\exp\{i\theta\} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + (-1)^n \frac{\theta^{2n}}{(2n)!} + \cdots$$

$$+i\left\{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right\}$$

$$= \cos\theta + i\sin\theta$$

where the last identification is from the expressions we obtained in our first course for the sine and cosine functions. Thus the definition we used above is same as the natural one defined just now. Further, if z has zero imaginary part, that is, z is real, then this definition coincides with what we learnt earlier.

One can prove Cauchy theorem on products of series and show, exactly as in the real case, that

$$\exp\{z+w\} = \exp\{z\}\exp\{w\}$$

justifying the use of the notation e^z for $\exp\{z\}$. This gives another proof of the De Moivre formula. But we need not depend on the unproved Cauchy rule.

Recall that integration is done by using the real and imaginary parts. Thus for $k \neq 0$,

$$\int_0^1 e^{2\pi ikt}dt = \int_0^1 \cos(2\pi kt)dt + i\int_0^1 \sin(2\pi kt)dt = 0.$$

$C[0,1]$ **as** C^n **for Huge** n**:**

Let us from now consider the space of complex valued continuous functions on $[0,1]$. This will be $C[0,1]$ during the remaining part of our discussion. This will replace C^n, as you will see.

We define some special functions on $[0,1]$ as follows.

$$e_n(t) \quad = \quad e^{2\pi int}, \quad n = 0, \pm1, \pm2\cdots.$$

The calculation of integral above can be recast in terms of these functions.

$$\int_0^1 e_n(t)\overline{e_m(t)}dt = 0; \quad if \ \ n \neq m; \quad = 1 \ \ if \ \ n = m.$$

We defined on C^n inner product

$$\langle z, w\rangle = \sum z_k\overline{w_k}; \quad z = (z_1, \cdots, z_n); \ \ w = (w_1, \cdots, w_n).$$

Analogously let us define an inner product on the vector spar $C[0,1]$ (remember complex valued continuous functions)

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$$

This is indeed an inner product:

linear in the first argument f,
conjugate linear in the second argument g;
$\langle f, f\rangle = \int |f|^2 \geq 0$
$\langle f, f\rangle = 0$ iff $f = 0$ (remember continuity of f)
$\langle f, g\rangle = \overline{\langle g, f\rangle}$ because $\int \overline{\varphi} = \overline{\int \varphi}$.

Moreover the integrals of exponentials we evaluated above, tells us that the family $\{e_n : -\infty < n < \infty\}$ is an orthonormal system.

The only thing is that in C^n we have a finite orthonormal basis. If you found n orthonormal vectors, then there are no more to extend this system. If you found only m orthonormal vectors and $m < n$ you can find more. You should actually find more because you can not express every vector as linear combination of just these m vectors.

In the present vector space we have found an infinite system of orthonormal vectors. It is not clear if there are any more that can be added to this list. Just as we expanded an vector in C^n in terms of an orthonormal system, we can try to expand a function in terms of this orthonormal system. Hopefully

$$f = \sum_{-\infty}^{\infty} c_k e_k.$$

Back to Fourier Series:

The question then is whether such an expansion is possible and what has it got to do with Fourier series. Well, actually, this is nothing but Fourier series. We started out with series of sines and cosines. We now have exponentials. But obviously $\{\sin nt; \cos nt\}$ can be expressed as linear combinations of $\{\exp(int); \exp(-int)\}$ and vice versa. Thus linear combination of exponentials and linear combination of sines and cosines are exactly the same.

So let us now turn our attention to the set up we have settled upon, namely the vector space $C[0,1]$ with inner product $\int f\overline{g}$ and the orthonormal basis $\{e_n : -\infty < n < \infty\}$.

Suppose we can express $f = \sum c_n e_n$; without even knowing the sense in which this infinite sum is to be interpreted. Then the first question is what should be the coefficients? Going by our understanding of the finite dimensional case we would guess

$$c_n = \langle f, e_n \rangle = \int_0^1 f(t)e^{-2\pi int}dt.$$

Notice that conjugate of e_n is e_{-n}.

Of course even if we did not know the finite dimensional case, as far as guess is concerned we would feel

$$\int f\overline{e_n} = \int (\sum c_k e_k)\overline{e_n} = \sum c_k \int e_k \overline{e_n} = c_n.$$

where we used the fact that the system (e_n) is orthonormal. Of course whether we can interchange the infinite sum with integral is unclear; any way the meaning of the infinite sum itself is unclear. This is only a thought process to guess what the coefficients should be.

Now we define Fourier series of a function $f \in C[0,1]$ to be the series

$$\sum \hat{f}_k e_k; \qquad \hat{f}_k = \int_0^1 f(x)e^{-2\pi ikx}dx.$$

The numbers $\hat{f}_k$ are called the Fourier coefficients of f; and more precisely this is the k-th Fourier coefficient.

Of course, yet there is no meaning for the infinite series. There are several ways of giving meaning. The right stage for this drama is the Lebesgue integral, but we shall not enter that stage. We shall continue to work with familiar continuous functions and familiar Riemann integral.

Towards giving a meaning to the series, let us define partial sums

$$S_N(x) = \sum_{-N}^{N} \hat{f}_k e^{2\pi ikx}; \quad N \geq 1.$$

Following usual procedure of giving a meaning to an infinite series, we ask if the above sequence of partial sums converges.

The fact that we have taken 'symmetric' partial sums will hurt you. But do not worry. We can deal other sums as well after the smoke clears. But in any case that should not be main issue.

Let us first think of uniform convergence. Whether point wise or uniform, one thing is clear. All e_n and all S_N are period one functions. So limit, when exists, is also of period one. Thus we are advised to restrict to period one functions — if we are aiming at uniform convergence.

Here is the first main theorem of the theory.

Theorem: Suppose f is a period one function with (period one) continuous derivative. Then S_N converges to f uniformly.

Let me add period one simply means $f(0) = f(1)$. Its derivative is period one means $f'(0) = f'(1)$. This is same as saying that when you extend f to all of R as period one functions then it is continuously differentiable on R. If you are restricting your attention to $[0, 1]$ then differentiability does not mean derivative is period one, even though original function is so. For example $f(x) = (x - 1/2)^2$ is a period one function, takes same values at zero and one. It is C^1 function on the interval $[0, 1]$. However $f'(0) = -1 \neq +1 = f'(1)$.

We start with some general observations, trying to imitate the finite dimensional case. Let $f \in C[0, 1]$ *not necessarily period one function.* not necessarily differentiable. Recall that now the space $C[0, 1]$ has metric d;

$$d^2(f, g) = \langle f - g, f - g\rangle = \int_0^1 |f - g|^2.$$

1^o. Does S_N capture 'all of f' in the span of $\{e_k : -N \leq k \leq N\}$? Yes.

Denote this subspace by L. Then we claim that $f - S_N \perp L$. to see this take any $h = \sum c_k e_k$ in L.

$$\langle f - S_N, \sum c_k e_k\rangle = \langle f, \sum c_k e_k\rangle - \langle \sum \hat{f}_k e_k, \sum c_k e_k\rangle$$

$$= \sum \overline{c_k} \langle f, e_k\rangle - \sum \hat{f}_k \overline{c_k} = 0$$

Remember we had exactly the same result with same proof in C^n.

2^o. S_N is the closest to f in the span of the vectors $\{e_k : -N \leq k \leq N\}$, this sub space is denoted by L above.

To see this take any $h = \sum c_k e_k$ in this subspace. Need to show

$$||f - S_N|| \leq ||f - h||.$$

To show this observe $f-h=(f-S_N)+(S_N-h)$ and $S_N-h\in L$ so that this is orthogonal to $f-S_N$ from 1^o. Thus pythogoras tells

$$||f-h||^2=||f-S_N||^2+||S_N-h||^2\geq||f-S_N||^2$$

with equality iff $S_N=h$.

In the above calculation, by pythogoras we mean, if $f\perp g$ then

$$\begin{aligned}||f+g||^2&=\langle f+g,f+g\rangle=\langle f,f\rangle+\langle f,g\rangle+\langle g,f\rangle+\langle g,g\rangle\\&=||f||^2+||g||^2\end{aligned}$$

3^o.

$$\sum_{-N}^{N}|\hat{f}_k|^2\leq||f||^2=\int_0^1|f(x)|^2dx.$$

This is known as *Bessel's inequality.*

This is immediate from the fact

$$f=(f-S_N)+S_N;\quad f-S_N\perp S_N$$

and Pythogoras to get

$$||f||^2=||f-S_N||^2+||S_N||^2;\quad ||S_N||^2\leq||f||^2$$

But

$$||S_N||^2=\langle\sum\hat{f}_ke_k,\sum\hat{f}_ke_k\rangle=\sum\hat{f}_k\overline{\hat{f}_k}=\sum|\hat{f}_k|^2.$$

Same result was in C^n. Now you go back and see 'direct computation' i mentioned there and how it does not make sense here.

4^o. For any $f\in C[0,1]$;

$$\sum_{-\infty}^{\infty}|\hat{f}_k|^2\leq||f||^2=\int_0^1|f|^2.$$

This immediately follows from the inequality for all finite sums obtained above.

Observe that convergence of the series tells us that for any $f\in C[0,1]$ $\hat{f}_k\to0$ as $k\to\pm\infty$.

5^o. Let now f and f' be period one functions. Then

$$\widehat{f'}_k=-2\pi ik\hat{f}_k.$$

This is a simple consequence of integration by parts.

$$\widehat{f'}_k = \int_0^1 f'(x)e^{-2k\pi ix}dx = f(x)e^{-2k\pi ix}\Big|_0^1 - \int_0^1 f(x)(-2k\pi i)e^{-2k\pi ix}dx.$$

The first term on right side is zero because the functions take same values at both end points.

Thus differentiating f is transformed to multiplication in the Fourier domain.

This is an extremely powerful result as the next observation shows.

6^0. Let f be as in the theorem. Then the sequence (S_N) is uniformly Cauchy.

Take $n < m$ positive integers, then

$$|S_n(x) - S_m(x)| = |\sum_{n<|k|\le m} \hat{f}_k e_k| \le |\sum_{n<|k|\le m} |\hat{f}_k|$$

$$\le \left|\sum_{n<|k|\le m} \widehat{f'}_k \frac{1}{2\pi ik}\right|$$

$$\le \frac{1}{2\pi}\sqrt{\sum_{n<|k|\le m} |\widehat{f'}_k|^2}\ \sqrt{\sum_{n<|k|\le m} 1/k^2}$$

$$\le \frac{1}{2\pi}\sqrt{\int |f'|^2}\ \sqrt{\sum_{|k|>n} 1/k^2}.$$

Denote $M = \sqrt{\int |f'|^2}/(\pi)$ we have

$$\sup_x |S_n(x) - S_m(x)| \le M\sqrt{\sum_{k>n} 1/k^2}$$

Since the series $\sum 1/k^2$ is convergent the tail sums can be made small and thus the right side above can be made small for all large values of n.

This shows that the sequence of functions (S_N) is Cauchy uniformly.

7^o. If we now show that the sequence S_N converges point wise to f we would have proved our theorem. To this end, let us understand the partial sums.

$$S_N(x) = \sum_{-N}^{N} \hat{f}_k e_k = \sum \int f(y)e^{-2\pi kiy}dy\ \ e^{2\pi ikx}$$

$$= \int f(y) \sum_{-N}^{N} e^{2\pi i k(x-y)} dy.$$

Let us denote

$$D_N(\theta) = \sum_{-N}^{N} e^{2\pi i k\theta}.$$

Then we have

$$S_N(x) = \int_0^1 f(y) D_N(x-y) dy.$$

Using formula for sum of finite geometric series,

$$D_N(\theta) = e^{-2\pi i N\theta} \frac{1 - e^{2\pi i(2N+1)\theta}}{1 - e^{2\pi i\theta}}.$$

When $\theta = 0$, this is to be interpreted as $(2N+1)$.

$$D_N(\theta) = \frac{e^{-2\pi i N\theta} - e^{2\pi i(N+1)\theta}}{1 - e^{2\pi i\theta}}$$

$$= \frac{e^{-2\pi i(N+1/2)\theta} - e^{2\pi i(N+1/2)\theta}}{e^{-\pi i\theta} - e^{\pi i\theta}}$$

$$= \frac{\sin \pi(2N+1)\theta}{\sin \pi\theta}.$$

As a consequence of the orthogonality of the exponential functions, we also get from the above summation,

$$\int_0^1 \frac{\sin \pi(2N+1)\theta}{\sin \pi\theta} d\theta = 1.$$

Thus

$$S_N(x) = \int_0^1 f(y) \frac{\sin \pi(2N+1)(x-y)}{\sin \pi(x-y)} dy$$

Change the variable, notice integrand is a period one function so that you need not worry about range of integration, any interval of length one would give the same answer.

$$S_N(x) = \int_{-1/2}^{1/2} f(x-y) \frac{\sin \pi(2N+1)y}{\sin \pi y} dy.$$

Let us from now on fix a point x. We shall show $S_N(x)$ converges to $f(x)$. Remembering that D_N integrates to one we see

$$S_N(x) - f(x) = \int_{-1/2}^{1/2} [f(x-y) - f(x)] \frac{\sin \pi(2N+1)y}{\sin \pi y} dy.$$

Observe that the function

$$\varphi(y) = \frac{f(x-y) - f(x)}{\sin \pi y}$$

is a nice continuous function on $[-1/2, 1/2]$. Its value at zero is

$$\varphi(0) = f'(x) \cdot \frac{1}{\pi} \cdot (-1).$$

Thus we can write

$$S_N(x) - f(x) = \int \varphi(y) \left[e^{\pi i(2N+1)y} - e^{-\pi i(2N+1)y} \right] \frac{1}{2i} dy$$

$$= \int \varphi(y) e^{\pi i y} \frac{1}{2i} \ e^{2N\pi i y} dy - \int \varphi(y) e^{-\pi i y} \frac{1}{2i} \ e^{-2N\pi i y} dy$$

The first term on right side is $(-N)$-th Fourier coefficient of some continuous function and hence converges to zero as $N \to \infty$. Similarly, second term is N-th Fourier coefficient of a continuous function which again converges to zero as $N \to \infty$.

This shows that for each x, $S_N(x) - f(x) \to 0$.

Since (S_N) is already shown to be uniformly Cauchy, we conclude that S_N converges to f uniformly.

This completes proof of the theorem.

8^o. Let f be any function as in the proof of the theorem, that is, both f and f' are continuous of period one.

$$S_N \to f \qquad \text{uniformly}$$

Thus

$$\overline{S_N} \to \overline{f} \qquad \text{uniformly}$$

Hence

$$|S_N|^2 = S_N \overline{S_N} \to |f|^2 \qquad \text{uniformly}$$

And thus finally

$$\int_0^1 |S_N|^2 \to \int_0^1 |f|^2.$$

But

$$||S_N||^2 = \sum_{-N}^{N} |\hat{f}_k|^2 \to \sum_{-\infty}^{\infty} |\hat{f}_k|^2.$$

so we conclude

$$\sum |\hat{f}_k|^2 = \int_0^1 |f|^2.$$

This is called *Plancherel equality.* remember we have proved an inequality earlier, for all functions f, not necessarily f of the theorem.

9^o Since polynomials are dense, it is reasonable to believe that the above equality remains to hold for all f, not necessarily for f satisfying the theorem. This is indeed true and we shall prove shortly. But why are we interested in this? Let me convince you of its importance.

10^o.

Consider the function

$$f(x) = x, \qquad 0 \le x \le 1.$$

This is a continuous function and of course not periodic, takes different values at zero and one. However from what we said above,

$$\sum |\hat{f}_k|^2 = \int_0^1 |f|^2.$$

Let us see what it means.

$$\int_0^1 x^2 dx = 1/3.$$

$$\hat{f}_0 = \int_0^1 x dx = 1/2.$$

For $k \neq 0$, integration by parts gives

$$\int_0^1 x e^{-2\pi i k x} dx = -\frac{1}{2\pi i k}$$

Thus we have

$$\frac{1}{4} + 2\sum_1^\infty \frac{1}{4\pi^2 k^2} = \frac{1}{3}$$

In other words

$$\sum_1^\infty \frac{1}{k^2} = \frac{\pi^2}{6}.$$

It is very interesting, in the very first course we showed that the series $\sum(1/k^2)$ converges, but we had to wait for an year to find out what the sum is.

One can use this method to evaluate sum of even powers of $1/k$. for example

$$\sum_1^\infty \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Try your hands on it.

Unfortunately, we still do not know sum of odd powers.

It remains to show that Plancherel equality hods for all $f \in C[0,1]$. This is not too difficult but needs some careful analysis.

The first step consists of understanding the map that sends f to its Fourier coefficients. Let $X = C[0,1]$, space of complex valued continuous functions on the interval $[0,1]$. We equip it with the metric

$$d(f,g) = \int_0^1 |f-g|^2.$$

Note we are not taking sup metric.

Let $Y = l_2$ be the space of all complex sequences $z = (z_n; -\infty < n < \infty)$ such that $\sum |z_n|^2 < \infty$. We considered this space earlier but there are two main differences. First, earlier we considered only real sequences. Now we are forced to consider complex sequences. So let us do, we have no choice! Second difference is that earlier we considered one sided sequences, but now we are forced to consider two sided sequences. that is, functions defined on Z rather than on N. We have no choice!

You pause for a moment and make sure that you understand. Luckily, a series of positive numbers converges iff the (finite) partial sums are bounded. Then the sup of the partial sums is the sum of the infinite series. This is so even if you have two sided sequences. We said $\sum |z_n|^2 < \infty$. You can interpret the infinite sum as limit of the sequence

$$s_N = \sum_{-N}^{N} |z_k|^2.$$

Even if you take, for example,

$$t_N = \sum_{-2N}^{3N} |z_|^2$$

then this sequence t_N converges and converges to the same limit as above. But, if you are getting confused, you can just keep in mind the symmetric sums s_N. It is OK at first attempt.

The Cauchy Schwarz inequality still holds. After all, the C-S inequality for infinite sums was obtained as limit from the finite case. The only problem is that we proved for real numbers. Suppose you have complex numbers $(a_k : 1 \leq k \leq n)$ and $(b_k : 1 \leq k \leq n)$,

$$|\sum a_k b_k| \leq \sum |a_k||b_k| \leq \sqrt{\sum |a_k|^2}\sqrt{\sum |b_k|^2}.$$

Here the first inequality is simply that mod of sum is smaller than sum of mod. The second inequality is usual C-S for real numbers, no more complex numbers!.

It is the C-S that shows

$$d(z, w) = \sqrt{\sum_{-\infty}^{\infty} |z_k - w_k|^2}$$

is a metric on the space l_2.

Exactly as in the case of the l_2 that we considered, this is also complete. Take a Cauchy sequence; show each coordinate converges; using that it is Cauchy in your metric show that this coordinatewise limit is actually in your space (show finite partial sums are bounded); show that the convergence takes place in the l_2-metric.

Define the map $T : X \to Y$ by

$$Tf = \{\hat{f}_k : -\infty < k < \infty\}$$

We already showed, right after Bessel inequality, that for $f \in C[0, 1]$, the sum $\sum |\hat{f}_k|^2 < \infty$. In other words, $Tf \in l_2$. Let D be the set of all $f \in C[0, 1]$ that satisfy the theorem. We shall show that D is dense in the metric d on the space X. Clearly, D is a linear subspace of X. Also

$$T(f - g) = Tf - Tg$$

What we proved amounts to saying that the map T is an isometry of D into l_2.

To proceed further let us recapitulate something that we discussed already once. Let X and Y be two metric spaces and Y be complete. Suppose D is a dense subset of X and T is an isometry of D to Y. Then we can extend T as an isometry of X to Y. Proof is trivial. Take any $x \in X$. Since D is dense, take $p_n \in D$ such that $p_n \to x$. Hence (p_n) is Cauchy; use T is isometry to conclude (Tp_n) is Cauchy in Y; use Y is complete to get its limit and declare it as Tx. This is good definition because if you take $q_n \to x$ in D; then $d(p_n, q_n) \to 0$; T being isometry $d(Tp_n, Tq_n) \to 0$; thus (Tp_n) and (Tq_n) have same limit. Finally, this extension is isometry because if you take x and y and $p_n \to x$ and $q_n \to y$ then using the fact that metric is 'continuous' we have

$$d(Tx, Ty) = \lim d(Tp_n, Tq_n) = \lim d(p_n, q_n) = d(x, y).$$

(Did you realize we are using the same symbol d for metric everywhere!)

Returning to our problem at hand, the Fourier coefficient map T from $D \subset C[0,1]$ to l_2 is an isometry and hence can be uniquely extended as an isometry on all of $C[0,1]$ to Y. Interestingly, we already have the existing map T defined on all of $C[0,1]$. Is this extension same as the existing map? If so it is an isometry already and this is exactly what we need!

We shall sort out this next time and complete this discussion.