Instructions: Full credit will be given only for precise answers. Make appropriate assumptions.

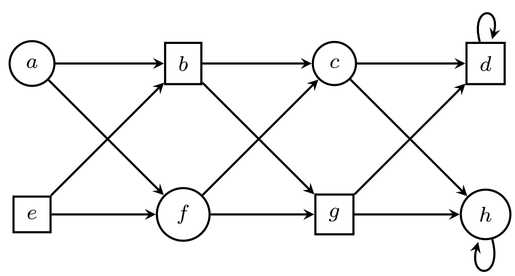
1. Let $\Sigma = \{0, 1\}$. Construct S1S formulas $\phi_1(X)$ and $\phi_2(X)$ for the languages L_1 and L_2 respectively:

$$L_1 = \Sigma^* 111 \Sigma^\omega$$
$$L_2 = (0110)^* 1^\omega$$

(A set $M \subseteq \mathbb{N}$ is an interpretation of $\phi_1(X)$ iff the characteristic vector of M is in L_1. Similarly for L_2).

(5 marks)

2. In the following reachability game, Player 0 wins plays that can reach the node with label d. Compute the winning regions for each player. What are their respective strategies? **(3 marks)**



3. Find a family of game arenas $(\mathcal{A}_n)_{n\geq 1}$ with designated nodes $(F_n)_{n\geq 1}$ for the Büchi winning condition, such that $Recur_0^{i+1}(F_n)$ is a strict subset of $Recur_0^{i}(F_n)$, that is: $Recur_0^{i+1}(F_n) \subset Recur_0^{i}(F_n)$ for $i = 1, \ldots, n$. Here $Recur_0^{i}(F_n)$ is the set of nodes from which Player 0 can force the play to take at least i visits to nodes in F_n. **(4 marks)**

4. a) Construct a non-deterministic Büchi tree automaton for the following tree language over $\{a, b\}$:

$$T_1 := \{\ t \in T^\omega_{\{a,b\}} \mid \text{there exists a path in } t \text{ labelled } ab^\omega\}$$

(in other words: root node should be labelled a and there is a path with all nodes (other than root) labelled b) **(4 marks)**

b) Can you construct a deterministic Büchi tree automaton for the above language? Justify.

(5 marks)

5. Let L be an ω-regular language of infinite words over an alphabet Σ. Construct a Muller tree automaton for the following language: **(4 marks)**

$$T_2 = \{\ t \in T^\omega_\Sigma \mid \text{ every path in } t \text{ is labelled by a word in } L\}$$

6. A directed graph can be thought of as a relational structure (V, E) where V is a finite set of elements, and $E \subseteq V \times V$ is a binary relation on the domain V. Properties in graphs can be expressed using Monadic Second Order logic (MSO) where the first order variables are quantified over V and the second order variables are over sets in V. For example, the following sentence is true on graphs that have a triangle:

$$\text{ContainsTriangle} := \exists x, y, z.\ (x \neq y \neq z) \wedge E(x, y) \wedge E(y, z) \wedge E(z, x)$$

The above formula is in fact a first-order formula. Using second order quantification allows to express more properties of graphs. Write MSO sentences that evaluate to true iff:

a) the graph is connected, **(4 marks)**

b) the graph is 3-colourable (vertices can be coloured so that vertices connected by an edge have different colours). **(4 marks)**

7. Let $\mathcal{A} = (V, E)$ be a game arena. Let $\chi : V \mapsto \{0, 1, 2, \ldots, c\}$ be a coloring function. Consider a *Streett* game $G = (\mathcal{A}, \chi, Acc)$ with $Acc = \{(E_1, F_1), \ldots, (E_m, F_m)\}$. Each $E_i \subseteq \{0, 1, 2, \ldots, c\}$ and $F_i \subseteq \{0, 1, 2, \ldots, c\}$. Recalling Streett acceptance: Player 0 wins a play ρ if for all $i = 1, \ldots, m$:

$$\mathrm{Inf}(\chi(\rho)) \cap F_i \neq \emptyset \;\Rightarrow\; \mathrm{Inf}(\chi(\rho)) \cap E_i \neq \emptyset$$

a) Give an example of a Streett game in which Player 0 needs at least 2 memory states to win: in other words any strategy automaton that represents a winning strategy for Player 0 will have at least 2 states. Give an informal (but complete) argument as to why she needs at least this much memory to win. **(6 marks)**

b) Is it possible to transform a (max) parity game $(\mathcal{A}, \chi)$ with arena $\mathcal{A}$ and colouring function χ to a Streett game $(\mathcal{A}, \chi, Acc)$ with the same arena and the colouring function? If yes, you need to give the acceptance condition Acc. If not, you need to give an argument why you think you cannot.

(4 marks)

8. This question is in the form of a proof. You need to fill in the missing details.

Let $\mathcal{A} = (V, E)$ be a game arena. Let V be partitioned into Player 0 nodes V_0 and Player 1 nodes V_1. Also assume that $\mathcal{A}$ does not have dead ends: that is, players always have a next move possible in the game. Let $\chi : V \mapsto \{0, 1, 2, \ldots, c\}$ be a coloring function. Consider a *Rabin* game $G = (\mathcal{A}, \chi, Acc)$ with $Acc = \{(E_1, F_1), \ldots, (E_m, F_m)\}$. Each $E_i \subseteq \{0, 1, 2, \ldots, c\}$ and $F_i \subseteq \{0, 1, 2, \ldots, c\}$. Recalling Rabin acceptance: Player 0 wins a play ρ if for there exists an $i \in \{1, \ldots, m\}$ such that:

$$\mathrm{Inf}(\chi(\rho)) \cap E_i = \emptyset \text{ and } \mathrm{Inf}(\chi(\rho)) \cap F_i \neq \emptyset$$

Claim: In Rabin games, Player 0 has a memoryless winning strategy in her winning region.

Proof of claim: Let W_0 and W_1 be winning regions of Player 0 and 1 respectively. Pick an arbitrary node $v \in V$. We will show:

$$(*) \quad \text{if Player 0 does not have a memoryless winning strategy from } v\text{, then } v \in W_1$$

We will prove (*) by induction on the number of edges n controlled by Player 0: that is, induction on $|E \cap V_0 \times V|$.

a) *Base case:* Prove that (*) is true when $n = |V_0|$. **(3 marks)**

b) *Inductive case:* Assume $n > |V_0|$. There exists a node $q \in V_0$ that has two exiting edges e_1 and e_2. Consider the Rabin game (with same Acc) over the arenas $\mathcal{A}_1 = (V, E - \{e_1\})$ and $\mathcal{A}_2 = (V, E - \{e_2\})$ with one of these edges removed in each. Pick a node $v \in V$.

   i) Show that if Player 0 does not have a memoryless winning strategy from v in the game $\mathcal{A}$, then she does not have a memoryless winning strategy from v in the two smaller games $\mathcal{A}_1$ and $\mathcal{A}_2$ as well.

   (3 marks)

   ii) Applying induction hypothesis on the smaller games, if Player 0 does not have memoryless winning strategies from v in $\mathcal{A}_1$ and $\mathcal{A}_2$, then Player 1 can win from v in both these games $\mathcal{A}_1$ and $\mathcal{A}_2$ (using some strategy). Now, show that if Player 1 can win from v in $\mathcal{A}_1$ and $\mathcal{A}_2$, then she can win from v in $\mathcal{A}$ as well. **(6 marks)**