We shall prove the implicit function theorem. Recall

Theorem (Implicit function theorem)

Let $f : \Omega \subset R^2 \to R$ be a C^1 function. Suppose $(a, b) \in \Omega$ Suppose $f(a, b) = 0$ and $f_2(a, b) \neq 0$. Then

(i) there is a rectangle $Q = (a-\delta, a+\delta) \times (b-\eta, b+\eta) \subset \Omega$ and a unique function φ defined on the interval $(a - \delta, a + \delta)$ whose graph is contained in the rectangle Q and such that $f(x, \varphi(x)) \equiv 0$.

(ii) This function is differentiable.

(iii)

$$\varphi'(x) = -\frac{f_1(x, \varphi(x))}{f_2(x, \varphi(x))}.$$

Let us assume without loss of generality that $f_2(a, b) = 2m > 0$. Since f is C^1, we can fix a rectangle $S = (a - \delta_1, a + \delta_1) \times (b - \eta, b + \eta)$ such that

$$[a - \delta_1, a + \delta_1] \times [b - \eta, b + \eta] \subset \Omega \text{ and } f_2(x, y) > m \text{ for } (x, y) \in S.$$

Here is the idea. Since $f_2 > 0$ on S, for each fixed $x \in (a - \delta_1, a + \delta_1)$, the function $y \mapsto f(x, y)$ is strictly increasing on $[b-\eta, b+\eta]$. If only we can ensure that it is of opposite sign at the end points, there will be exactly only number y in between where $f(x, y)$ takes the value zero and that number will be our $\varphi(x)$. this plan is executed by taking a suitable δ smaller than δ_1. This choice is made after seeing (or getting an idea of) the values $f(x, y)$ at the end points $y = b - \eta$ and $y = b + \eta$.

Choose $M > 0$ so that $|f_1(x, y)| < M$ and $|f(x, y)| < M$ on S. This is possible because these are continuous functions on the closed rectangle.

Let us take one x, $a - \delta_1 < x < a + \delta_1$. Using $f(a, b) = 0$ we have

$$f(x, b+\eta) = f(x, b+\eta) - f(x, b) + f(x, b) - f(a, b) = f_2(P_2)\eta + f_1(P_1)(x - a)$$

by Mean value theorem (one variable), where P_2 and P_1 are points on appropriate lines, but within the rectangle we have. Thus

$$f_1(P_1) < M, a - \delta_1 < x < a \Rightarrow f_1(P_1)(x - a) \geq -M(a - x)$$

whereas

$$f_1(P_1) > -M, a < x < a + \delta_1 \Rightarrow f_1(P_1)(x - a) \geq -M(x - a)$$

Thus in any case

$$f(x, b+\eta) \geq m\eta - M|x-a|.$$

Let us choose $\delta > 0$ smaller than δ_1 and also smaller than $m\eta/M$. Then,

$$|x-a| < \delta \Rightarrow M|x-a| < m\eta \Rightarrow f(x, b+\eta) > 0.$$

Similarly, let us take an x, $a-\delta_1 < x < a+\delta_1$. Using $f(a,b)=0$ we have exactly as above,

$$f(x, b-\eta) = f(x, b-\eta) - f(x,b) + f(x,b) - f(a,b) = f_2(P_2)(-\eta) + f_1(P_1)(x-a)$$

so that

$$f_1(P_1) > -M, a-\delta_1 < x < a \Rightarrow f_1(P_1)(x-a) \leq -M(a-x)$$

whereas

$$f_1(P_1) < -M, a < x < a+\delta_1 \Rightarrow f_1(P_1)(x-a) \leq -M(x-a)$$

Thus in any case

$$f(x, b-\eta) \leq -m\eta + M|x-a|.$$

$$|x-a| < \delta < m\eta/M \Rightarrow M|x-a| < m\eta \Rightarrow f(x, b-\eta) < 0.$$

Thus with this choice of δ we have achieved what we wanted. Thus we have a function φ whose value at $x \in (a-\delta, a+\delta)$ is the unique $y \in (b-\eta, b+\eta)$ with $f(x,y)=0$. As a result we have $f(x, \varphi(x)) \equiv 0$.

shall now show φ is a continuous function on the interval $(a-\delta, a+\delta)$. Let us take two points x and $x+h$ in this interval. Then by the mean value theorem (two variable)

$$0 = f(x+h, \varphi(x+h)) - f(x, \varphi(x)) = f'(P) \cdot (h, \varphi(x+h) - \varphi(x)).$$

that is,

$$f_1(P)h + f_2(P)[\varphi(x+h) - \varphi(x)] = 0.$$

Thus

$$|\varphi(x+h) - \varphi(x)| \leq \frac{M}{2m}|h|.$$

This shows uniform continuity of the function φ.

We shall now show that φ is differentiable. The same calculation above gives

$$\frac{\varphi(x+h) - \varphi(x)}{h} = -\frac{f_1(P)}{f_2(P)}.$$

Of course the point P above depends on h, though not visible in the notation. It is on the line joining $(x+h, \varphi(x+h))$ and $(x, \varphi(x))$. As $h \to 0$ we see that this point converges to $(x, \varphi(x))$, we already know that φ is continuous. Thus

$$\lim_{h \to 0} \frac{\varphi(x+h) - \varphi(x)}{h} = -\frac{f_1(x, \varphi(x))}{f_2(x, \varphi(x))}.$$

Remember that $f_2 > 0$ for all points under consideration.

Of course, the formula above shows that φ is C^1 as well.

Uniqueness is clear in this rectangle $(a-\delta, a+\delta) \times (b-\eta, b+\eta)$ because there is only one point y for each x satisfying $f(x,y) = 0$.

This completes proof of the theorem.

Sometimes this is also stated as follows. One does not say unique function with graph contained in the rectangle. One states as follows: there is an interval $(a-\delta, a+\delta)$ and a unique function φ on this interval with

(i) $\varphi(a) = b$ and $f(x, \varphi(x)) = 0$ for all x in this interval.
Further this function satisfies conditions (ii) and (iii) above.

Uniform continuity:

We are trying to understand whether the new knowledge gained about functions of several variables tells us some new things about functions of one variable which we could not have seen last semester. Yes, sometimes a function of one variable may itself arise from functions of two variables. Before proceeding further, we need to understand uniform continuity.

Suppose that $S \subset R^2$ is a closed bounded set and $f : C \to R$ be a continuous function. Then f is uniformly continuous. That is, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$a, b \in S; \quad ||a - b|| < \delta \Rightarrow |f(a) - f(b)| < \epsilon.$$

Recall that continuity is also defined exactly the same way, but first we defined continuity at a point a and f is continuous if it is continuous at every point a. Thus for a continuous function, given a point a and $\epsilon > 0$, there is a $\delta > 0$ which may depend not only on ϵ but also on the point a so that

$$b \in S; \quad ||a - b|| < \delta \Rightarrow |f(a) - f(b)| < \epsilon.$$

Thus uniform continuity demands that given $\epsilon > 0$ we should be able to find one $\delta > 0$ that works for all points a.

We have seen continuous functions that are not uniformly continuous in case of R. similar examples can be constructed in R^2 and so on. The above theorem was proved for R and the same proof works even in R^2. here it is.

Let $\epsilon > 0$ be given. Suppose that we can not find $\delta > 0$ as stated. Then for each $n \geq 1$, we can find two points $a_n, b_n \in S$ so that $||a_n - b_n|| < 1/n$ but $|f(a) - f(b)| \geq \epsilon$. By taking subsequence if necessary, we can assume that the sequences (a_n) and (b_n) converge. But then they converge to the same point $v \in S$. But then continuity of f leads to $\lim\limits_n |f(a_n) - f(b_n)| = 0$ whereas each of these quantities are at least ϵ..

Maximum: $\max\limits_y f(x, y)$

Let $f(x, y)$ be a continuous function on a rectangle $S = [a, b] \times [c, d]$. Put

$$\varphi(x) = \max\{f(x, y) : c \leq y \leq d\}; \quad x \in [a, b].$$

then φ is a continuous function.

In fact given $\epsilon > 0$, using uniform continuity of f, select $\delta > 0$ such that

$$P, Q \in S; \quad ||P - Q|| < \delta \Rightarrow |f(P) - f(Q)| < \epsilon/2.$$

We claim that

$$x_1, x_2 \in [a, b]; \quad |x_1 - x_2| < \delta \Rightarrow |\varphi(x_1) - \varphi(x_2)| < \epsilon.$$

Indeed, for any y, choosing $P = (x_1, y)$ and $Q = (x_2, y)$ we see

$$f(x_2, y) - \epsilon/2 \leq f(x_1, y) \leq f(x_2, y) + \epsilon/2.$$

Taking supremum through out we get

$$\varphi(x_2) - \epsilon/2 \leq \varphi(x_1) \leq \varphi(x_2) + \epsilon/2.$$

That is

$$|\varphi(x_1) - \varphi(x_2)| < \epsilon.$$

This result can be interpreted as follows. We know that supremum of a finite number of continuous functions is continuous. We also now that this need not be true for a sequence of continuous functions. Here we have a family of continuous functions, namely one function $x \mapsto f(x, y)$ for each $y \in [c, d]$. We are asserting that supremum of this bunch of functions is again a continuous function.

integral of a function of two variables:

Let f be a continuous function on $S = [a, b] \times [c, d]$. Put

$$\varphi(x) = \int_c^d f(x, y)dy.$$

Then φ is a continuous function on $[a, b]$.

Given $\epsilon > 0$, choose $\delta > 0$ as above, using uniform continuity, so that

$$||P - Q|| < \delta \Rightarrow |f(P) - f(Q)| < \epsilon/(d - c).$$

If $|x_1 - x_2| < \delta$ then,

$$|\int_c^d f(x_1, y)dy - \int_c^d f(x_2, y)dy| \leq \int_c^d |f(x_1, y) - f(x_2, y)|dy < \epsilon.$$

This can be interpreted as follows. Sum of two continuous functions is continuous. But sum of a sequence of continuous functions need not be continuous. What we are saying here is the following. We have a bunch of continuous functions $x \mapsto f(x, y)$ one for each y. Their 'continuous sum' which you think of integral, is continuous.

differentiation under integral:

Let f be a continuous function on $S = [a, b] \times [c, d]$. Suppose that f_1 is a continuous function on S. Let

$$\varphi(x) = \int_c^d f(x, y)dy.$$

Then φ is C^1 and

$$\varphi'(x) = \int_c^d f_1(x, y)dy.$$

In symbols

$$\frac{d\varphi}{dx} = \int_c^d \frac{\partial f}{\partial x}(x, y)dy.$$

or

$$\frac{d}{dx}\int_c^d f(x,y)dy = \int_c^d \frac{\partial f}{\partial x}(x,y)dy.$$

This has the following interpretation. We know that sum of a finite number of differentiable functions is again differentiable and derivative of sum equals sum of derivative. We are here saying continuous analogue of this statement. Derivative of integral equals integral of derivative. This is also referred to, as the symbols suggest, interchange of the order of derivative and integration.

proof is not difficult. Since f_1 is uniformly continuous, given $\epsilon > 0$, choose $\delta > 0$ as earlier using $\epsilon/(d-c)$. Now take any x. Let $|h| < \delta$.

$$\left|\frac{\varphi(x+h)-\varphi(x)}{h} - \int_c^d f_1(x,y)dy\right|$$

$$= \left|\int_c^d [\frac{f(x+h,y)-f(x,y)}{h} - f_1(x,y)]dy\right|$$

But by mean value theorem (one variable) the fraction in the integrand equals $f_1(P)$ for some P. Since P is on the line joining (x,y) and $(x+h,y)$ and since $|h| < \delta$; we conclude that the integrand is at most $\epsilon/(d-c)$. This is true for each y to complete the proof.

Change of order of integration:

Let f be a continuous fundtion on $S = [a,b] \times [c,d]$. Then

$$\int_c^d \left[\int_a^b f(x,y)dx\right] dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx.$$

Note that both sides make sense because we already know that after one integration the function you get is again continuous function (of the remaining variable) and hence can be integrated.

These integrals are called repeated integrals. Thus if you are doing 're-peated integration' of a continuous function on a closed bounded rectangle, then the order does not matter.

A natural question should occur to you. Can we not imitate integration we learnt and adapt it to functions of two variables? For example divide the rectangle into smaller ones, calculate the upper and lower sums and so on? Yes, can and will be done. These are called double integrals.

To prove the theorem, put

$$\Phi(u,y) = \int_a^u f(x,y)dx; \quad a \le u \le b; \quad c \le y \le d$$

Fundamental theorem of calculus (one variable, anyway, this is the only one we have at this stage) tells us that (for each fixed y) Φ is differentiable w.r.t. x and

$$\frac{\partial}{\partial u}\Phi(u,y) = f(u,y).$$

In other words Φ and Φ_1 are continuous functions on S.

Using the earlier theorem we conclude that

$$\frac{d}{du}\int_c^d \Phi(u,y)dy = \int_c^d f(u,y)dy.$$

Integrating both sides from a to b we get

$$\int_c^d \Phi(b,y)dy - \int_c^d \Phi(a,y)dy = \int_a^b \int_c^d f(u,y)dydu$$

Noting $\Phi(a,y) = 0$ and substituting $\Phi(b,y)$ we get the stated equation.

All this is fine, but the most important cases are when the integrand is unbounded or when the range of integration is infinite — in other words improper integrals. In such cases the above results would not apply. However, we can use these special cases to prove more general theorems.

But why are we doing all this? Well, these results enable us to understand some functions better, enable us to evaluate some integrals which can not be evaluated by usual methods etc. For example let us consider the function

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt; \quad x \in (0,\infty).$$

We know that this is a well defined function, that is, for every $x > 0$ the integral does give us a number. It is pertinent to ask if this is a differentiable function and if so whether

$$\Gamma'(x) = \int_0^\infty e^{-t}t^{x-1}\log t \;\; dt; \quad x \in (0,\infty).$$

What about second derivative? There are several other interesting integrals that appear in practice.

At some stage, to understand things better we have specialised to functions of two variables. We should keep in mind that this is done only to facilitate understanding and these theorems hold for any number of variables.

There are several directions in which our course can proceed.

We can discuss analogues of the above theorems for improper integrals and understand their use.

We have mentioned about double integral. We can discuss its development parallel to what we have learnt last semester and its relation with repeated integrals we talked about. This brings us to discover change of variables formula. After all, just as last semester we had areas in mind, we can now ask abut finding volumes.

We can use our knowledge to understand geometry, nature of curves and surfaces.

Amidst all these we should not forget one important thing: We knew for one variable C^1 function f, if $f'(a) \neq 0$ then there is a small interval $I = (a - \delta, a + \delta)$ in which the function is one to one, takes you to another interval J and you can define inverse function g on this interval J and $g'(y) = 1/f'(x)$ if $f(x) = y$. What is its analogue for functions of two variables?

And so on!