holomorphic functions:

We have seen last time that if $f : C \to C$ is complex differentiable, then Cauchy-Riemann equations are satisfied by the real and imaginary parts of the function. We shall now show that conversely if two real functions u and v are C^1 functions satisfying the Cauchy Riemann equations then $f = u + iv$ is complex differentiable.

Functions which are complex differentiable are called holomorphic functions or analytic functions. As mentioned earlier, if $f : C \to C$ is differentiable at every point of a region $\Omega \subset C$ then it is differentiable any number of times. Not only that it has a power series expansion around every point. This means the following. if $z_0 \in \omega$ then there is an $r > 0$ such that the ball $S(z_0, r) \subset \Omega$ and there are numbers c_0 such that

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots; \qquad z \in S(z_0, r).$$

Functions having this property; namely given a point in the domain there is a neighbourhood around it where the function has power series representation; are called analytic functions. Thus differentiable functions are not only infinitely differentiable, but also have such power series representation. This is for $f : C \to C$.

For functions of real variables such results are not true. There are functions $f(x, y)$ of two variables which are differentiable just as many times as we want and no more. Even if it is infinitely differentiable, such power series expansions need not exist.

So let $\Omega \subset C$ be an open set and u, v be two real valued C^1 functions on Ω such that the Cauchy Riemann equations hold

$$u_1(x, y) = v_2(x, y); \quad u_2(x, y) = -v_1(x, y) \quad (x, y) \in \Omega.$$

We now show that

$$f(x + iy) = u(x, y) + iv(x, y)$$

is a holomorphic function.

Let us fix a $z \in C$. By the mean value theorem applied to u, there is some point ξ on the line joining the two points $z+h$ and z; that is the points $(x+h_1, y+h_2)$ and (x, y) such that

$$u(x+h_1, y+h_2) - u(x,y) = (u_1(\xi), u_2(\xi)) \cdot (h_1, h_2)$$

$$= u_1(z)h_1 + u_2(z)h_2 + \varphi(h)$$

where

$$\varphi(h) = (u_1(\xi) - u_1(z), u_2(\xi) - u_2(z)) \cdot (h_1, h_2)$$

so that

$$|\varphi(h)| \leq ||\nabla u(\xi) - \nabla u(z)|| \; |h|$$

Since u is C^1 and ξ is on the line joining $z+h$ and z we see that

$$h \to 0 \quad \Rightarrow \quad \varphi(h)/|h| \;\to\; 0.$$

similarly

$$v(x+h_1, y+h_2) - v(x,y) = u_1(z)h_1 + u_2(z)h_2 + \psi(h)$$

where

$$\psi(h)/|h| \;\to\; 0.$$

Thus,

$$f(z+h) - f(z) = u(z+h) - u(z) + i[v(z+h) - v(z)]$$

$$= u_1(z)h_1 + u_2(z)h_2 + \varphi(h) + i[v_1(z)h_1 + v_2(z)h_2 + \varphi(h)]$$

$$= [u_1(z) + iv_1(z)]h_1 + [v_2(z) - iu_2(z)]ih_2 + \varphi(h) + \psi(h)$$

using Cauchy Riemann equations

$$= [u_1(z) + iv_1(z)](h_1 + ih_2) + \varphi(h) + \psi(h)$$

Hence

$$\frac{f(z+h) - f(z)}{h} = u_1(z) + iv_1(z) + \frac{\varphi(h) + \psi(h)}{h}$$

since the second term on the right converges to zero as $h \to 0$ we conclude that f is complex differentiable with

$$f'(z) = u_1(z) + iv_1(z).$$

We can regard f as a function from R^2 to R^2, namely,

$$f(x,y) = (u(x,y), v(x,y)).$$

If f is complex differentiable then as a function from R^2 to R^2, it is differentiable and the derivative is given by

$$f'(x, y) = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1(x, y) & -v_1(x, y) \\ v_1(x, y) & u_1(x, y) \end{pmatrix}$$

where we have used he C-R equations. This is non-singular matrix unless both $f'(z) = 0$.

These simple results already have non-trivial consequences giving a glimpse into complex analysis. For instance if $f = u + iv$ and $g = u + iw$ are holomorphic on C (or on any connected open set) then v and w differ by a constant. This is because $f - g$ is holomorphic and its real part is zero and C-R equations tell imaginary part of $f - g$ also has zero partial derivatives and hence must be constant. Thus the real part (u) of a holomorphic function uniquely determines its imaginary part (v); of course, up to a constant.

similarly the imaginary part also determines its real part up to a constant.

In particular, if f is holomorphic and is purely real then it must be a constant. That is, if range f is contained in x-axis or y-axis, then f must be constant.

Can a holomorphic function (on C or on a connected open set) have range contained in the unit circle? That is $|f| \equiv 1$. No, unless it is a constant. If at some point $f'(z)$ is non-zero, then regarding f as a function from R^2 to R^2 combined with inverse function theorem tells that range f contains non-empty open sets. Of course, if the derivative is zero at all points then it is a constant.

Polynomials $P(z)$ are holomorphic. The function

$$f(z) = \overline{z},$$

conjugate of z is not holomorphic.

curves again:

Let us consider planar curves. let

$$\varphi : [a, b] \to R^2$$

be a curve. According to our definition, curves are continuous functions.

The curve φ is said to be simple if it is one-to-one except possibly on $\{a, b\}$. equivalently, if

$$s \neq t \in [a, b]; \varphi(s) = \varphi(t) \Rightarrow \{s, t\} = \{a, b\}.$$

Thus when you trace a simple curve, you do not pass through any point second time except when you reach the finish and finish with the starting point.

The curve φ is closed if $\varphi(a) = \varphi(b)$. Thus a closed curve ends at the starting point.

For example circle is a closed curve. If you trace it only once; $\varphi(t) = (\cos t, \sin t); 0 \leq t \leq 2\pi$; then it is simple closed curve. You can think of many examples of curves which are not simple or not closed.

If you think of simple closed curves, like, circle, triangle, polygon, and so on; you see that the plane is divided into three parts: points on the curve, outside the curve, inside the curve. Of course we used the word outside/inside by visual feeling. We can not exactly define these terms in general. Of course, if you take concrete curves like the ones mentioned above you can, in each specific case, precisely define.

Here is a theorem which is intuitively obvious but is non-trivial to prove. This is called Jordan curve theorem. Let φ be a simple closed plane curve. Then we can express

$$R^2 \;=\; I \;\cup\; \Gamma \;\cup\; E.$$

where (i) the union is disjoint; (ii) Γ is range of φ, that is, it is the curve; (iii) I and E are connected sets. Further such a decomposition is unique. Further exactly one of I and E is bounded and the other unbounded. If we think of I as the bounded part, then it is called the interior of the curve. The set E, unbounded component is called the exterior.

We have discussed C^1 curves φ and proved that they have length, given by the formula

$$L = \int_a^b ||\varphi'(t)||dt.$$

Many times we come across curves that are piece-wise smooth, but not smooth. A curve φ defined on $[a, b]$ is said to be piece-wise C^1 if the following holds. There are finitely many points

$$a = a_0 < a_1 < a_2 < \cdots < a_k = b$$

such that φ is C^1 on each piece $[a_i, a_{i+1}]$. Thus we demand left derivative at a_{i+1} and right derivative at a_i for each i. You can easily give examples (rectangle, triangle etc).

even for a piece-wise C^1 curve, length exists and is given by the same formula as above. The only thing you should take note is that at the points a_i the integrand possibly has two values (or has no value, depending on how you think). But it makes no difference. We have seen last semester that such bounded functions which are continuous except for finitely many discontinuities are integrable.

vector calculus again:

The concept of divergence $divF$ and curl $curlF$ are useful in discussing flow of fluids. Apparently, divergence measures tendency for the flow to dissipate/diverge in its plane of motion. On the other hand curl explains the tendency of the flow to move out of its plane of motion (like when it forms a whirlpool etc).

I shall only explain a nice interpretation of vector product for two vectors in R^3. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two vectors then their scalar product or inner product

$$|u \cdot v| = ||u||\ ||v||\ \cos\theta$$

where θ is the angle between the vectors. Similarly the vector product

$$||u \times v|| = ||u||\ ||v||\ |\sin\theta|.$$

This follows from

$$||u \times v||^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \quad - \quad (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$= ||u||^2\ ||v||^2\ \ [1 - (u \cdot v)^2].$$

normal integral again:

The theorems we learnt regarding differentiation under integral sign (Bounded interval) can be used to give a third method of evaluating the normal integral.

Consider the function

$$f(x) \;=\; \left(\int_0^x e^{-t^2/2}dt\right)^2 \;+\; 2\int_0^1 \frac{e^{-x^2(t^2+1)/2}}{t^2+1}dt.$$

Our theorems allow us to differentiate under the integral sign giving us, after differentiation $f' = 0$ so that f is a constant function. But you see $f(0) = \pi/2$. Thus f is the constant function $\pi/2$. But

$$\lim_{x\to\infty} f(x) = \left(\int_0^\infty e^{-t^2/2}dt\right)^2$$

giving us

$$\int_0^x e^{-t^2/2}dt = \sqrt{\pi/2}.$$

In the above calculation we used that the second integral converges to zero as $x \to \infty$. Of course the integrand converges to zero for every t, but this is not enough reason to conclude that the integral converges to zero. However the integrand decreases to zero point wise. In this case you can see directly that the integrand uniformly decreases to zero, it is smaller than $(1/2)\exp\{-x^2\}$.

This completes the justification for the second term converging to zero and hence evaluation of the normal integral.

It is interesting to note that if you have a sequence of continuous functions on $[0,1]$ that decrease to zero point wise then they do so uniformly. This is not difficult and is seen as follows.

Let $f_n \downarrow 0$ point wise. Let $\epsilon > 0$ be given. since the functions are decreasing, it is enough to show one N such that $f_N(x) < \epsilon$ for all x. Take any $x \in [0,1]$. since $f_n(x) \downarrow 0$, get $n(x)$ such that $f_{n(x)}(x) < \epsilon$. But this function $f_{n(x)}$ is a continuous function. So get an interval $I(x)$, such that $x \in I(x)$ and $f_{n(x)}(y) < \epsilon$ for all points y of $[0,1]$ which are in this interval $I(x)$. All these open intervals cover the compact $[0,1]$. Get finitely many of these intervals that cover, say $I(x_1), I(x_2), \cdots, I(x_k)$. it is easy to see that $N = \max\{n(x_1), n(x_2), \cdots, n(x_k)\}$.

This theorem is Dini's theorem.

uniform convergence of integrals:

We now make another attempt on understanding differentiation under the integral sign, when the integrals are over infinite interval. If we have a bounded interval,then uniform convergence was useful in establishing change of order of integration with integration/differentiation. Just now we saw how such things help us.

Let $f(x,y)$ be a continuous function on $[0,\infty)\times[c,d]$. Suppose that for each $y\in[c,d]$ the integral

$$\int_0^\infty f(x,y)dy \qquad (\spadesuit)$$

converges. Remember, this means

$$\lim_{A\to\infty}\int_0^A f(x,y)dx$$

converges to a number. This means that given $\epsilon>0$, here is a number A_0 such that

$$A>A_0 \;\;\Rightarrow\;\; |\int_A^\infty f(x,y)dx|<\epsilon.$$

We say that the integerl $(\spadesuit)$ converges uniformly over $[c,d]$ if given $\epsilon>0$, there is $A_0>0$ such that

$$A>A_0,\;\; y\in[c,d] \;\;\Rightarrow\;\; |\int_A^\infty f(x,y)dx|<\epsilon.$$

In other words the A_0 does not depend on the number y. Equivalently, the 'tail areas' (?) are uniformly small.

Since the condition that the tail area should be small involves again integral over infinite interval, usually the above condition is stated in the following equivalent form. the advantage is that it involves integral over finite interval. the complication is that you need to bring in A and B, two characters.

$$B>A>A_0,\;\; y\in[c,d] \;\;\Rightarrow\;\; |\int_A^B f(x,y)dx|<\epsilon.$$

Here are three useful theorems.

1. (**continuity**)

Suppose that $f:[0,\infty)\times[c,d]\to R$ is continuous. Suppose that $\int_0^\infty f(x,y)dx$ converges uniformly. Then the function

$$\varphi(y)=\int_0^\infty f(x,y)dx; \qquad y\in[c,d]$$

is c continuous function.

2. (**change of order of integration**)
Same conditions as above. Then

$$\int_c^d \varphi(y)dy = \int_0^\infty \left(\int_c^d f(x,y)dy\right) dx.$$

equivalently

$$\int_c^d \left(\int_0^\infty f(x,y)dx\right) dy = \int_0^\infty \left(\int_c^d f(x,y)dy\right) dx.$$

3. (**change of order of integration and differentiation**)

Same conditions as above. let us assume that for each x the function $y \mapsto f(x,y)$ is C^1 function on $[c,d]$ with derivative $g(x,y)$ and assume that the integral $\int_0^\infty g(x,y)dx$ is uniformly convergent. Then φ is differentiable and

$$\varphi'(y) = \int_0^\infty g(x,y)dx.$$

Equivalently,

$$\frac{d}{dy}\int_0^\infty f(x,y)dx = \int_0^\infty \frac{\partial f}{\partial y}(x,y)dx.$$

In other words, you can push the differentiation under the integral sign. Since the left side is a function of one variable y, we use d/dy. Since the integrand on right side is a function of two variables, we use $\partial/\partial y$.

First let us see some uses of these results.

Normal distribution again:

Evaluate

$$\varphi(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cos(tx)\, dx. \qquad (\clubsuit)$$

This integral arises in several areas of mathematics. Of course it arises in Probability and is called the characteristic function of the standard normal variable. strictly speaking the following is the characteristic function.

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, e^{itx}\, dx$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cos(tx)\, dx + i\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sin(tx)\, dx.$$

We have used the fact $\exp(i\theta) = \cos\theta + i\sin\theta$ for real numbers θ. The second integral on the right side exists (integrand being continuous and dominated by the normal integrand). But integrand being odd function the integral is zero. Thus only the first term remains.

This integral arises in Fourier analysis and is called the Fourier transform of the normal density function.

So how do we evaluate the integral. if only some one assures us that it van be differentiated under integral sign, then differentiating under the integral sign and evaluating the resulting integral by parts we see

$$\varphi'(t) = -t\varphi(t); \quad \varphi'(t) + t\varphi(t) = 0$$

multiplying by $\exp\{t^2/2\}$ we see

$$\left[e^{t^2/2}\varphi(t)\right]' = 0 : \quad \varphi(t) = Ce^{-t^2/2}.$$

Since $\varphi(0)$ can be explicitly evaluated and seen to be one we finally get

$$\varphi(t) = e^{-t^2/2}.$$

Heat equation:

Imagine an infinite rod, think of it as real line R. I supply a certain amount of heat, say $\varphi(y)$ at the point y of the rod. We assume that the function φ is a bounded continuous function on R. If you have any reservations about my supplying heat at *every* point of the *infinite* rod, you can assume that the function φ is zero outside a bounded interval.

How does it diffuse over time? How does it distributed over the rod? Thus, explain the amount of heat at time $t > 0$ at the point x of the rod. The answer is the following. Let $u(t, x)$ denote the amount of heat at time t at point x of the rod. Then

$$u(t, x) \text{ is a continuous function on } [0, \infty) \times R.$$

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t, x); \quad (t, x) \in (0, \infty) \times R.$$
$$u(0, x) = \varphi(x).$$

The equation

$$u_t = \frac{1}{2} u_{xx}$$

is called *heat equation* already derived by Newton. I have taken 1/2 on the right side, but generally it is taken as some constant c.

It is not difficult to verify that if

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} \quad (t,x) \in (0,\infty) \times R$$

satisfies the heat equation If you draw the normal curves for various values of t, you observe that as $t \downarrow 0$ the curves get more and more concentrated around zero. Thus it is plausible that this function gives the heat distribution if initial supply was unit amount at the point $y = 0$. Thus if you supply unit amount at the point y then you expect the heat distribution over time to be given by

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{(x-y)^2/2t} \quad (t,x) \in (0,\infty) \times R$$

If you supply amount $\varphi(y)$ at y you expect the distribution at time t to be

$$\int_{-\infty}^{\infty} p(t,x,y)\varphi(y)dy.$$

Thus the suggestion is

$$u(t,x) = \int_{-\infty}^{\infty} p(t,x,y)\varphi(y)dy; \quad u(0,x) = \varphi(x).$$

solves the problem of heat conduction. Yes it is true. Note that for each y the function $p(t,x,y)$ satisfies the heat equation as a function of $(t,x) \in (0,\infty) \times R$. That is

$$p_t = \frac{1}{2} p_{xx}.$$

If only someone allows you differentiate under the integral sign, then you see

$$u_t = \int p_t\, \varphi \qquad u_{xx} = \int p_{xx}\, \varphi$$

and hence conclude that

$$u_t = \frac{1}{2}\, u_{xx}.$$

Yes, the theorems stated above allow you to happily do this. Some more work needs to be done to see that $u(t,x)$ converges to $\varphi(x_0)$ as $(t,x) \to (0,x_0)$. We shall not do.

the main purpose of all this discussion is to explain to you why it is useful if some one tells you that you can differentiate before integrating (when you actually are suppose to differentiate after integrating.)