integrability:

we shall now prove the following result:

Q is a rectangle $[a, b] \times [c, d]$ and $f : Q \to R$ is a bounded function whose set of discontinuity points is a small set.
Then f is integrable.

Let $\epsilon > 0$. We shall exhibit Π such that $U(\Pi) - L(\Pi) < \epsilon$. Tis is done in a few simple steps. fix $M > 1$ such that $|f(p) \leq M$ for all points $p \in Q$. Let $\alpha = (b - a)(d - c)$, area of the rectangle Q.

First consider the set

$$D = \left\{ p \in Q : O(p) \geq \frac{\epsilon}{4M\alpha}. \right\}$$

we know that D is a closed bounded set, that is, compact. Since it is a subset of a small set, it is also small, we can cover it by finitely many open rectangles $\mathcal{S}_0$ whose total area is at most $\epsilon/4M$.

Take each open rectangle in $\mathcal{S}_0$, consider the corresponding closed rectangle and denote this family by $\mathcal{S}$. This is a finite family of closed rectangles with total area still at most $\epsilon/4M$. remember that each point of D is in the interior of one of these rectangles. we assume that all our rectangles are contained in Q, otherwise take the closed rectangle and intersect with Q. It is again a closed rectangle.

Consider the usual product partition starting from $\mathcal{S}$. That is, take all the corners of rectangles of $\mathcal{S}$, their x-coordinates will give a partition η_1 of $[a, b]$ and their y-coordinates gives a partition η_2 of $[c, d]$ and let $\eta = \eta_1 \times \eta_2$ be the product partition. Clearly, every rectangle in $\mathcal{S}$ is union of rectangles from η.

This partition has two types of rectangles. Type 1: part of a rectangle of $\mathcal{S}$. Type 2; rectangle that does not overlap with any rectangle of $\mathcal{S}$. Whatever partition Π we produce later on, it will be finer than the present partition η. Thus every rectangle $R \in \Pi$ will be contained in some rectangle of η. Let us say a rectangle of Π is type I if it is contained in a type 1 rectangle of η, otherwise Type II. Thus if we denote

$$A = \sum \ \{ \ [\sup_R f - \inf_R f] \, a(R) : \ \ R \in \Pi, \ \ \text{Type I}\}$$

$$B = \sum \ \{ \ [\sup_R f - \inf_R f] \, a(R) : \ R \in \Pi, \ \ \text{Type II}\}$$

$$U(\Pi) - L(\Pi) = A + B.$$

let us observe one thing. No matter what our future partition Π is, we have $A < \epsilon/2$. Because in each rectangle of type I, we can bound the sup minus inf by $2M$. Thus the sum corresponding to the type I rectangles is at most $2M$ times total area of those rectangles. But their total area is at most $\epsilon/4M$. Thus

$$A \leq 2M \frac{\epsilon}{4M} = \frac{\epsilon}{2}.$$

Let us consider a rectangle T of η of type II. At each point of T we have oscillation at most $\epsilon/4M\alpha$. Observe a subtle point. Each point of D is in the interior of one of the rectangles of $\mathcal{S}$. Thus if you take a rectangle of type II, then even at the boundary points of this rectangle we have oscillation smaller than $\epsilon/4M\alpha$. (if it were larger, the point would be in the interior of one of those rectangles etc). Thus by one of the previous theorems there is a partition of T, say $\pi(T)$ such that

$$U(\pi(T)) - L(\pi(T)) < \frac{\epsilon}{4M\alpha} a(T)$$

. Get such a partition for each T of type II. Consider the product partition made up of all the rectangles of all these $\pi(T)$ as T ranges over type II rectangles and all rectangles in η. Remember this means the following. Take all x-coordinates of all corners of all these rectangles mentioned and similarly the y-coordinates and consider the product partition Π.

Clearly this Π is finer than η. From the work we did above, we only need to show that that $B < \epsilon/2$. again keep a subtle point in mind. if you take the 'trace' of this grand partition on T above, you may not get back $\pi(T)$ because the other corners also influence this grand partition. Think about it. But of course the trace of this Π on T will be finer than $\pi(T)$ and hence by the property of upper and lower sums we have

$$\sum \ \{ \ [\sup_R f - \inf_R f] \, a(R) : \ R \in \Pi, \ \ R \subset T\}$$

$$\leq \ U(\pi(T)) - L(\pi(T)) \ \ \leq \ \ \frac{\epsilon}{4M\alpha} a(T).$$

as a consequence

$$B = \sum \{ \ [\sup_R f - \inf_R f] \ a(R) : \ R \in \Pi, \ \ \text{Type II}\}$$

$$= \sum_{T \text{ type II}} \ \sum \{ \ [\sup_R f - \inf_R f] \ a(R) : \ R \in \Pi, \ \ R \subset T\}$$

$$\leq \sum_{T \text{ type II}} U(\pi(T)) - L(\pi(T)) \ \ \leq \sum_{T \text{ type II}} \frac{\epsilon}{4M\alpha} a(T).$$

$$\leq \ \frac{\epsilon}{4M\alpha}\alpha \ \ \leq \ \frac{\epsilon}{2}.$$

This completes the proof.

Let f and g be two (bounded) functions on a (bounded) rectangle Q and assume that $D = \{(x, y) : f(x, y) \neq g(x, y)\}$ is contained in a small compact set. Then f is integrable iff g is integrable. More over when they are integrable, their integrals are same.

Proof is very simple. Incidentally the hypothesis implies that D itself is small. But the compact containment tells us that given any $\epsilon > 0$ you can cover the set by finitely many rectangles of small total area.

Suppose that f is integrable. Let $\epsilon > 0$ be given. Let $|f| < M$ and $|g| < M$. Take any partition Π of Q such that $U(\Pi, f) - L(\Pi, f) < \epsilon$. cover D by finitely many open rectangles of total area smaller than $\epsilon/(2M|Q|)$, where $|Q|$ is area of the rectangle Q. Take a product partition η finer than these finitely many rectangles and Π. When you calculate U or L then the summand that participates in the sum is same for both f and g in all rectangles except those that are involved in covering D. But for each of the rectangles T involved in covering D the summand is at most $2M|T|$ and hence their sum is at most $\epsilon/2$. Thus $U(f)$ and $U(g)$ differ by at most ϵ. Same argument shows that $L(f)$ and $L(g)$ also differ by at most ϵ. Thus in particular, $U(g) - L(g) < 4\epsilon$. Since $\epsilon > 0$ is arbitrary, this shows that g is integrable.

This also shows that integrals are same too. (why?)

definition of integral:

So far we have been discussing a very very special case of integration. We are dealing with integral for bounded functions defined on a bounded rectangle (with sides parallel to the usual axes). We have defined upper and lower sums by taking product partitions with sides parallel to the axes. These are the simplest.

The whole thing appears very very unsatisfactory. However when we complete the discussion you see that you can take your function on any kind of (reasonable) bounded region, not necessarily rectangle. you can also take any kind of reasonable partition of the region. I am using the adjective reasonable because we need to definitely put some conditions, but the condition would not be serious; in the sense you have to work hard to find cases not satisfying the conditions we put!

By the way, do keep in mind, we already know that we can take partitions into rectangles with sides parallel to the axes, not necessarily product partitions.. You get the same sup of all lower sums and you get the same inf of all upper sums.

So now let us implement the idea described before introducing small sets. Take a bounded set S and bounded real function f on S. Take any rectangle $Q \supset S$. This is possible because S is bounded. Define g on Q by $g(p) = f(p)$ if the point $p \in S$ and for points $p \in Q - S$ put $g(p) = 0$.

Say that f *is integrable on* S if g is integrable on Q and in that case declare value of the integral

$$\int_S f = \int_Q g.$$

The first question to be addressed is whether this definition depends on the Q taken. if you take a bigger rectangle $Q' \supset Q$, then you see that it makes no difference. You can express $Q' - Q$ as union of non-overlapping rectangles, one of them being Q. (First rigorously prove that if a rectangle is contained in another then the corresponding sides are contained in one another and proceed).

We already had a theorem: A function is integrable on a rectangle which is union of non-overlapping rectangles iff it is integrable on each of these rectangles and then integral is sum of the integrals. apply this theorem to see integrability of f as well as the value of integral remains same.

Now if you take two different rectangles $Q \supset S$ and $Q' \supset S$, possibly one not contained in the other, you can arrive at the same answer by taking another bigger rectangle which includes both Q and Q' and comparing both with that rectangle.

The second thing to be attended to is whether it gives the same answer as earlier in case the set S is already a rectangle. This is easily settled because you can take the original rectangle itself as the Q containing S.

The third question is whether many functions are integrable and whether integral has properties that we would like, linearity etc. We shall discuss this now.

Even though we made the definition for functions defined on an arbitrary set S, we shall be interested in concrete sets. For example f be a continuous function on the disc $\{(x,y) : x^2 + y^2 \leq 1\}$, is it integrable? Or f is a continuous function on a closed triangle or quadrilateral which may not be a rectangle. is it integrable? all these questions are answered rather easily. if you take any rectangle which contains the disc or triangle or whatever and define g on the rectangle as suggested, namely, define zero for the new points, then it is easy to see that this function has all its discontinuity points contained the boundary of the disc/triangle/quadrilateral. Thus the only thing one needs to verify is that these sets are small.

The set $D = \{(x,y) : 0 \leq x = y \leq 1\}$ is a small set. Indeed, the set

$$\bigcup_1^n \ [(k-1)/n,\ k/n] \ \times \ [(k-1)/n,\ k/n]$$

is a union of rectangles, contains D and sum of areas of these rectangles is $1/n$. Thus D is small. This is prototype of proof.

Let $\varphi : [0,1] \to R$ be a continuous function. Then its graph

$$G \ = \ \{(x,y) : \ \ 0 \leq x \leq 1; \ \ \varphi(x) = y\}$$

and

$$H \ = \ \{(x,y) : \ \ 0 \leq y \leq 1; \ \ \varphi(y) = x\}$$

are small sets. This is seen as follows. let $\epsilon > 0$ be given. get $n \geq 1$ such that

$$|x_1 - x_2| \leq 1/n \Rightarrow |\varphi(x) - \varphi(y)| < \epsilon/2$$

Denote $y_k = \varphi(k/n)$. Then

$$G \subset \bigcup_1^n \ [(k-1)/n,\ k/n] \ \times \ [y_k - \epsilon/2,\ y_k + \epsilon/2].$$

a finite union of rectangles whose total area is at most ϵ. Similar argument applies for H.

You can see that the interval could be any closed bounded interval, not necessarily $[0,1]$. Either you can repeat this proof or use that a finite union of small sets is small.

In particular you get that boundaries of rectangles, discs, triangles, quadrilaterals are all small. There are open sets whose boundaries are not small, but you need to work a little hard. There are also simple open sets whose boundaries are not small. Here simple means their boundaries are given as image of a continuous function on the unit interval (a simple closed curve). However to construct such things you need to work very vey hard. Thus the open sets you come across have boundaries small.

Here are simple facts that follow from properties of integrals on rectangles and properties of small sets.

(1) if f_1 and f_2 are integrable on S so is $f_1 + f_2$ and

$$\int_S (f_1 + f_2) = \int_S f_1 + \int_S f_2.$$

$39f$ is integrable and

$$\int_S (39f) = 39 \int_S f.$$

(2) Let S be a bounded open set whose boundary is small. suppose that f is a bounded continuous function on S. Then f is integrable.

If you extend f to a rectangle, then the set of discontinuities are small.

Let us say that a set S has *area* in case the function $f \equiv 1$ (defined on S) is integrable. In that case we put

$$a(S) = \int_S 1.$$

We denote area by $|S|$ also.

(3) If V is a bounded open set with small boundary, then it has area. simply because the function 1 on V is continuous and (2) above takes care.

Such sets arise often and let us give a name. an open set is *good* if it is bounded and its boundary is small.

(4) if V is a good open set and if $\overline{V} = V \cup \partial V$ then both have areas and $|V| = |\overline{V}|$.

this is because they differ on a small set and an earlier theorem takes care.

(5) if $f \leq g$ are both integrable on S then

$$\int_S f \leq \int_S g.$$

(6) Let V be good open set and let $m \leq f \leq M$ on V, then

$$m\ a(S) \leq \int_S f \leq M\ a(S).$$

This follows from the above. We already knew f is integrable.

(7) Let S_1 and S_2 be disjoint sets and f_1 and f_2 defined on S_1, S_2 respectively are integral. Define

$$S = S_1 \cup S_2; \quad f = f_1 \text{ on } S_1; = f_2 \text{ on } S_2.$$

Then f is integrable on S and

$$\int_S f = \int_{S_1} f_1 + \int_{S_2} f_2.$$

This follows by taking large rectangle that includes both S_1 and S_2 and applying known results for integrals on rectangles. Observe that you are already told that f_1 and f_2 are integrable and that S_1 and S_2 are disjoint.

(8) V is a good open set and f a bounded continuous function on $\overline{V}$. Then f is integrable on V (this means restriction of f to V is integrable on V) and f is integrable on $\overline{V}$ and

$$\int_V f = \int_{\overline{V}} f$$

This is because when you put both in a rectangle, they differ on a small set. Go by the rule book. Take large rectangle Q that includes $\overline{V}$. when you calculate $\int\limits_V f$ you take g on Q to be zero outside V. When you calculate $\int\limits_{\overline{V}} f$ you take g' to be zero outside $\overline{V}$.

(9) Let V_1 and V_2 be good open sets, then so is their union and

$$|V_1 \cup V_2| = |V_1| \cup |V_2|.$$

This follows from (7).

Thus area adds up for disjoint open sets.

Let V be an open set with small boundary. A finite collection of open sets Π is said to be a *good partition* of V, if they are disjoint open subsets of V each having a small boundary and their closures cover V. By closure of an open set W we mean $W \cup \partial W$.

if you feel uncomfortable with this definition you can consider $\overline{V}$ to start with. Then the family $\{\overline{W} : W \in \Pi\}$ is indeed a partition of $\overline{V}$. Theses sets are non-overlapping with union equal to $\overline{V}$.

We shall now show that you can take any nice open set and any good partition of it to calculate integrals. This removes the unnatural conditions of taking partitions with rectangles sides parallel to the axes.

We need a definition. For a bounded set A, diameter of A is defined by

$$d(A) = \sup\{d(p, q) : p \in A, q \in A\}.$$

here $d(p, q)$ is the distance between the two points p and q, that is $||p - q||$. if you take a disc with diameter a then the diameter of the set consisting of the disc, as defined above, is indeed a, verify this.

For a collection Π of sets,

$$||\Pi|| = \sup\{d(A) : A \in \Pi\}.$$

(10). Let V be a good open set. f be a continuous function on $\overline{V}$. Given $\epsilon > 0$, there is an $\delta > 0$ such that for any good partition Π of V with $||\Pi| < \delta$

$$|U(\Pi, f) - \int_V f| < \epsilon; \quad |L(\Pi, f) - \int_V f| < \epsilon.$$

In other words even if you take finer and finer partitions with sets you like and calculate upper or lower sums you will still get the integral we got. But you need to take good partitions. After all, you need to calculate the sup of the function and multiply by the area, so you need areas for your sets.

Otherwise you can not calculate the sums.

if you feel uncomfortable with the function being given on $\overline{V}$ and integrals being talked about are on V, you can take integrals also on $\overline{V}$. an earlier theorem tells you both are same.

Proof is simple. Using uniform continuity of the continuous function f on the compact set $\overline{V}$ take $\delta > 0$ so that

$$p, q \in \overline{V}; \;\; d(p,q) < \delta \to |f(p) - f(q) < \frac{\epsilon}{2|V|}.$$

Let now be Π be any good partition. We have enough theorems above to justify each of the following equalities.

$$L(\Pi, f) = \sum_{T \in \Pi} m_T |T| \leq \sum_{T \in \Pi} \int_T f = \int_{\cup \Pi} f = \int_V f$$

Last equality is from the fact that $\cup\Pi$ and V differ by a small set, namely at most union of all the ∂T for $T \in \Pi$ put together. similarly

$$U(\Pi, f) = \sum_{T \in \Pi} M_T |T| \geq \sum_{T \in \Pi} \int_T f = \int_{\cup \Pi} f = \int_V f$$

finally

$$U - L \leq \frac{\epsilon}{2|V|} |V| = \epsilon/2.$$

This completes the proof.

Thus you can use some rectangles with sides not necessarily parallel to the axes, some sets could be interior of triangles and so on. There is no restriction.

We shall prove one theorem that will allow us to reduce all the double integrals to integrals of one variable 'at a time'. this is analogue of the following theorem we proved: if f is continuous on a closed rectangle, than the repeated integrals are equal and in fact they equal the double integral.

Theorem: Let Q be a rectangle $[a, b] \times [c, d]$ and f be a bounded integrable function. Define for each x,

$$H(x) = \underline{\int_c^d} f(x, y) dy; \qquad G(x) = \overline{\int_c^d} f(x, y) dy.$$

Then G, H are integrable on $[a,b]$ and

$$\int_Q f = \int_a^b G(x)dx = \int_a^b H(x)dx.$$

The notation

$$\overline{\int_c^d} \varphi(y)dy.$$

is the upper integral of the function φ, it is the lower bound of all upper sums of the function. Since our hypothesis is only that the function f is integrable on Q and it does not imply that for each x the function $y \mapsto f(x,y)$ is integrable we need to take upper integral. similarly lower integral is the sup of all lower sums.

Also you can consider the other iterated integrals too, that is, lower and upper integrals w.r.t. x first. The corresponding statement is also be true.

Proof is simple. Let $\epsilon > 0$. Since f is integrable, take a product partition $\pi_1 \times \pi_2 = \Pi$ such that

$$U(f,\Pi) - L(f,\Pi) < \epsilon.$$

let us see what will be $U(H,\pi_1)$. Take a rectangle $T \times S \in \Pi$ Then

$$m_{T\times S}(f) \leq f(x,y); \quad (x,y) \in T \times S.$$

$$m_{T\times S}(f)|S| \leq \underline{\int_S} f(x,y)dy.$$

$$\sum_{S\in\pi_2} m_{T\times S}(f)|S| \leq \sum_{S\in\pi_2} \underline{\int_S} f(x,y)dy \leq \underline{\int_c^d} f(x,y)dy.$$

(Justify this last inequality) Hence

$$\sum_{S\in\pi_2} m_{T\times S}(f)|S| \leq H(x); \quad x \in T$$

$$\sum_{S\in\pi_2} m_{T\times S}(f)|S| \leq m_T(H)$$

$$\sum_{T\in\pi_1} \sum_{S\in\pi_2} m_{T\times S}(f)|S||T| \leq \sum_{T\in\pi_1} m_T(H)|T|$$

$$L(f,\Pi) \leq L(H,\pi_1)$$

Similarly

$$U(f,\Pi_1) \geq U(H,\pi_1).$$

Thus

$$L(f,\Pi) \le L(H,\pi_1) \le U(H,\pi_1) \le U(f,\Pi).$$

First of all this shows, since $\epsilon > 0$ is arbitrary and the two extremities differ by at most ϵ, that H is integrable. The same inequalities show that

$$\int_Q f = \int_a^b H$$

Similarly argument holds for G, completing the proof.

In practice we have a continuous function $f(x,y)$ defined on a region of the following type

$$S = \{(x,y) : a \le x \le b; \varphi(x) \le \psi(x).\}$$

where φ and ψ are continuous functions defined on the interval $[a,b]$ Thus the boundary of S consists of the graphs of φ, ψ and the two vertical lines at a and b. here it is assumed that $\varphi(x) \le \psi(x)$ for all $x \in [a,b]$. Since the boundary of S is small and f is continuous we first conclude that f is integrable.

To integrate we can apply the previous theorem. Of course, you need not complicate life because for each x this function $y \mapsto f(x,y)$ is integrable. To be more precise, if you go by the rule book, you will put S in a rectangle, apply previous result, then if you look at the vertical line at x the function $y \mapsto f(x,y)$ is continuous except possibly at the two points $y = \varphi(x)$ and $y = \psi(x)$. this is integrable. You need not make fuss about upper and lower integrals.

thus we conclude

$$\int_S f = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)dy \right] dx.$$

The main point is that double integral is reduced to integrating one variable at a time, something we learnt last semester.

Let us work out one example: Find the volume of the ellipsoid,

$$\{(x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$$

Here $a,b,c > 0$. Consider the region in R^2

$$S = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$$

and define the function

$$f(x,y) = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

The ellipsoid is the region bounded by the function $-f(x,y)$ and $f(x,y)$ on the region S. Thus the volume required is found by calculating

$$2\int_S f$$

This is done by the above method and we get

$$\frac{4}{3}\pi abc.$$

In particular, if we have the sphere of radius r , that is $a = b = c = r$ we get

$$\frac{4}{3}\pi r^3.$$

Change of variables:

We shall now proceed to analogue of the change of variable formula.

Recall that in one dimension it states the following. Let V be a bounded interval. Let φ be a C^1 function on V, which is one-to-one and onto $\varphi(V)$ another bounded interval. Let f be a bounded continuous function on $\varphi(V)$. Then

$$\int_V f(\varphi(x))|\varphi'(x)|dx = \int_{\varphi(V)} f(y)dy.$$

of course we did not put modulus sign, stated it when φ is increasing and when it is decreasing separately. In practice, this translates to: put $y = \varphi(x)$ so that $dy = \varphi'(x)dx$.

The exact same formula remains true even in R^2. In one dimension, we had an easy proof of this formula taking recourse to the chain rule of differentiation and fundamental theorem of calculus. Here we do have chain rule, but at this moment we do not have fundamental theorem of calculus. We have to take recourse to a different method.

In one dimensions, a small interval around x is transformed to an interval around $y = \varphi(x)$ and the length of this interval is 'approximately' $\varphi'(x)dx$.

So first we need to understand how areas change under mappings. Of course the simplest mappings are linear mappings. Just bear in mind that R^2 is column vectors. For typographical convenience we are showing as rows. usually books put a transpose, but we are not taxing ourselves with this. This may be confusing, but as long as you know what you are talking about, it will not be confusing.

Consider the map: *interchange coordinates*

$$T(x, y) = (y, x)$$

This is given by the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Assuming you will not get confused, we are using the same symbol for the linear map as well as its matrix representation. If you take any rectangle Q (sides parallel to the axes, to start with) then

$$|T(Q)| = |Q|$$

In other words the area is multiplied by $|T|$, the absolute value of the determinant of the matrix representing T.

Consider the map: *multiply a coordinate*

$$T(x, y) = (31x, y).$$

This is given by the matrix

$$T = \begin{pmatrix} 31 & 0 \\ 0 & 1 \end{pmatrix}$$

if you take a rectangle Q (sides parallel to the axes, to start with)

$$|T(Q)| = 31|Q|.$$

again the area is multiplied by $|T|$, absolute value of the determinant of the matrix representing T. Try multiplication by -31 too.

Consider the map: *add one coordinate to the first one*

$$T(x, y) = (x + y, y).$$

This is given by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

We see for rectangle (sides parallel etc)

$$|T(Q)| = |Q|$$

again the area gets multiplied by $|T|$. of course this is not as obvious as the earlier ones. take first the unit square, that is the rectangle $[0, 1] \times [0, 1]$, verify that it is transformed to a parallelogram. Then take other rectangles.

It is believable that this should be true for any linear transformation. Let us consider only non-singular transformations because these are the only things we will be interested in. (In any case if you take a singular linear transformation, the statement we are going to make is true and trivial.)

Fact: If T is a non-singular linear transformation of R^2 to itself with matrix representation T, then for any bounded rectangle with sides parallel to the axes

$$|T(Q)| = |T||Q|.$$

This is actually done because any non-singular transformation is a composition of the above three types!