We shall complete our discussion on uniform convergence of integrals and close the chapter. I hope you have got a feeling for the applications I have outlined last time, namely, to evaluate the characteristic function of the normal random variable, solving the heat equation. Of course, apart from these you can think of continuity of the gamma function and its derivatives; which we explained earlier.

While discussing power series, especially its continuity and term by term differentiation we have understood how crucial it was to have uniform convergence and how it immediately provides the answers. Just to impress upon you that there are no new ideas to treat integrals, I shall repeat discussion of series also as we go along.

uniform convergence:

So let f_n be continuous functions defined on $[c, d]$. Recall that the series $\sum f_n$ converges to φ pointwise if for every number $y \in [c, d]$ the series of numbers $\sum f_n(y)$ converges to the number $\varphi(y)$. This means that given y, the partial sums $s_n(y) = \sum_1^n f_i(y)$ converge to $\varphi(y)$. This, in turn, means that given a point y and $\epsilon > 0$, there is an integer N such that $|s_n(y) - \varphi(y)| < \epsilon$ for all $n \geq N$. Of course this integer N depends on the number ϵ and also on the point y.

We say that $\sum f_n$ converges uniformly to φ if given $\epsilon > 0$, there is an N such that $|s_n(y) - \varphi(y)| < \epsilon$ for all $n \geq N$ and also all $y \in [c, d]$. Thus the integer N does not depend on any y in the interval. There is one N that works for all y. This is the spirit of uniformity.

Similarly, let $f(x, y)$ be a continuous function defined on $[0\infty) \times [c, d]$. Recall that the integral $\int_0^\infty f(x, y)dx$ converges to $\varphi(y)$ point wise if for every number $y \in [c, d]$ the integral $\int_0^\infty f(x, y)dx$ converges to the number $\varphi(y)$. This means that given y, the 'partial' integrals $s_A(y) = \int_0^A f(x, y)dx$ converge to $\varphi(y)$ as $A \to \infty$. This, in turn, means that given a point y and $\epsilon > 0$, there is an A_0 such that $|s_A(y) - \varphi(y)| < \epsilon$ for all $A \geq A_0$. Of course this A_0

depends on the number ϵ and also on the point y.

We say that $\int\limits_0^\infty f(x,y)dx$ converges uniformly to φ if given $\epsilon > 0$, there is an A_0 such that $|s_A(y) - \varphi(y)| < \epsilon$ for all $A \geq A_0$ and also all $y \in [c,d]$. Thus the number A_0 does not depend on any y in the interval. There is one A_0 that works for all y. This is the spirit of uniformity as said above.

Let us return to series and use the earlier notation. Assume $\sum f_n$ converge to φ uniformly. Let $\epsilon > 0$ be given and N be as above. then we have

$$n \geq N \Rightarrow |s_n(y) - \varphi(y)| < \epsilon.$$

This implies

$$n > m \geq N \Rightarrow |s_n(y) - s_m(y)| = |s_n(y) - \varphi(y)| + |\varphi(y) - s_m(y)| < 2\epsilon.$$

In other words

$$n > m \geq N \Rightarrow \left| \sum_{n+1}^{m} f_i(y) \right| < 2\epsilon.$$

Since $\sum\limits_1^\infty f_i(y)$ converges, so do all the sums $\sum\limits_m^\infty f_i(y)$. Thus the above inequality also implies that

$$n > N \Rightarrow \left| \sum_{n}^{\infty} f_i(y) \right| \leq 2\epsilon.$$

Returning to integrals let us continue with earlier notation. Assume $\int\limits_0^\infty f(x,y)dx$ converge to φ uniformly. Let $\epsilon > 0$ be given and A_0 be as above. then we have

$$A \geq A_0 \Rightarrow |s_A(y) - \varphi(y)| < \epsilon.$$

This implies

$$B > A \geq A_0 \Rightarrow |s_B(y) - s_A(y)| = |s_B(y) - \varphi(y)| + |\varphi(y) - s_A(y)| < 2\epsilon.$$

In other words

$$B > A \geq A_0 \Rightarrow \left| \int_A^B f(x,y)dx \right| < 2\epsilon.$$

Since $\int\limits_0^\infty f(x,y)dx$ converges, so do all the integrals $\int\limits_A^\infty f(x,y)dx$. Thus the above inequality also implies that

$$A > A_0 \Rightarrow \left| \int_A^\infty f(x,y)dx \right| \leq 2\epsilon.$$

continuity:

Suppose that the series of continuous functions $\sum f_i$ converges to φ uniformly on $[c, d]$. We show φ is continuous. We show uniform continuity of φ. let $\epsilon > 0$ be given. Choose N so that

$$n \geq N \Rightarrow |s_n(y) - \varphi(y)| < \epsilon/3; \qquad \forall y.$$

Let us not bother on $n \geq N$, but just consider this integer N. Since s_N is a finite sum of continuous functions it is continuous and hence uniformly continuous on $[c, d]$. So choose $\delta > 0$ so that

$$|y_1 - y_2| < \delta \Rightarrow |s_N(y_1) - s_N(y_2)| < \epsilon/3.$$

If we take this δ then

$$|y_1 - y_2| < \delta \Rightarrow |\varphi(y_1) - \varphi(y_2)|$$

$$\leq |\varphi(y_1) - s_N(y_1)| + |s_N(y_1) - s_N(y_2)| + |s_N(y_2) - \varphi(y_2)|$$

$$\leq \epsilon.$$

Let us now return to integrals. Suppose that we have continuous functions $f(x, y)$ as above with $\int\limits_0^\infty f(x, y)dx$ converging to φ uniformly on $[c, d]$. We show φ is continuous. We show uniform continuity of φ. let $\epsilon > 0$ be given. Choose A_0 so that

$$A \geq A_0 \Rightarrow |s_A(y) - \varphi(y)| < \epsilon/3; \qquad \forall y.$$

Let us not bother on $A \geq A_0$, but just fix just one number $A \geq A_0$. Since

$$s_A(y) = \int_0^A f(x, y)dx$$

is integral over a finite interval, we know from earlier theorems that s_A is a continuous function and hence uniformly continuous on $[c, d]$. So choose $\delta > 0$ so that

$$|y_1 - y_2| < \delta \Rightarrow |s_A(y_1) - s_A(y_2)| < \epsilon/3.$$

If we take this δ then

$$|y_1 - y_2| < \delta \Rightarrow |\varphi(y_1) - \varphi(y_2)|$$

$$\leq |\varphi(y_1) - s_A(y_1)| + |s_A(y_1) - s_A(y_2)| + |s_A(y_2) - \varphi(y_2)|$$

$$\leq \epsilon.$$

This shows that the uniformly convergent improper integral defines a continuous function (if f is continuous).

differentiation:

Let us first consider series. Let us now assume that f_i are C^1 functions on $[c,d]$ with derivative g_i. Assume that $\sum f_i$ converges to φ uniformly and $\sum g_i$ converges to ψ uniformly. then φ is differentiable and $\varphi' = \psi$. In other words derivative of sum equals sum of derivatives.

Of course we do not need $\sum f_i$ to converge uniformly, enough if it converges point wise so that we have a function φ to talk about. But uniform convergence of the series $\sum g_i$ is important. But this remark can be ignored because usually you have uniform convergence of both the series $\sum f_i$ and $\sum g_i$.

Let us fix a point y_0. Let us fix $\epsilon > 0$. Need to show $\delta > 0$ so that

$$0 < |h| < \delta \Rightarrow \left| \frac{\varphi(y_0 + h) - \varphi(y_0)}{h} - \psi(y_0) \right| < \epsilon.$$

(while reading you should assume that y_0 as well as $y_0 + h \in [c,d]$.

First fix N so that

$$n > m \geq N \Rightarrow \left| \sum_m^n g_i(y) \right| < \epsilon/3; \quad \forall y.$$

In particular note that

$$\left| \sum_{N+1}^{\infty} g_i(y) \right| < \epsilon/3.$$

Let us concentrate on this N. Since s_N is a finite sum of C^1 functions, it is differentiable. So fix $\delta > 0$ such that

$$0 < |h| < \delta \Rightarrow \left| \frac{s_N(y_0 + h) - s_N(y_0)}{h} - \sum_1^N g_i(y_0) \right| < \epsilon/3.$$

Now

$$0 < |h| < \delta \Rightarrow \left| \frac{\varphi(y_0 + h) - \varphi(y_0)}{h} - \psi(y_0) \right|$$

$$\leq \left| \frac{s_N(y_0 + h) - s_N(y_0)}{h} - \sum_1^N g_i(y_0) \right|$$

$$+\left|\frac{\sum\limits_{N+1}^{\infty} f_i(y_0+h) - \sum\limits_{N+1}^{\infty} f_i(y_0)}{h}\right| + \left|\sum_{N+1}^{\infty} g_i(y_0)\right|$$

First and last terms are at most $\epsilon/3$ by choice of δ and N respectively. Regarding the middle term observe that the mean value theorem applied to the C^1 function $\sum\limits_{N+1}^{m} f_i$ we get

$$\left|\frac{\sum\limits_{N+1}^{m} f_i(y_0+h) - \sum\limits_{N+1}^{m} f_i(y_0)}{h}\right| = \left|\sum_{N+1}^{m} g_i(\theta)\right| < \epsilon/3.$$

This being true for every $m > N$ we conclude that the middle term is also at most $\epsilon/3$ completing the proof.

You will now see that exactly the same proof works for integrals.

Let us now assume that there is a continuous function g on $[0, \infty) \times [c, d]$ such that for each x as a function of y it is derivative of $y \mapsto f(x, y)$. Assume that $\int\limits_0^{\infty} f(x, y)dx$ converges to φ uniformly and $\int\limits_0^{\infty} g(x, y)dx$ converges to ψ uniformly. then φ is differentiable and $\varphi' = \psi$. In other words derivative of integral equals integral of derivative.

As in the case of series, we do not need $\int_0^{\infty} f(x, y)dx$ to converge uniformly, enough if it converges point wise so that we have a function φ to talk about. But uniform convergence of $\int\limits_0^{\infty} g(x, y)dx$ is important. But this remark can be ignored because usually you have uniform convergence of both integrals.

Let us fix a point y_0. Let us fix $\epsilon > 0$. Need to show $\delta > 0$ so that

$$0 < |h| < \delta \Rightarrow \left|\frac{\varphi(y_0+h) - \varphi(y_0)}{h} - \psi(y_0)\right| < \epsilon.$$

(we assume that $y_0 + h \in [c, d]$).

First fix N so that

$$B > A \geq N \Rightarrow \left|\int_A^B g(x, y)dx\right| < \epsilon/3; \quad \forall y.$$

In particular note that

$$\left|\int_N^\infty g(x,y)dx\right| < \epsilon/3.$$

Let us concentrate on this N. Since s_N is integral over finite interval, from the results we proved earlier, it is differentiable. So fix $\delta > 0$ such that

$$0 < |h| < \delta \Rightarrow \left|\frac{s_N(y_0+h) - s_N(y_0)}{h} - \int_0^N g(x,y_0)dx\right| < \epsilon/3.$$

Now

$$0 < |h| < \delta \Rightarrow \left|\frac{\varphi(y_0+h) - \varphi(y_0)}{h} - \psi(y_0)\right|$$

$$\leq \left|\frac{s_N(y_0+h) - s_N(y_0)}{h} - \int_0^N g(x,y_0)dx\right|$$

$$+ \left|\frac{\int\limits_N^\infty f(x,y_0+h)dx - \int\limits_N^\infty f(x,y_0)dx}{h}\right| + \left|\int_N^\infty g(x,y_0)dx\right|$$

First and last terms are at most $\epsilon/3$ by choice of δ and N respectively. Regarding the middle term observe that the mean value theorem applied to the C^1 function $\int\limits_N^B f(x,y)dx$ (note that range of integration is finite) we get

$$\left|\frac{\int\limits_N^B f(x,y_0+h)dx - \int\limits_N^B f(x,y_0)dx}{h}\right| = \left|\int_N^B g(x,\xi)dx\right| < \epsilon/3.$$

This being true for every $B > N$ we conclude that the middle term in the earlier string of terms is also at most $\epsilon/3$ completing the proof.

integration:

With the same hypothesis as in continuity assume that the series $\sum f_i$ uniformly converges to φ. Then

$$\sum_i \int_c^d f_i(y)dy = \int_c^d \varphi(y)dy = \int_c^d \sum_i f_i(y)dy.$$

In other words infinite sum and integral can be interchanged. Similarly with the same hypothesis as in continuity,

$$\int_0^\infty \left(\int_c^d f(x,y)dy\right) dx = \int_c^d \varphi(y)dy = \int_c^d \left(\int_0^\infty f(x,y)dx\right) dy.$$

The integrals can be interchanged. Proof is similar to the above, you only need to make the tail sum (or tail integral) smaller than $\epsilon/(d-c)$ and argue carefully.

criterion for uniform convergence:

Consider series $\sum f_n$. If there are numbers M_n such that $|f_n| \leq M_n$ for each n and $\sum M_n$ converges then the series $\sum f_n$ converges uniformly. Given $\epsilon > 0$ you only need to choose N so that $\sum\limits_{N}^{\infty} M_n < \epsilon$. This is known as Weierstrass M-test.

Similarly, suppose we have continuous function $f(x, y)$ on $[0, \infty) \times [c, d]$. Suppose that there is a function $M(x)$ such that $|f(x, y)| \leq M(x)$ for every (x, y) and $\int\limits_{0}^{\infty} M(x)dx$ converges. then the integral $\int\limits_{0}^{\infty} f(x, y)dx$ converges uniformly.

GOOD LUCK