Normal integral again:

Here is a tricky way of calculating normal integral. Let us put

$$a_n = \int_0^\infty e^{-t^2/2}\ t^n\ \ dt; \qquad n = 0, 1, 2, \cdots.$$

We do not know a_0 but can explain all others using it. Integration by parts gives

$$a_1 = 1; \quad a_n = (n-1)a_{n-2} \quad n = 2, 3, 4, \cdots.$$

This immediately gives

$$a_{2m} = (2m-1)(2m-3)\cdots 1 a_0.$$

$$a_{2m+1} = 2m(2m-2)\cdots 2\cdot 1.$$

Now comes a high school idea. Note that for any λ we have

$$\int_0^\infty e^{-t^2/2}\ t^n\ (\lambda + t)^2\ \ dt > 0$$

simply because the integrand is positive. That is, for every λ

$$a_{n+2} + 2\lambda a_{n+1} + \lambda^2 a_n > 0$$

Thus

$$a_{n+1} \le \sqrt{a_n a_{n+2}}.$$

In particular, for every m we have

$$a_{2m} < \sqrt{a_{2m-1}a_{2m+1}}.$$

giving us

$$a_0 \le \sqrt{2m}\frac{(2m-2)(2m-4)\cdots 2\cdot 1}{(2m-1)(2m-3)\cdots 1}$$

This is true for every m and taking limits and appealing to Walli we get

$$a_0 \le \sqrt{\pi/2}.$$

We also have

$$a_{2m+1} < \sqrt{a_{2m}a_{2m+2}}.$$

Analogous to the above, on simplification, this inequality gives us

$$a_0 \geq \sqrt{\pi/2}.$$

These two inequalities give us

$$\int_0^\infty e^{-t^2/2} \ dt = \sqrt{\frac{\pi}{2}}.$$

From this we get

$$\int_{-\infty}^\infty e^{-t^2/2} \ dt = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}.$$

Again returning back to the integral on the positive side, we substitute $t^2/2 = u$ so that $tdt = du$ or $dt = du/\sqrt{2u}$ we see

$$\int_0^\infty e^{-u}u^{-1/2}du = \sqrt{\pi}.$$

That is

$$\Gamma(1/2) = \sqrt{\pi}.$$

We know that for integers $n > 1$, $\Gamma(n+1) = n!$. We can use the above result to calculate gamma values of half integers. Recall that

$$\Gamma(a+1) = a\Gamma(a).$$

Thus for example

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}.$$

Volume of unit ball:

Let us consider, for $R > 0$, the ball

$$B_R = \Big\{(x_1, x_2, \cdots, x_n) : \sum x_i^2 \leq R\Big\}$$

Let $V_n(R)$ denote the volume of this ball in R^n. That is

$$V_n(R) = \int_{B_R} 1 \ dx_1 \ dx_2 \ \cdots dx_n.$$

First let us note one thing.

$$V_n(R) = R^n V_n; \qquad V_n = V_n(1).$$

This is because the linear transform $Tx = Rx$ takes unit ball onto the R-ball.

To calculate $V_n(R)$ let us integrate the variables $(x_2, x_3, \cdots, x_n)$. Clearly they range over the set

$$\sum_3^n x_i^2 \leq R^2 - x_1^2 - x_2^2.$$

Thus if you integrate these $(n-2)$ variables over this set we get

$$\left(\sqrt{R^2 - x_1^2 - x_2^2}\right)^{n-2} V_{n-2}.$$

Note that V_{n-2} is a number and does not depend on x_1, x_2. Now let us integrate w.r.t. x_1, x_2. These variables range over

$$S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq R^2\}.$$

Thus

$$V_n(R) = V_{n-2} \int_S \left(R^2 - x_1^2 - x_2^2\right)^{(n-2)/2} dx_1 dx_2.$$

Changing to polar coordinates

$$x = r\cos\theta; y = r\sin\theta; \quad 0 \leq r \leq R;\ 0 < \theta < 2\pi.$$

Noting that the Jacobian is r we get

$$V_n(R) = \int_{r=0}^{R} \int_{\theta=0}^{2\pi} (R^2 - r^2)^{(n-2)/2} r\ d\theta\ dr.$$

The presence of $r\ dr$ allows us the substitution $r^2 = u$ and we have

$$V_n(R) = \frac{2\pi}{n} R^n V_n = \frac{2\pi R^2}{n} V_{n-2}(R).$$

Note that $V_1(R) = 2R$ and $V_2(R) = \pi R^2$. Thus the above relation gives us

$$V_{2k}(R) = \frac{\pi^k}{k!} R^{2k}$$

$$V_{2k+1} = \frac{2^{k+1}\pi^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)} R^{2k+1}.$$

As far as unit ball is concerned you can combine the even and odd formulae to get

$$V_n = \pi^{n/2} \Big/ \Gamma(\frac{n}{2} + 1)\ .$$

It is interesting to note that V_k converges to zero as $k \to \infty$. As the dimension grows, the volume of any fixed ball shrinks.

You can use the above result to calculate the volume of the ellipse. Fix strictly positive numbers $a_1, a_2, \cdots, a_n$. Consider the region

$$E = \{(x_1, x_2, \cdots, x_n) : \sum \frac{x_i^2}{a_i^2} \leq 1\}.$$

Clearly the unit ball is mapped onto this by the transformation

$$(x_1, x_2, \cdots, x_n) \mapsto (a_1 x_1, a_2 x_2, \cdots, a_n x_n).$$

Thus the Jacobian rule tells us that

$$|E| = a_1 a_2 \cdots a_n V_n.$$

Dirichlet integral:

We know that

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \beta(a,b)$$

Here and in what follows the parameters a, b or a_i are strictly positive numbers.

Let

$$S_n = \{(x_1, x_2, \cdots, x_n) : \quad x_i > 0 \ \ \forall \ \ i; \ \ \sum x_i < 1\}$$

This is called simplex. Fix numbers $a_1, a_2, \cdots, a_{n+1}$ all strictly positive.

We wish to calculate

$$I_n = \int_{S_n} x_1^{a_1-1} \, x_2^{a_2-1} \ \cdots x_n^{a_n-1} \, (1 - \sum x_i)^{a_{n+1}-1} \ \ dx.$$

(see how we changed x_1, x_2 to x, y and a_1, a_2, a_3 to a, b, c)

Let us consider $n = 3$.

$$I_3 = \int_{S_3} x^{a-1} y^{b-1} (1-x-y)^{c-1} dx dy$$

$$= \int_0^1 x^{a-1} \left[\int_0^{1-x} y^{b-1}(1-x-y)^{c-1} dy \right] dx$$

If you substitute $y = (1-x)u$ in the y-integral and simplify we get

$$I_3 = \beta(b,c)\beta(a, b+c)$$

This is not in recognizable (and symmetric) form. Let us now use a result proved earlier

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

We get

$$I_3 = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}.$$

You can show by induction

$$I_n = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{n+1})}{\Gamma(\sum a_i)}.$$

This is called Dirichlet integral.

Normal integral again:

Let us define on R^n,

$$\varphi(x) = (2\pi)^{-n/2} \;\; e^{x^t x}.$$

More precisely,

$$\varphi(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2} \exp\left\{\frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)\right\}.$$

This is called the standard normal density in n variables.

$$\int_{R^n} \varphi(x) dx = 1.$$

We used the abbreviation $dx = dx_1\, dx_2 \cdots dx_n$. This is because the integrand 'splits' and allows you to integrate variables one after the other (all integrals giving you the value one).

You should remember that vectors in R^n are column vectors. Thus $x^t x$ is a number. Suppose we take a vector $\mu \in R^n$. Then

$$\int_{R^n} \varphi(x-\mu) dx = 1.$$

That is

$$\int_{R^n} \frac{1}{(\sqrt{2\pi})^n} e^{-\{(x_1-\mu_1)^2 + (x_2-\mu_2)^2 + \cdots\}/2} dx = 1.$$

Here again the integral splits. Or you can make change of variable $y = x - \mu$.

Let us take a symmetric positive definite matrix Σ. Then

$$\int_{R^n} \frac{1}{(\sqrt{2\pi})^n \sqrt{|\det \Sigma|}} \quad e^{x^t \Sigma^{-1} x} \quad dx = 1.$$

To prove this you need some matrix theory. You know that there is a (necessarily non-singular) matrix A such that

$$A^{-1}\Sigma A = D; \qquad (i.e.) \quad \Sigma = ADA^{-1}.$$

where D is diagonal matrix. (You know $\underline{A}^{-1}\Sigma\underline{A} = D$, I am just renaming. Let us start with this.

First observe that the above equation says

$$\Sigma A = AD$$

Let v be the first column of A. Then the first column of left side equals Σv. If λ_1 is the first diagonal entry of the diagonal matrix D, then the first column of the right side equals $\lambda_1 v$.

In other words the jth column, say v^j of A, gives you eigen vector for the eigen value corresponding to j-th diagonal entry of D, say λ_j . Since A is invertible these eigen vectors form a basis.

In other words the above representation of Σ gives you the eigen values and basis of eigen vectors.

Also if $\lambda_1 \neq \lambda_2$ are two distinct eigen values (distinct diagonal entries of D) then the corresponding columns v_1, v_2 are orthogonal. Indeed using that Σ is symmetric,

$$\lambda_1\langle v_1, v_2\rangle = \langle \lambda_1 v_1, v_2\rangle = \langle \Sigma v_1, v_2\rangle = \langle v_1, \Sigma v_2\rangle = \langle v_1, \lambda_2 v_2\rangle = \lambda_2\langle v_1, v_2\rangle$$

Since $\lambda_1 \neq \lambda_2$ we conclude that

$$\langle v_1, v_2\rangle = 0.$$

If a λ repeats, say $\lambda_1, \lambda_3, \lambda_7$ are all equal to λ, then v^1, v^3, v^7 gives you a three dimensional subspace S (remember columns of A form a basis and hence they are independent vectors). Since for all these vectors $\Sigma v = \lambda v$ we see that this is true for all vectors in this three dimensional subspace S. If you replace these vectors by choosing three orthogonal vectors from this subspace S then the above equation still remains true. Multiply and see. In other words if you take orthogonal vectors w^1, w^3, w^7 from S and replace

v^1, v^3, v^7 columns of A by these w's then the matrix A is still non-singular and the equation $\Sigma A = AD$ still remains true.

Finally, if you multiply each column by norm of that vector then also the equation above remains true. All this amounts to saying that you can safely assume that the columns of A are orthogonal and each column is a unit vector. But then a direct multiplication shows that $A^t A = I$, the identity matrix. In other words we have an orthogonal matrix A with

$$\Sigma A = AD; \qquad A^t A = AA^t = I$$

Define $B = A\sqrt{D}A^{-1}$ where $\sqrt{D}$ is the diagonal matrix with entry-wise square roots. Remember Σ being positive definite, these λ are positive.

In other words Σ is the matrix which sends the vector v^j to $\lambda_j v^j$ and B sends v^j to $\sqrt{\lambda_j}\, v^j$.

Easy to see that B commutes with Σ. Hence B^t commutes with $\Sigma^t = \Sigma$. Also $B^t B = \Sigma$, multiply and see.

Finally, returning to our integral, if we substitute

$$x = By$$

then,

$$x^t \Sigma^{-1} x = y^t B^t \Sigma^{-1} B y = y^t y$$

where we used that B and B^t commute with Σ and $B^t B = \Sigma$.

Also the Jacobian $dx = |B| dy = \sqrt{|\Sigma|} dy$. In other words this integral is reduced to the previous one without Σ.

We can combine both the processes, namely of introducing μ and Σ. Let as earlier $\mu \in R^n$ and Σ be a positive definete matrix. Then

$$\int_{R^n} \frac{1}{(\sqrt{2\pi})^n \sqrt{|\det \Sigma|}} \ e^{-(x-\mu)^t \Sigma^{-1} (x-\mu)/2} \ dx = 1.$$

You can do it in two steps. First substitute $z = x - \mu$ and then $z = By$.

volume of simplex:

To find the volume of the region, called simplex,

$$S = \{(x_1, x_2, \cdots, x_n) : x_i \geq 0 \ \forall \ i; \ \sum x_i \leq 1.\}$$

This is simple
$$v_n = \int_S 1dx$$
Range is
$$0 \le x_1 \le 1; \quad 0 \le x_2 \le 1 - x_1; \quad 0 \le x_3 \le 1 - x_1 - x_2; \cdots$$
$$0 \le x_n \le 1 - x_1 - x_2 - \cdots - x_{n-1}.$$
You can successively integrate x_n and then x_{n-1} etc to get sucessively
$$(1 - \sum_1^{n-1} x_i); \quad (1 - \sum_1^{n-2} x_i)^2/2!; \quad (1 - \sum_1^{n-3} x_i)^3/3! \cdots \cdots 1/n!.$$
Thus the volume is $1/n!$ Of course, you use induction.

As suggested by one of you, you can do it neatly using Dirichlet integral. In a sense, this is special case of the Dirichlet integral where all the a_i are one.

You can use the above result to find volume of the following simplex. Fix strictly positive numbers $a_1, a_2, \cdots a_n$. Consider the region,
$$\{(x_1, x_2, \cdots, x_n) : \forall \ i \ \ x_i > 0; \ \ \sum \frac{x_i}{a_i} \le 1\}.$$

Higher dimensions:

We started with general R^n norm, convergence of sequences and so on. Continuity and differentiability were also discussed in general and soon after those definitions we specialized to R^2 and sometimes to R^3. This is only to get a better feel and actually see things. all the results remain true in general. We shall mention only some.

You should get a full and clear picture. There is no need to be able to write complete proofs. The philosophy is simple. If you understand R^2 and R^3; both results as well as proofs; then you can carry out the details in R^n too.

<u>Inverse function theorem:</u>

suppose $\Omega \subset R^n$ is an open set and $f : \Omega \to R^n$ be a C^1 function. Let $a \in \Omega$ with $f'(a)$ non-singular. Recall that $f(x)$ being an n-tuple we can write
$$f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$$

$f'(x)$ is the matrix

$$\begin{pmatrix} \nabla f_1(x) \\ \nabla f_2(x) \\ \cdot \\ \cdot \\ \nabla f_n(x) \end{pmatrix}$$

or equivalently the matrix

$$(D_j f_i(x) : 1 \leq i, j \leq n)$$

Thus the matrix $f'(a)$ is non-singular. then there is an open set $V \subset \Omega$ such that $a \in V$ and an open set $W \subset R^n$ such that the following is true:

(i) f is one-one on V onto W.

(ii) The inverse map $g : W \to V$ is C^1 map.

(iii) For $y = f(x) \in W$ we have $g'(y) = [f'(x)]^{-1}$.

Proof goes along similar lines as in two dimensions.

<u>implicit function theorem:</u>

Suppose $\Omega \subset R^n \times R^m$ is an open set and $f : \Omega \to R^m$ is an C^1 function. Let $(a, b) \in \Omega$. The notation is $a \in R^n$ and $b \in R^m$. suppose that $f(a, b) = 0$. suppose that $f_2(a, b)$ is non-singular. here f_2 is the derivative w.r.t. the last set of m coordinates.

More precisely, let us write, still using the notation, $(x.y)$ for points of Ω with the understanding $x \in R^n$ and $y \in R^m$;

$$f(x, y) = (f_1(x, y), f_2(x, y), \cdots, f_m(x, y))$$

Then $f_2(a, b)$ is the $m \times m$ matrix

$$\begin{pmatrix} \nabla_y f_1(a, b) \\ \nabla_y f_2(a, b) \\ \cdot \\ \cdot \\ \nabla_y f_m(a, b) \end{pmatrix}$$

Then there is an open set $V \subset R^n$; open set $W \subset R^m$ such that $(a, b) \in V \times W \subset \Omega$ and an C^1 function $\varphi : V \to W$ such that $f(x, \varphi(x)) = 0$ for all $x \in V$.

In fact matters are so arranged that for each $x \in V$ there is just one $y \in W$ such that $f(x,y) = 0$ and this unique y is defined as $\varphi(x)$.

The proof we saw in the case $m = n = 1$ is a hands on proof and did not use the inverse function theorem. However the proof for R^n uses inverse function theorem.

integration:

We define (rspa) rectangles with sides parallel to the axes to be a set of the form

$$Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

These are also called boxes or cubes. Volume of this set is the number

$$|Q| = \prod_1^n (b_i - a_i).$$

Let f be a bounded function on such a Q. we partition each side $[a_i, b_i]$ and take the product partition of Q. Calculate inf and sup in each set of the partition, multiply by the volume of the set and add up. These are $L(\pi)$ and $U(\pi)$; lower and upper sums. The inf of upper sums and sup of lower sums are calculated. when they are equal we say f is integrable and this common value is the integral.

A set is small if you can cover it by countably many rspa with total volume as small as desired. All the theorems that we had earlier, remain valid: iterated integration, integrability of continuous functions; relation of integrability to smallness of the set of discontinuities.

We define volumes of more general sets as integral of the constant function one and so on. We define integrals over sets more general than Q. The proofs are nearly same. We say nearly because you need to use induction.

One can then proceed to integrals of unbounded functions or of bounded functions on an unbounded set or unbounded functions on unbounded sets etc.

Jacobian rule:

Let T be a non-singular linear transform of R^n to itself. If A is a bounded set, then TA is bounded. If A is open, then TA is open. Boundaries are preserved, that is, $T(\partial A) = \partial(TA)$. If A is small then so is TA.

Thus bounded open sets with small boundaries are transformed to bounded open sets with small boundaries. and the formula $|TV| = |T||V|$ remains valid.

The Jacobian rule remains valid. Here it is.

Let $\Omega \subset R^n$ be a bounded open set. Let $T : \Omega \to R^n$ be a one-to-one C^1 map with non-singular derivative at every point of Ω.

At every point $x \in \Omega$ let $|T'(x)|$ denote the modulus of the determinant of the derivative matrix. Recall if

$$T(x) = (T_1(x), T_2(x), \cdots, T_n(x))$$

$$T'(x) = ((D_j T_i(x)))_{1 \leq i,j \leq n}$$

Then the following are true:

(i) For any rectangle $Q = \prod[a_i, b_i] \subset \Omega$; TQ has small boundary and

$$|TQ| = \int_Q |T'|.$$

that is

$$|TQ| = \int\int_Q |T'(x)|dx.$$

Here $dx = dx_1 dx_2 \cdots dx_n$.

(ii) (when (i) holds for a T) For any open set $V \subset \Omega$ with small boundary; TV is an open set with small boundary and

$$|TV| = \int_V |T'|.$$

that is,

$$|TV| = \int\int_V |T'(x)|.$$

(iii) (whenever (i) and (ii) hold for a T) For any bounded continuous function f on TV,

$$\int_{TV} f = \int_V f|T|$$

that is,

$$\int_{TV} f(u)du = \int_V f(T(x))|T'(x)|dx.$$