Prelude:

Last semester we understood some aspects of the set R of real numbers —- rational numbers, irrational numbers, sequences, their convergence, series, absolute convergence, products of series and so on. We understood some aspects of functions defined on R taking values in R —- continuity, derivatives, power series, integration of bounded functions over bounded intervals, integration when the function is not bounded or when the interval is not bounded and so on.

As a by-product of the analysis we saw several interesting facts. Functions which have enough derivatives can be expanded in powers of x (Taylor expansion), Fundamental theorem on power series says they define continuous functions which can be differentiated term by term just as we do for polynomials, continuous function on a closed bounded interval can be uniformly approximated by a polynomial as close as desired (Weierstrass); $n!$ can be explained in simpler terms but using complicated numbers like π and e (Stirling); π can be expressed as a (infinite) product of simple ratios of integers (Walli) and so on.

We devised methods to compute simple integrals —- fundamental theorem of calculus allowed us to recognize differentiation and integration as inverse operations in a precise sense. This helped us to convert product rule of differentiation into 'integration by parts', chain rule of differentiation into 'method of substitution'. We devised methods to compute complicated integrals like $\int\limits_0^\infty \frac{\sin x}{x} dx$.

At the same time, you should keep in mind, that there are several problems that we have not discussed. For example, given a bounded interval, what exactly are the conditions for a bounded function to be Riemann integrable? What reasonable conditions are needed so that we can differentiate a series of functions term by term? And so on. Questions like the first one are theoretical in character. However questions like the second are of immense practical use. We shall discuss some of those later in this course.

It would be a good idea for you to do a quick review of what we did so far.

Euclidean spaces:

This semester we shall learn about functions of several variables. First we make clear to ourselves what we mean by 'several variables'. We shall consider the set

$$R^d = \{(x_1, \cdots, x_d) : \quad x_i \in R \quad 1 \leq i \leq d\}.$$

That is, the set of d-tuples of real numbers. Here $d \geq 1$ is an integer. Of course, the case $d = 1$ corresponds to the set of one-tuples, thus it is really no different from the set of real numbers. These spaces are called Euclidean spaces. R^d is called d-dimensional Euclidean space.

We have picturized R as a line; plotted all real numbers on the line —- after arbitrarily marking zero and marking one to its right (then nothing else is in our hands, other numbers have specific places on the line). Every number corresponds to a point on the line and every point on the line corresponds to a real number. In a similar fashion, we can picturize R^2 and you have already done so in high-school (actually we also did when we drew graphs of functions and calculated areas, but we did not stress this aspect). Here it is.

The paper or board is the picture corresponding to R^2. You draw two perpendicular lines: the horizontal line is called x-axis and vertical line is called y-axis. Their point of intersection is taken as the pair $(0, 0)$. Now you think of the two lines as copies of the real line. Plot all numbers on the horizontal line after fixing the place for 1. Similarly on the y-axis, plot all numbers. Just as we have fixed the right side of zero as positive numbers, we fix (just a convention, after all we have to follow something or the other) numbers 'above' zero to be positive on the y-axis. We follow the same units in both axes (it is pleasing).

The pair $(4, 3)$ is plotted on the paper as follows: Start from $(0, 0)$ move on the x axis to 4, then move three units up, mark this point as $(4, 3)$. Similarly, every pair $(a, b) \in R^2$, whether the numbers a and b are positive or not, is identified with a point on the paper. Conversely, every point on the paper corresponds to a point in R^2.

We could not only picturize, but also 'draw' R and R^2. You can not draw R^3 but can imagine as follows. Think of three lines from where you stand: Two lines on the floor they are x-axis and y-axis and the third line is yourself.

You are standing at $(0, 0, 0)$. If you want to plot the point $(4, 7, 3)$ go four units right on the floor and from there go seven units to the front and from there go up three units above, that is the point $(4, 7, 3)$. Of course if the z coordinate is negative, the point is on the other side of the floor!

The reason for this detailed discussion is that you should start making mental pictures for the cases $d = 2, 3$. We consider $d > 1$. You can read our analysis by thinking $d = 1$ too and it remains true. But we said $d > 1$ because we shall be explaining all the concepts using what we already know about real numbers, namely, the case $d = 1$.

We would like to now understand sequences. A sequence is a function defined on natural numbers with values in R^d. Instead of thinking of it as a function f we think of a sequence by its values at one, at two and so on. Think of them as the first term, second term etc of the sequence. We write the sequence , as earlier, as (x^n) or $(x^n : n \geq 1)$ or $(x^n)_{n\geq 1}$. We are using n as super fix, rather than suffix. This is because we used suffix to denote coordinates. Thus $x^n = (x^n_1, x^n_2, \cdots, x^n_d)$.

Incidentally, there is nothing new in the concept of function. In fact we discussed functions from a set A to a set B. But just keep in mind the following. If $f(x) = \pm\sqrt{x}$ on the interval $(0, 1)$ then it is *not* a function. However $g(x) = +\sqrt{x}$ and $h(x) = -\sqrt{x}$ are functions. of course, we made a convention that $\sqrt{x}$ means $+\sqrt{x}$ (just as, 2 means +2).

Returning to sequences and paralleling the earlier development, we wish to say that a sequence converges to a point $x \in R^d$ if the terms of the sequence are getting closer to the point x. Thus we first need to understand what is meant by 'close'.

Norm and Distance:

We define norm on R^d by

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}; \quad x = (x_1, x_2, \cdots, x_d).$$

Thus, square the coordinates and add them up and then take squareroot. Of course, if $d = 1$ this turns out to be just the familiar modulus. Thus, sometimes we would write $|x|$ instead of $||x||$. We think of norm as the distance of the point from the origin 0, the point with all coordinates zero. The reason for this definition comes from Pythagoras theorem. Imagine the case

$d = 2$. If you have a point (x, y), you can make the right angled triangle $(0, 0); (x, 0); (x, y)$ and apply Pythagoras theorem.

As you have noticed, we denoted point in R^2 by (x, y) and not (x_1, x_2). This is how we are used to in school and so we continue. But this has the disadvantage that unless you are alert, you may think that $x = (x_1, x_2)$, $y = (y_1, y_2)$ are two points rather than understanding that x y are real numbers and the pair (x, y) is the point we are talking about.

In R we felt that the distance from 4 to 10 is same as the distance from 0 to $10 - 4 = 6$. Same philosophy we adapt here too. For two points $u, v \in R^d$, the distance between them is

$$d(u, v) = ||u - v|| = \sqrt{\sum_1^d (u_j - v_j)^2}.$$

Squaring is complicated operation where as linear operations are simple to understand and manipulate. It is pleasing to note that the norm is indeed driven by a linear operation. Define

$$u \cdot v = \sum_j u_j v_j; \quad u, v \in R^d.$$

This is called inner product between u and v, sometimes also denoted $\langle u, v \rangle$. It is linear in each argument when the other argument is fixed. Now $||u||$ is nothing but $u \cdot u$. You are probably familiar with these concepts from your linear algebra course. In fact we do need all that material as we go along. We have already used the vector space structure when we used $u - v$ above.

Here are some properties of norm and inner product.
Theorem 1:
(**A**) Norm has the following properties. Here $u, v, w \in R^d$ and $\alpha \in R$.
(i) $||u||$ is a real number; $||u|| \geq 0$; $||u|| = 0$ iff $u = 0$.
(ii) $|u \cdot v| \leq ||u||||v||$ (Cauchy-Schwarz inequality)
(iii) $||\alpha u|| = |\alpha|||u||$ and $||u + v|| \leq ||u|| + ||v||$

(**B**) Distance satisfies the following
$d(u, v) \geq 0$ (positive);
$d(u, v) = d(v, u)$ (symmetric);
$d(u, v) = 0 \leftrightarrow u = v$:
$d(u, v) \leq d(u, w) + d(w, v)$ (triangle inequality)
$d(u + w, v + w) = d(u, v)$ (translation invariant)

$d(\alpha u, \alpha v) = \alpha d(u, v)$ if $\alpha > 0$.

All the properties can be easily verified. We only recall proof of (ii). If $u_1, \cdots, u_k$ and $v_1, \cdots, v_k$ are non-negative real numbers then the following quadratic in λ is always non-negative,

$$\sum (u_j - \lambda v_j)^2 = (\sum v_j^2)\lambda^2 - (2\sum u_j v_j)\lambda + (\sum u_j^2) \geq 0.$$

The fact that its discriminant is nonpositive is precisely the C-S inequality.

To get familiarity with distance, let us get a feel for the following. Given a point u what are all the points which are at a distance smaller than one from u? In case of real line, we have already noted that this distance is same as $|u - v|$. Thus given a number u, the set of points that are at a distance smaller than 1 is just the interval $(u - 1, u + 1)$.

In case $u \in R^2$, your high school familiarity tells you that this set consists precisely points in the interior of the circle with radius 1 and centered at u. In case $u \in R^3$ it is the set of all points in the interior of the sphere with radius one and centered at u. In a sense, this is the meaning of circle and sphere.

convergence of sequences:

Returning to sequences, we say that a sequence (x^n) of points in R^d converges to a point x in R^d, in case the points of the sequence are getting closer and closer to the point x. More precisely, given $\epsilon > 0$, there is a N such that $d(x^n, x) < \epsilon$ for all $n \geq N$. We denote this by $x^n \to x$.

It is useful to explain this notion in terms of sequences of real numbers familiar to us.

Theorem: Let (x^n) be a sequence in R^d and $a \in R^d$. Then the following are equivalent.

(i) $x^n \to a$.

(ii) given $\epsilon > 0$, there is N such that $|x_j^n - a_j| < \epsilon$ for each j; $1 \leq j \leq d$ and for each $n \geq N$.

(iii) For each j, $x_j^n \to a_j$.

Most of you could guess the proof, I suggest you practice writing proof.

This will immediately tell us several things. If $u^n \to u$ and $v^n \to v$, then $(u^n + v^n) \to (u + v)$. Also if we have real numbers $\alpha^n \to \alpha$, then $\alpha^n u^n \to \alpha u$.

In other words the vector space operations are respected by this notion of convergence.

Thus for example the sequence $(1/n, 1 + 2^{-n}) \to (0, 1)$.

The sequence $(e^{-n} \sin n, e^{-n} \cos n) \to (0, 0)$. You plot this sequence and see that it spirals around the origin, getting closer to $(0, 0)$. Intuitively speaking, it is not heading in any fixed direction.

continuous functions:

A function $f : R^d \to R$ is continuous at a point a if for points close to a functional values are close to $f(a)$. More, precisely, given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. A function is continuous if it is continuous at every point.

The concept of continuity makes sense, and is useful, for functions not necessarily defined on all of R. suppose f is defined on a set $S \subset R^d$ and $a \in S$. We say that f is continuous at a if the following happens: given $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \in S$.

Just as in the case of real line, we have the following result.

Theorem: Let f be defined on $S \subset R^d$ and $a \in S$. the following are equivalent.

(i) f is continuous at a.

(ii) $f(x_n) \to f(a)$ whenever (x_n) is a sequence of points in S and $x_n \to a$.

You should be careful, superficial appearance may be deceptive. Consider the function

$$f(x, y) = \frac{xy}{x^2 + y^2}; \quad (x, y) \neq (0, 0); \quad f(0, 0) = 0.$$

If you fix any number x, say, 4 or π or zero, whatever, then $y \mapsto f(\pi, y)$ is a function of one variable. Similarly when you fix a value of y, say, $\sqrt{2}$, then $x \mapsto f(x, \sqrt{2})$ is a function of one variable. In the present example these are all continuous functions. However the function f is not a continuous function on R^2. Reason: calculate $f(1/n, 1/n)$ and see.

Does the intermediate value theorem hold? Yes. Suppose f is a function defined on all of R^d. Let $f(a) = 4$ and $f(b) = 20$. Is there a point c such that $f(c) = 15$? Yes. Join a and b by straight line. In other words consider the set $\{\lambda a + (1-\lambda)b : 0 \leq \lambda \leq 1\}$. This is not part of real line, but none-the-less looks like a line segment and there must be a point on this line with value of f as desired. In fact, if we define $g(\lambda) = f(\lambda a + (1-\lambda)b)$, then g is a continuous function on the interval $[0, 1]$ and from what you have learnt earlier, there must be a number λ with $g(\lambda) = 15$ giving what we wanted.

But what happens if the function is not defined on all of R? If you can draw paths joining points, then the result should be true. Yes. We take this occasion to develop some set theoretic terminology.

Open, Closed, Connected sets:

For a point $a \in R^d$ and a real number $r \geq 0$, we put

$$B(a,r) = \{x \in R^d : ||x-a|| < r\}. \quad \overline{B}(a,r) = \{x \in R^d : ||x-a|| \leq r\}.$$

These are called the open ball and closed ball respectively, with center a and radius r. As already seen earlier, in R this amounts to the intervals $(a-r, a+r)$ and $[a-r, a+r]$ respectively. In case of R^2, $B(a,r)$ is precisely points inside the circle with centre a radius r. And $\overline{B}(a,r)$ is precisely set of points inside as well as points on the circle.

A set $V \subset R^d$ is open if whenever $a \in V$, some space around a is also in the set V. More precisely, there is an $r > 0$ such that $B(a,r) \subset V$. For example in R, the set of rational numbers, the set $[0, 1]$ are not open where as the set $(0, 1)$ is open.

Clearly union of any number of open sets is open. In fact if a point a is in the union, then it must be in one of those sets and then some ball around is contained in that set and hence in the union. Also intersection of finitely many open sets is open. In fact if a is in the intersection then it is in all of them so some ball around a is contained in each of them, take minimum of those finitely many radii to see that this ball around a is contained in the intersection.

Each of the sets $(-1/n, +1/n)$ are open sets and their intersection is just the singleton $\{0\}$ which is not an open set.

Let $S \subset R^d$ and $a \in R^d$. We say that a is a limit point of the set S if every $B(a,r)$ contains a point of S other than a. We say that a set $C \subset R^d$ is closed if all limit points of S are in S. In other words the set is 'closed under limits'.

Just as in case of R we can show that point a is a limit point of S iff there is a sequence (x_n) such that each $x_n \neq a$ and $x_n \to a$.

A set is closed iff its complement is open. Let us prove this. Let A be the set. suppose A is open. If $a \in A$ then there is $r > 0$ such that $B(a, r) \subset A$. In other words, it has no point of A^c. Thus if you take a point of A it can not be a limit point of A^c. As a result all limit pouts of A^c are already in A^c. Thus A^c is closed.

Conversely, let us assume that A^c is closed. To show that A is open, fix $a \in A$. If we can not show $r > 0$ such that $B(a, r) \subset A$, it means that every $B(a, r)$ contains points of A^c. This is precisely the statement that a is a limit point of A^c, see definition. But then A^c does not contain its limit point a, contradicting that A^c is closed.

This shows that closed sets are precisely complements of open sets. There are sets which are neither closed nor open. For example consider the set $[0, 1)$.

A set $S \subset R^d$ is connected if whenever you cut it into two pieces, there is a point which is on the boundary of both pieces. More precisely, if $S = A \cup B$ where $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ then there is a point $x \in S$ such that x is a limit point of A and also x is a limit point of B.

For example very interval contained in R is connected and actually these are the only connected sets in R. Recall, interval means whenever there are two numbers in the set, everything in between is also in the set.

Returning to intermediate value property, here is the precise result.

Let $S \subset R^d$ be a connected set and $f : S \to R$ be continuous. Then f has intermediate value property.

Indeed, let $\alpha < \gamma < \beta$ be three numbers and α, β are both in the range of f. let us define

$$A = \{x \in S : f(x) \leq \gamma\}, \quad B = \{x \in S : f(x) \geq \gamma\}$$

Since α and β are in the range of f it is clear that these sets are non-empty. Also $S = A \cup B$.

The set A contains all its limit points. Indeed let $x_n \in A$ for all n and $x_n \to x$. Then $f(x_n) \leq \gamma$ for all n and $f(x_n) \to f(x)$. Hence $f(x) \leq \gamma$. Thus

$x \in A$. Similarly B contains all its limit points.

Thus there is no point of S which is a limit point of both A and B. In case $A \cap B = \emptyset$ we have a contradiction for the connectedness of S. Thus there is a point common to both A and B. Clearly value of f at such a point equals γ. This proves the intermediate value property.

The argument that we gave earlier using paths is also interesting and let us show the following interesting fact. Suppose that V is a connected open set. Then any two points of V can be connected by a path which lies entirely in V. We have seen several open sets, U-shaped; star shaped, open sets with holes and so on where it was tricky to join points by a path. The fact that we can do at all is a miracle and definitely needs proof. Also, this involves absolutely no complicated maths. If you omit 'open' it is no longer true in general. Of course if you omit 'connected' it is never true.

Well, what do we mean by path? A path is a continuous function $\gamma : [0, 1] \to R^d$. This is called path joining the points $\gamma(0)$ and $\gamma(1)$. The path lies entirely in a set if all values of γ belong to that set.

Returning to our problem, let V be an open connected set. Let us agree temporarily to say $x \sim y$ if we can join x and y by a path entirely lying in V. Fix any point $a \in V$ you like. Let

$$A = \{x \in V : x \sim a\}; \quad B = \{x \in V : x \not\sim a\}.$$

Clearly A and B are disjoint. Also A is non-empty. Indeed, using the fact that V is open take $r > 0$ with $B(a, r) \subset V$. Every point in this ball can be joined to a by straight line which lies entirely in V. Further $A \cup B = V$.

We show A is open. Let $x \in A$. Take $r > 0$ such that $B(x, r) \subset V$, possible because V is open. Every point of this ball can be joined to a because you join to x and then draw a straight line from x to the point in the ball. In other words, this entire ball is contained in A. Thus A is open. In particular, no point of A can be limit point of B.

We show that B is open. Let $x \in B$. Take $r > 0$ with $B(x, r) \subset V$. If any point of this ball can be joined to a, then we can join that end point to x by straight line contradicting the fact that x can not be joined to a. In other words this entire ball is contained in B. Thus B is open. In particular, no point of B can be limit point of A.

If both the sets A, B are non-empty, then connectedness of V is contradicted. Since A is already known to be non-empty, we conclude that B must be empty. This proves our result.

There are many interesting properties and facts about these three concepts; open, closed, connected sets. We can not afford to spend much time on these matters; even if we do, some of you may find it rather abstract. We pick them up when needed.

Differentiation:

The idea of derivative is to understand the rate at which a function is changing; or to find the best linear function that approximates 'near' a given point or to understand velocity etc.

Let $f : R^2 \to R$ be given and $a = (a_1, a_2) \in R^2$.

Remembering our expertise on R, we can try to get functions of one variable using f. This is easily done. We can restrict the function to the horizontal line and study. Thus put,

$$g(x) = f(x, a_2).$$

This is a function of one variable. If it has a derivative at the point a_1, it is reasonable to feel that this is the rate of change at a when you consider the horizontal line, that is, the direction of x-axis. If so, then we look at

$$\lim_{h \to 0} \frac{f(a_1 + h, a_2) - f(a_1, a_2)}{h} = f_x(a_1, a_2).$$

When this limit exists, it is called partial derivative of f w.r.t. x at the point a. This has several notations

$$f_x(a); \quad \frac{\partial}{\partial x} f(a); \quad \frac{\partial f}{\partial x}(a); \quad f_1(a); \quad D_x f(a).$$

Or, one could try to understand the function along the y-axis; in other words, see if the function $y \mapsto f(a_1, y)$ has a derivative at a_2. It is reasonable to feel that this is the rate of change at a when you consider the vertical line, that is the direction of y-axis. If so, then we consider

$$\lim_{k \to 0} \frac{f(a_1, a_2 + k) - f(a_1, a_2)}{k} = f_y(a_1, a_2).$$

This is called (when it exists) partial derivative of f w.r.t. y at the point a. This has several notations

$$f_y(a); \quad \frac{\partial}{\partial y} f(a); \quad \frac{\partial f}{\partial y}(a); \quad f_2(a); \quad D_y f(a).$$

Interestingly, even if these partial derivatives, vertical and horizontal, boh exist at a point, the function need not be continuous. Try the example given earlier $xy/(x^2+y^2)$. Moreover, there are several other directions, lines passing through the given point and we can talk about the rate of change along all those lines. Situation appears unmanageable.

Let us see what we are lead to if we take the other attitude: what is the best linear fit?