Normal integral:

The following integral appears in several contexts.

$$I = \int_{-\infty}^{\infty} e^{-t^2/2} dt.$$

We can calculate this integral by using some tricks. But the simplest is to valculate

$$I^2 = \int_{R^2} e^{-(x^2+y^2)/2} dx dy.$$

Note that by iterated integral process, if you integrate w.r.t. y first you get I and then you integrate w.r.t. x to get I^2.

We shall use the jacobian rule. Put

$$x = r\cos\theta; \quad y = r\sin\theta \qquad (\spadesuit)$$

A little give and take is needed here because this transformation is not really C^1, in fact not even continuous. Let us first precisely define the transformation.

Given any point (x, y) different from zero, we set $r = +\sqrt{x^2 + y^2}$. Thus the given point is $r(x/r, y/r)$ We know from last semester, there is a unique angle $\theta \in [0, 2\pi)$ such that

$$\frac{x}{r} = \cos\theta; \qquad \frac{y}{r} = \sin\theta.$$

Thus given any point different from $(0, 0)$ in R^2 there is a unique pair of numbers $r > 0$ and $0 \leq \theta < 2\pi$ satisfying $(\spadesuit)$. These are called polar coordinates of the cartesian point (x, y).

Thus given a point P, the number r is the distance of the point P from the origin and θ is the angle determined by the positive x-axis and the line joining origin to P.
to the point.

For the point $(0, 0)$ we can and should take $r = 0$ but any θ would do. For other points we have a unique choice. It is easy to show uniqueness.

If a sequence (P_n) of points from the fourth quadrant approach a point P on the x-axis, then $\theta(P_n)$ approaches 2π where as $\theta(P) = 0$.

Let us consider the open set

$$\Omega = R^2 - \{(x, y) : y + 0, x \geq 0\}$$

We remove the non-negative x-axis from R^2. Now it is easy to see that

$$T(x, y) = (r, \theta); \quad \Omega \mapsto (0, \infty) \times (0, 2\pi)$$

is one-one and is in fact C^1 map. Also Jacobian at a polar point (r, θ) is given by

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$|T'| = r.$$

Remember that the set removed is a small set and hence if we integrate over Ω instead of R^2 we still gate I^2. Thus the integral reduces to

$$I^2 = \int_\Omega = \int_0^\infty \int_0^{2\pi} e^{-r^2/2} r dr d\theta$$

Integrate w.r.t. —theta and then w.r.t r to get

$$I^2 = 2\pi, \qquad I = \sqrt{2\pi}$$

Thus we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1.$$

This integrand is called standard normal density.

The way we evaluated is the integral is not right. We treated the whole space as if we have a bounded region. We should actually calculate over bounded regions and take limit. Thus take $R > 0$ and consider

$$\Omega(R) = \{(x, y) : x^2 + y^2 \leq R^2\}$$

We calculate the integral over this region. We have a nice bounded continuous function on this bounded region. From earlier theorems you can give up a little bit, this integral is same as integral over the region with non-negative x-axis (the part which is in the region) removed. You need to calculate integral, after transforming to polar coordinates, over the region

$$\{(r, \theta) : 0 < r < R; 0 < \theta < 2\pi.\}$$

This can be easily done and you can take the limit to get the same answer as above.

Even this is not really right. Imagine calculating integral of the function $f(x, y) = x$ over this region. We will get zero. Are we then going to say integral of f over R^2 is zero?

We should first show that the integral exists and only then proceed to calculate the integral Since our integrand is positive, we can afford not to do this. If over one sequence of regions increasing to R^2 the integrals of the positive continuous integrand are bounded, then the integral exists and you can choose your own convenient regions increasing to R^2 to calculate the integral

We have observed in the process

$$\int_{R^2} e^{-(x^2+y^2)/2} dx dy = 2\pi.$$

suppose you now take $\mu = (\mu_1, \mu_2) \in R^2$. Then we have

$$\int_{R^2} e^{-\{(x-\mu_1)^2+(y-\mu_2)^2\}/2} dx dy = 2\pi.$$

This follows from the Jacobian rule again. Simply change the variables. This can be rewritten as

$$\int_{R^2} e^{-(x-\mu)^t(x-\mu)/2} dx = 2\pi.$$

Remember vectors in R^2 are column vectors. Also we use the notation $x = (x_1, x_2)$ for pints of R^2 rather than (x, y). Also we use dx for $dx_1 dx_2$.

Now suppose that you have a symmetric 2×2 positive definite matrix Σ. Then

$$\int_{R^2} e^{-(x-\mu)^t\Sigma^{-1}(x-\mu)/2} dx = 2\pi\sqrt{|\Sigma|}.$$

This again follows from Jacobian rule. First, Get a symmetric positive definite matrix B with $B^t B = B^2 = \Sigma$.

First change the variables $u_1 = x_1 - \mu_1$ and $u_2 = x_2 - \mu_2$. that is $u = x - \mu$ then change $u = Bv$. Jacobian is B and $|B| = \sqrt{|\Sigma|}$

Beta and Gamma:

Recall that for numbers $a, b > 0$

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

and

$$\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx$$

We shall now show

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Start with

$$\Gamma(a)\Gamma(b) = \int_\Omega e^{-(x+y)}x^{a-1}y^{b-1}dxdy$$

where

$$\Omega = (0,\infty) \times (0,\infty).$$

This is true because if you perform the integration w.r.t. y first and then w.r.t. x you get it. Now we shall change the variable

$$x + y = u; y = y$$

The range set is

$$\Omega' = \{(u,v) : 0 < y < u < \infty\}$$

In other words, Ω' is an open set and on that there is the C^1 map $T(u,y) = (u-y,y)$ which will take you to Ω and you are integrating a function on Ω Here the Jacobian is one. Thus

$$\Gamma(a)\Gamma(b) = \int_{\Omega'} e^{-u}(u-y)^{a-1}y^{b-1}dudy$$

$$= \int_0^\infty e^{-u}\left[\int_0^u (u-y)^{a-1}y^{b-1}dy\right]du$$

Integrate w.r.t. y by substituting $y = vu$ so that dy v du$and range of$viszero to 1. You get

$$\Gamma(a)\Gamma(b) = \beta(a,b)\Gamma(a+b).$$

See how all the theorems are at work.

We have simply written the range of (u,y) without explanation. here it is. x, y range over Ω Thus first of all

$$0 < u < \infty; 0 < y < \infty$$

But not every such u, y come from a point of Ω if you want this pair to come from Ω, since the only pair from which it can come is $(u-y, y)$ e must have

$$0 < u - y < \infty; 0 < y < \infty$$

These four inequalities will tell us $0 < u < \infty$. And if you take such a u, then y ranges over

$$0 < y < \infty; 0 < u - y < \infty$$

If you want both these to be satisfied we must have

$$0 < y < u.$$

Of course here again the integrand could be unbounded and actually we should go through integrals over (ϵ, ∞). But we do not do.

$\int ||\mathbf{x}||^{-\alpha}$:

Let us consider $\alpha > 0$. We want to find out if the following integral exists.

$$\int_{\Omega} ||x||^{-\alpha} dx$$

where

$$\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 < 1.\}$$

If you take $0 < \epsilon < 1$ and

$$\Omega_\epsilon = \{(x_1, x_2) : \epsilon^2 < x_1^2 + x_2^2 < 1.\}$$

then change to polar coordinates (you need to remove the line segment $\{(x_1, 0) : \epsilon < x_1 < 1\}$) we get

$$\int_{\Omega_\epsilon} = \int_\epsilon^1 \int_0^{2\pi} r^{-\alpha} r dr d\theta.$$

You can explicittly calculate this integral and see that a limit as $\epsilon \to 0$ exists iff $\alpha < 2$.