1. Suppose f and g are continuous functions on an interval $[a, b]$. Suppose that $g(x) \geq 0$ for every x. Show that there is a number $\theta \in (a, b)$ such that
$$\int_a^b f(x)g(x)dx = f(\theta)\int_a^b g(x)dx.$$
[In case $\int g = 0$ argue that $g \equiv 0$ and any theta would do. If not, consider the ratio of the two integrals and intermediate value theorem for f. Remember to show $a < \theta < b$.] This is called *mean value theorem for integrals.*

Let f be a function on an interval $[a, b]$ which is n times continuously differentiable. Show that for $a \leq x \leq b$,
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt. \qquad (\spadesuit)$$
[integration by parts]

The above equality is stated as follows
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n;$$
where
$$R_n = \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt.$$
R_n is called the remainder term. Thus ($\spadesuit$) is called *Taylor expansion with integral form of remainder.*

Use appropriate g in the mean value theorem to show
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + f^{(n)}(\theta)\frac{(x-a)^n}{n!}.$$
Here θ is a number between a and x. This is called *Taylor expansion with Lagrange form of remainder.*

Use appropriate g in mean value theorem ($\spadesuit$) to show

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + f^{(n)}(\theta)\frac{(x-\theta)^{n-1}}{(n-1)!}(x-a).$$

Here θ is a number between a and x. This is called *Taylor expansion with Cauchy form of remainder.*

2. Consider R^3.

   Can you imagine the set of points

$$\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1\}.$$

   For each fixed z, its (x,y)-section is either an ellipse or a single point or empty set depending on whether $|z| < 1$ or $|z| = 1$ or $|z| > 1$. (hold on, what is *section*?) This is called an ellipsoid. More generally, for fixed numbers a, b, c (none zero) the set of points satisfying $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ is called an ellipsoid.

   Can you imagine the set of points

$$\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1\}.$$

   For each fixed z, its (x,y)-section is an ellipse. It is smallest when $z = 0$, keeps on becoming bigger as $|z|$ increases. This is called a hyperboloid of one sheet. More generally, for fixed numbers a, b, c (none zero) the set of points satisfying $(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$ is called a hyperboloid of one sheet.

   Can you imagine the set of points

$$\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} - z^2 = -1\}.$$

   For each fixed z its (x,y)-section is either an ellipse or a single point or empty set depending on whether $|z| > 1$ or $|z| = 1$ or $|z| < 1$. In particular the set is in two parts, one in the region ($z \geq 1$) and another in the region ($z \leq -1$). This is called hyperboloid of two sheets. More generally, for fixed numbers a, b, c (none zero) the set of points satisfying $(x/a)^2 + (y/b)^2 - (z/c)^2 = -1$ is called a hyperboloid of two sheets.

3. Define the function f on R^2 as follows. If x and y have same sign then $f(x,y) = xy$ and otherwise $f(x,y)$ is zero. Understand the function. What is its value at $(0,5)$? at $(-3,0)$?

   is this a continuous function?

   Where is the function $f(x,y) = \sqrt{x} - \sqrt{y}$ defined? is it continuous at those points?

   Let $f(x,y)$ equal 1 on the two axes and zero for points not on the axes. Where is this function continuous?

   $f(x,y) = (x^2+y^2)/(x^2-y^2)$. Where is this defined? Is it continuous there?

   $f(x_1,x_2,x_3,x_4) = \sin x_1 \cos x_2 - x_4 e^{x_3}$ is a continuous function on R^4.

4. Let A be an $n \times n$ matrix. Define f on R^n by $f(x) = Ax$. Is it continuous?

   Let us think of points in R^n as column vectors and let x^t denote transpose of x; so x^t is a row vector. Define $f(x) = x^t Ax$. is it continuous function?

5. Let $S \subset R^n$. Let $C(S)$ denote the set of all real valued continuous functions on S. Sow that this is a linear space. show that product of two continuous functions is again continuous.

   Suppose that I have n^2 continuous functions on S denoted as f_{ij} for $1 \le i,j \le n$. Define $f(x)$ to be the determinant of the matrix $((f_{ij}(x)))$ for each $x \in S$. Show that f is a continuous function on S.

   Let f be a continuous function on R^n. Let π be a permutation of the set $\{1,2,\cdots,n\}$. Define a function g on R^n by

   $$g(x_1,x_2,\cdots,x_n) = f(x_{\pi(1)},x_{\pi(2)},\cdots,x_{\pi(n)}).$$

   Show g is continuous.

   Let f be a continuous function on R^n. Let $f_1,\cdots,f_n$ be continuous functions on R to R. Define g on R^n by

   $$g(x_1,x_2,\cdots,x_n) = f(f_1(x_1),f_2(x_2),\cdots f_n(x_n)).$$

   then g is a continuous function on R^n. More generally, let $f_1,f_2,\cdots,f_m$ be continuous functions on R^n and f be a continuous function on R^m. Define g on R^n by

   $$g(x) = f(f_1(x),f_2(x),\cdots,f_m(x)); \quad x \in R^n.$$

   Show g is continuous on R^n.

6. Suppose f is a continuous function on R^5. Let us fix 2 real numbers a_1, a_2. Define a function on R^3 by

$$g(x_1, x_2, x_3) = f(x_1, a_1, x_2, x_3, a_2).$$

Show that g is a continuous function on R^3. do you think the numbers 3 and 5 are special or they an be replaced by any integers $1 \leq m < n$. How does such a generalization read?

7. Suppose f is a function on R^3 and g is a function on R^4. We define a function h on R^7 as follows (think of it as *external product*).

$$h(x_1, x_2, \cdots x_7) = f(x_1, x_2, x_3)g(x_4, \cdots, x_7).$$

Show that h is a continuous function on R^7. do you think 3 and 4 are special or you can replace them by any integers $m \geq 1$ and $n \geq 1$. How does such a generalization read?

8. This problem concerns the way of thinking of our spaces themselves.

Let $(x, y) \in R^2$ and $\sqrt{x^2 + y^2} = 1$. Show that there is a number $\theta \in [0, 2\pi)$ such that $\cos\theta = x; \sin\theta = y$.

Show that every non-zero vector $(x, y) \in R^2$ can be uniquely represented as $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ via $x = r\cos\theta, y = r\sin\theta$. These numbers (r, θ) are called the polar coordinates of the point whose Cartesian coordinates are (x, y).

Show that every non-zero vector $(x, y, z) \in R^3$ can be uniquely represented as $(r, \theta, z) \in (0, \infty) \times [0, 2\pi) \times R$ via $x = r\cos\theta, y = r\sin\theta, z = z$. These numbers (r, θ, z) are called cylindrical coordinates of the point (x, y, z).

Show that every non-zero vector $(x, , y, z) \in R^3$ can be uniquely represented as $(r, \theta, \phi) \in (0, \infty) \times [0, \pi] \times [0, 2\pi)$ via

$$x = r\cos\phi\sin\theta;\ y = r\sin\phi\sin\theta;\ z = r\cos\theta.$$

These numbers (r, θ, ϕ) are called spherical coordinates of (x, y, z).

9. Last time we said that behind the norm is the 'linear' inner product. Let us verify that inner product is actually linear (actually bilinear or linear in each argument).

   Show $\langle cx, y\rangle = c\langle x, y\rangle = \langle x, cy\rangle$.

   Show that $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

   Show the polarisation identity:

$$\frac{||x+y||^2 - ||x-y||^2}{4} = \langle x, y\rangle.$$

10. Argue rigorously that the following set is an open set.

$$\{x \in R^d : ||x-a|| < r\}.$$

   Here $a \in R^d$ and $r > 0$.

11. Let $A \subset R^d$ (you think of R^2 and proceed till the finish and then look back).

   Given a point $x \in R^d$ there are exactly three possibilities:

   either there is an $r > 0$ such that $B(x, r) \subset A$,

   or there is an $r > 0$ such that $B(x, r) \subset A^c$,

   or $B(x, r)$ intersects both A and A^c for ever $r > 0$.

   In the first case say that x is an interior point of A, in the second case say that x is an exterior point of A, in the third case say that x is a boundary point of A.

   Show that the set A^o of interior points of A is an open set. Show that it is the largest open set contained in A.

   Show that the set of exterior points of A is an open set. It is the largest open set disjoint with A.

   Show that the set ∂A of boundary points of A is a closed set.

   Show that $\overline{A} = A \cup \partial A$ is a closed set. Show that it is the smallest closed set containing A. This is called closure of A.

12. If $A \subset R$ is the set of rational numbers, calculate A^o and ∂A.

    Do the same if $A = \{(x, y) : \;\; x^2 + y^2 < 1\} \subset R^2$.

    Do the same if $A = \{(x, y) : \;\; x > 0, y > 0\} \subset R^2$.

13. I have a closed subset of the real line. I know that every rational number is in my set. What do you think my set could be?

14. Let $f : R^2 \to R$. Suppose that $f(x, y_1) = f(x, y_2)$ for any three real numbers x, y_1, y_2.

    Show that there is a function $g : R \to R$ such that $f(x, y) = g(x)$ for every (x, y).

15. Let $f(x, y)$ be one or zero according as y is rational or not. Does f_1 exist? what is it? Is f continuous at any point?

16. Find derivatives of the following functions.

    $f(x, y) = x^y$ defined on $\{(x, y) : x > 0, y > 0\}$.

    $f(x, y, z) = x^{(y^z)}$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y, z) = (x^y)^z$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y, z) = x^{(y+z)}$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y, z) = (x + y)^z$ defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

    $f(x, y) = \sin(x \cos y)$.

    $f(x, y, z) = \sin(x \cos[y \sin z])$.

    $f(x, y, z) = \sin(xyz)$

    $f(x, y, z) = \sin x + \sin y + \sin z$.

17. Let $n \geq 3$. Put

    $$f(x_1, x_2, \cdots, x_n) = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^{(n-2)/2}}.$$

    Show

    $$f_{11} + f_{22} + \cdots + f_{nn} \equiv 0.$$

18. Let

    $$f(x, y) = e^x \cos y; \qquad g(x, y) = e^x \sin y.$$

    Show $f_{11} + f_{22} \equiv 0$ and similar for g.

19. Find if limit as $(x, y) \to (0, 0)$ exists for the following and find when the limit exists.

$$f(x,y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}; \qquad g(x,y) = \frac{\sin(x^4 + y^4)}{x^2 + y^2}$$

$$f(x,y) = \frac{\sin(x^2 + y^2)}{x^4 + y^4}; \qquad g(x,y) = \frac{e^{-1/(x^2+y^2)}}{x^4 + y^4}$$

20. Suppose that ξ and η are C^1 functions on R to R. Find the derivatives of the following.

$$f(x,y) = \xi(x + y); \qquad g(x,y) = \eta(xy).$$

$$f(x,y) = \xi(x)\eta(y); \qquad g(x,y) = \eta(x).$$

21. $f : R^2 \to R$. Assume that the two partial derivatives exist and $f_1 \equiv 0$ and $f_2 \equiv 0$. Show that f is a constant.

[sometimes I feel that some of you are not serious and are violating the trust put on you by the Institute. If it persists, and if you use your freedom not to attend classes, I would have to use my freedom not to allow you to sit for my exam. But let us hope sense prevails and I do not have to do this.]

We have finally changed our attitude towards derivative, it is not a number, not a vector, but a linear transformation — the best that you can think of for your function near your point of interest. It takes time to get used to this idea. Do not worry. But what you should realize is that we came to the conclusion in order to bring some order in a chaotic situation. I am not giving many problems in this set so that you have time to settle down to the new ideas.

22. The following functions have the interesting property that they are continuous 'along any line' at $(0,0)$, but they are not continuous at $(0,0)$.

$$f(x,y) = \frac{x^4y^4}{(x^2+y^4)^3}, \quad (x,y) \neq (0,0); \quad f(0,0) = 0.$$

$$g(x,y) = \frac{x^2}{x^2+y^2-x}, \quad (x,y) \neq (0,0); \quad f(0,0) = 0.$$

(I have not defined the phrase 'along any line'; what could it mean?)

23. Sometimes a function can be expressed both in cartesian coordinates and also in polar coordinates. for example the function $u(x,y) = x^2+y^2$ is 'same as' $u(r,\theta) = r^2$. It would be confusing at first sight, using the same notation u. But let us indulge in such a confusion.

Express cartesian coordinate derivatives in terms of polar coordinate derivatives:

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}.$$

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

$$u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2} u_\theta^2.$$

$$u_{xx} = u_{rr}\cos^2\theta + u_{\theta\theta}\frac{\sin^2\theta}{r} - 2u_{r\theta}\frac{\cos\theta\sin\theta}{r} +$$

$$u_r\frac{\sin^2\theta}{r} + 2u_\theta\frac{\cos\theta\sin\theta}{r^2}.$$

$$u_{yy} = u_{rr}\sin^2\theta + u_{\theta\theta}\frac{\cos^2\theta}{r} + 2u_{r\theta}\frac{\cos\theta\sin\theta}{r} +$$

$$u_r\frac{\cos^2\theta}{r} - 2u_\theta\frac{\cos\theta\sin\theta}{r^2}.$$

$$u_{xy} = u_{rr}\cos\theta\sin\theta - u_{\theta\theta}\frac{\cos\theta\sin\theta}{r^2} + u_{r\theta}\frac{\cos^2\theta - \sin^2\theta}{r} -$$

$$u_r\frac{\sin\theta\cos\theta}{r} + u_\theta\frac{\sin^2\theta - \cos^2\theta}{r^2}.$$

$$u_{xx} + u_{yy} = u_{rr} + u_{\theta\theta}\frac{1}{r^2} = u_r\frac{1}{r}.$$

$$= \frac{1}{r^2}\left\{r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right\}.$$

Here is how you express polar coordinate derivatives in tees of cartesian coordinate derivatives.

$$u_r = u_x\cos\theta + u_y\sin\theta.$$

$$u_\theta = -u_x r\sin\theta + u_y r\cos\theta.$$

24. f and g are two C^2 functions on R. Put $h(x,y) = f(x-y) + g(x+y)$. Show $h_{xx} - h_{yy} = 0$.

25. When coordinates x, y are changed to

$$\xi = a + \alpha x + \beta y; \quad \eta = b - \beta x + \alpha y;$$

where $\alpha^2 + \beta^2 = 1$; the function $u(x,y)$ is changed to $U(\xi,\eta)$. Show

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 = u_x u_{yy} - u_{xy}^2.$$

As I keep on mentioning, it is the ideas and the ability to think and grasp what is going on that plays crucial role. The mathematics is not difficult, probably imitation (with proper notation) of one variable results. This is so at the initial stages of the development but gradually it changes we will have things that are not imitations of one variable results.

Though most of the problems below are stated for R^n. It is alright if you do not solve it for general n. But you should work out the two cases $n = 2$ and $n = 3$ and also get a feel. However, you must understand what it says for n.

26. There are two C^1 functions $f : R \to R^n$ and $g : R \to R^n$. Define

$$h(t) = \langle f(t), g(t) \rangle; \quad t \in R.$$

Calculate $h'(t)$.

27. Let f be a C^1 function from R to R^n. Suppose $||f(t)|| = 1$ for each t. Show that $f'(t) \cdot f(t) \equiv 0$

In other words the function is orthogonal to its gradient at every point.

For example, $f(t) = (\cos t, \sin t)$ is such a function.

28. Suppose f is a real valued C^1 function of two variables; g, h are real valued C^1 functions of one variable, then I define real valued function of three variables:

$$F(x, y, z) = f(g(x + y), h(y + z)).$$

Calculate F'.

29. I have three real valued C^1-functions: f is a function of one variable, g is a function of two variables, h is a function of three variables. I have a C^1 function φ of four variables. I define a function of three variables;

$$F(x, y, z) = \varphi(x, f(x), g(x, y), h(x, y, z))$$

find F'.

30. Suppose I have n^2 many C^1 functions $f_{ij}(t)$ defined on R to R. Here $1 \leq i \leq n$ and $1 \leq j \leq n$.

    I consider the $n \times n$ matrix $A(t) = (\ f_{ij}(t)\ )$. Consider the real valued function $F(t)$ defined on the real line by the formula

    $$F(t) = \mathrm{Det}A(t).$$

    Show that F is C^1 function and $F'(t)$ is sum of determinants of n matrices $\{A_i :\leq i \leq n\}$ where A_i is obtained by replacing the i-th row of $A(t)$ by derivative of that row; more precisely, replace the i-th row of $A(t)$ by

    $$(f'_{i1}(t), f'_{i2}(t), \cdots, f'_{in}(t)).$$

31. This exercise is long. Take fifteen/twenty minutes to understand. To solve, you need five minutes.

    It is difficult to keep on writing R^{n^2} all the time. Let us use an abbreviation just for this exercise: $\mathbf{d} = \mathbf{n^2}$.

    The space of $n \times n$ matrices can be thought of as R^d in the following way: First row followed by second row etc. More precisely if A is $n \times n$ matrix (a_{ij}) then it is the point in R^d defined as

    $$(a_{11}, a_{12}, \cdots, a_{1n},\ \ a_{21}, a_{22}, \cdots, a_{2n},\ \ \cdots,\ \ a_{n1}, a_{n2}, \cdots, a_{nn}).$$

    We regard R^d as column vectors, but for this exercise and for typographical considerations imagine rows. It is possible to stick to column vectors but that will confuse you.

    When you read a matrix, you read first row and then second row etc. This is precisely how we identified it as a point of R^d.

    Let me say it differently to familiarize you. If you have matrix with rows $r_1, r_2, \cdots, r_n$ then the point of R^d to which it corresponds is

    $$(r_1, r_2, \cdots, r_n).$$

    Be careful. Here r_i is not a number it is an n-tuple of numbers, the i-th row of A, thus all these n many n-tuples of numbers make up an n^2-tuple.

    Similarly, if you have a point x of R^d it corresponds to the matrix whose first row is the first n entries of x, second row is the next n entries of x and so on. Define

    $$F(x) = \mathrm{Det}(x); \qquad x \in R^d.$$

Thus when we wrote Det, we are regarding the x as a $n \times n$ matrix. Then show that F is C^1 (defined on R^d to R).

If you take a point $a \in R^d$, then the derivative of F at a is a linear transformation $F'(a) : R^d \to R$. What is it? If you take $x \in R^d$ then value of the linear transformation $F'(a)$ at the point x is the sum of n determinants of matrices $A_1, A_2, \cdots A_n$ where A_i has all rows except the i row same as those of a, but the i-th row is the i-th row of x.

To say differently, if

$$a = (a_1, a_2, \cdots, a_n); \quad x = (x_1, x_2, \cdots, x_n) \in R^d$$

(pause, do you understand what are these a's and x's?) then

$$F'(a)(x) = \sum_1^n \mathrm{Det}(a_1, \cdots, a_{i-1}, x_i, a_{i+1} \cdots, a_n).$$

32. z is a function of two variables (x, y) and I know $xy + yz - zx \equiv 2$.

    Using implicit function rules calculate z_1 and z_2, the partial derivatives. Also explicitly solve for the function z and calculate.

33. Find a number θ such that $0 < \theta < 1$ and

$$f\left(1, \frac{1}{2}, \frac{1}{3}\right) = f_x\left(\theta, \frac{\theta}{2}, \frac{\theta}{3}\right) + \frac{1}{2} f_y\left(\theta, \frac{\theta}{2}, \frac{\theta}{3}\right) + \frac{1}{3} f_z\left(\theta, \frac{\theta}{2}, \frac{\theta}{3}\right)$$

    when

$$f(x, y, z) = xyz.$$

$$f(x, y, z) = xy + yz + zx.$$

34. Let $\Omega \subset R^2$ be a region which has the following property:

$$(x,y) \in \Omega; \lambda > 0 \Rightarrow (\lambda x, \lambda y) \in \Omega.$$

For example any quadrant or all of R^2 satisfies this condition. If you do not like this, think of R^2. Let n be an integer.

A function $f : R^2 \to R$ is *homogeneous* of degree n if the following holds:

$$(x,y) \in \Omega; \lambda > 0 \Rightarrow f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

Show that the following functions are homogeneous and find the order.

$$f(x,y) = ||(x,y)||; \qquad \Omega = R^2.$$

$$f(x,y) = \log(y/x); \qquad \text{First quadrant, axes not included}$$

$$f(x,y) = ax^2 + bxy + cy^2.$$

Let f be real valued C^1 function on R^2.

If f homogeneous of degree n, show that $xf_1 + yf_2 = nf$.

(Hint: differentiate w.r.t. λ, the equation that defines homogeneity.)

If $xf_1 + yf_2 = nf$ then show that f is homogeneous of degree n.

(Hint: show $\varphi(\lambda) = f(\lambda a, \lambda b)$ satisfies $\lambda\varphi'(\lambda) = n\varphi(\lambda)$ and $\varphi(\lambda)\lambda^{-n}$ is a constant.)

35. Find derivative of

$$f(x) = \int_{x^2}^{x^3} \frac{1}{x+t} dt.$$

Evaluate integral and differentiate and also do by using the formula derived by us.

$$f(x) = \int_{-x}^{x} \frac{1}{x^2+t+1} dt. \qquad f(x) = \int_{x^2}^{-sinx} e^{xt} dt.$$

$$f(x) = \log\left(\int_0^{x^2} \frac{\sin xt}{t} dt\right)$$

36. Calculate $f_1(x,y)$ and $f_2(x,y)$

$$f(x,y) = \int_{xy}^{x^2+y} \frac{\sin(t+y)}{t^2+y^2} dt.$$

37. Calculate

$$\lim_{x\to 0} \frac{1}{x^3} \int_0^x (x-t)^2 f(t) dt.$$

38. This is something we should have done last semester. Recall, we showed that the integral

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt$$

is finite for $x > 0; y > 0$. Value of the integral was denoted by $\beta(x,y)$. show

$$\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} \, dt.$$

(Put $t/(1+t) = u$)

Show

$$\beta(x,y) = 2\int_0^{\pi/2} (\sin t)^{2x-1} (\cos t)^{2y-1} \, dt.$$

Show that for any two integers $x \geq 1$ and $y \geq 1$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Actually this is true for any $x, y > 0$ but its proof should wait till we develop integration of functions of two variables.

39.

$$\frac{\partial^n e^{xy}}{\partial x^m \partial y^{n-m}}(0,0) = \begin{cases} 0 & \text{if} \quad 2m \neq n \\ m! & \text{if} \quad 2m = n \end{cases}$$

40. Consider the complement of the closed fourth quadrant in R^2, that is U = complement of the set $\{(x,y) : x \geq 0, y \leq 0\}$.
Let f be be a real valued continuous function on U such that both partial derivatives are zero. Show f is a constant.

Suppose that U is the set of all points (x,y) such that both x, y are strictly positive or bot are strictly negative. Show that U is an open set. Suppose that both partial derivatives are zero. Can you conclude that f is a constant function?

41. suppose that U is an open set in R^2 and $f : U \to R^2$ is a C^1 function. Assume that f is C^1, f is one-to-one, $f')(x$ is non-singular for all x.

Show that range of f is again an open set. In fact show that whenever $V \subset U$ is an open set then $f(V)$ is an open set.

42. This exercise is simple but you should work out fully.

We consider R^2. In my mind I think of (x, y) as $x + iy$ where $i = \sqrt{-1}$. So what?

Define addition on R^2 as usual

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

We define multiplication on R^2 as follows.

$$(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

We made it clear in the class that you can not divide one vector by another vector when you are considering R^n with $n \geq 2$. There is going to be confusion and chaos if we keep on using the same symbol R^2. Let use the symbol C to denote the same set R^2 when we consider the above multiplication.

Thus as a set C is same as R^2, but when we use C it means that we are allowed to use the operation of multiplication. We then refer to elements of C as complex numbers. If $z = (x, y)$ is a complex number we also refer to x as the real part of z and y as the imaginary part of z (instead of calling them as first and second coordinates of z).

Show the following in C. Let $z_1 = (x_1, y_1)$ $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3)$.

$$z_1 \times z_2 = z_2 \times z_1.$$

Thus multiplication is commutative.

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3.$$

Thus multiplication is distributive w.r.t. addition. Let e stand for the complex number $(1, 0)$. Of course 0 stands for $(0, 0)$.

$$z \times e = z \quad \forall z.$$

Thus e is the identity element for multiplication.

$$(\forall z,\ z \neq 0)\ (\exists!\ w)\ zw = e.$$

Thus every non-zero complex number has an (multiplicative) inverse.

We define modulus $|z|$ of a complex number $z = (x, y)$ as norm of (x, y), more precisely, $|z| = \sqrt{x^2 + y^2}$. Show that the usual rules hold:

$$|z_1 + z_2| \leq |z_1| + |z_2|; \quad |z_1 \times z_2| = |z_1||z_2|.$$

We want to identify a complex number whose imaginary part is zero with real number, namely, the real part of that number. In other words the x-axis is identified with real numbers. Thus when we write 'the complex number 5' we mean $(5, 0)$.

Show that the complex multiplication we defined above, when restricted to x-axis coincides with usual multiplication of real numbers.

43. In the following repeated integrals, draw a picture of the region in R^2 where the integration is being carried out. Change the order of integration. There is nothing for you to evaluate.

$$\int_0^1 \left[\int_{x^2}^1 f(x,y)dy\right] dx. \qquad \int_{-2}^1 \left[\int_x^{x^2} f(x,y)dy\right] dx.$$

$$\int_{1/3}^{2/3} \left[\int_{y^2}^{\sqrt{y}} f(x,y)dx\right] dy.$$

44. Show

$$\int_a^b \left[\int_a^x f(x,y)dy\right] dx = \int_a^b \left[\int_y^b f(x,y)dx\right] dy.$$

This is called Dirichlet's formula.

Show

$$2! \int_a^b f(x) \left[\int_x^b f(y)dy\right] dx = \left[\int_a^b f(x)dx\right]^2.$$

Expressing this repeated integral as double integral will help.

45. Consider the solid bounded by the planes $z = x + a$; $z = -x - a$; and the cylinder $x^2 + y^2 = a^2$. Express its volume as a double integral.

Consider the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ where a, b, c are positive numbers. Express its volume as a double integral and evaluate.

46. You are given n points $\{(x_i, y_i) : 1 \le i \le n\}$ in R^2. You should fit the best straight line. That is straight line $y = ax + b$ so that

$$f(a,b) = \sum_1^n (ax_i + b - y_i)^2$$

is minimum. This is called method of least squares.

47. f and g are C^1 functions on R^n to R. Assume that g does not take the value zero. Show

$$\nabla(f/g) = \frac{g\nabla f \ - \ f\nabla g}{g^2}.$$

48. f is a C^1 function on R^3 to R. At every point (x, y, z) the vectors $\nabla f(x, y, z)$ and (x, y, z) are parallel. Show that

$$f(0, 0, z) \equiv f(0, 0, -z).$$

49. Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be points of R^3. Define

$$f(x) = \langle x \times a, x \times b \rangle; \qquad x \in R^3.$$

Here $\times$ denotes vector product. show

$$\nabla f(x) = a \times (x \times b) + b \times (x \times a).$$

50. Let F be a C^2 function and G be a C^1 function on R. Let c be a real number. Define a function $f(x, t)$ on R^2 by

$$f(x,t) = \frac{F(x+ct) - F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds. \qquad (*)$$

Show that the function satisfies

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}. \qquad (**)$$

and

$$f(x, 0) = F(x); \quad f_2(x, 0) = G(x). \qquad (***)$$

The (partial) differential equation (**) is called one-dimensional wave equation. The solution (*) is called D'Alembert's solution. The conditions (***) are called initial conditions.

51. Find stationary points (zero gradient) for the following functions and classify if they are maxima or minima or neither.
$f(x, y) = x^2 + (y - 1)^2$.
$f(x, y) = x^2 - (y - 1)^2$.
$f(x, y) = \sin x \sin y \sin(x + y)$.
$f(x, y) = \sin x \cosh y$.

52. If $f : R^3 \to R^3$ be C^1 function, $f = (f_1, f_2, f_3)$, divergence of f is defined by

$$div(f) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \nabla \cdot f.$$

Show $div(f + g) = div(f) + div(g)$. if $\varphi : R^3 \to R$ be C^1, then show $div(\varphi f) = \varphi(div(f)) + \nabla\varphi \cdot f$.

53. calculate the following integrals:

(a) $\int\int_{\Omega} e^{x+y} dxdy$ over $\Omega = \{|x| + |y| \leq 1\}$.

(b) $\int\int_{\Omega} x^2y^2 dxdy$ over $x \geq 0, y \geq 0, xy = 1, xy = 2, y = x, y = 4x$.

(c) $\int\int_{\Omega} (x^2 + y^2) dxdy$ over $\Omega = \{|x| \leq 1, |y| \leq 1\}$.

(d) $\int\int_{\Omega} (3x + y) dxdy$ over $\Omega = \{x \geq 0, y \geq 0, 4x^2 + y^2 \leq 36\}$.

54. In each of the following, a region Ω is given, a transformation $(x, y) \mapsto (u, v)$ is given. Show that it is one-one on Ω; describe the transformed region in two ways

(i) possible values of u and for every u, the possible values of v;

(ii) possible values of v and for every v, the possible values of u.

(a) $\Omega = [0, 1] \times [0, 1]$; $u = x + y$, $v = x - y$.

(b) $\Omega = [-1, +1] \times [-1, +1]$; u, v as above.

(c) $\Omega = [0, 1] \times [0, 1]$; $u = x + y$, $v = x$.

(d) $\Omega = (0, \infty) \times (0, \infty)$; $u = xy$, $v = y$.

(e) $\Omega = (0, \infty) \times (0, \infty)$; $u = xy$, $v = x/y$.

(f) $\Omega = (0, \infty) \times (0, \infty)$; $u = x + y$, $v = y$.

(g) $\Omega = (0, \infty) \times (0, \infty)$; $u = x + y$, $v = y/(x + y)$.

(h) $\Omega = (0, \infty) \times (0, \infty)$; $u = x^2 + y^2$, $v = y$.

(i) $\Omega = (0, \infty) \times (0, \infty)$; $u = x^2/y^2$, $v = y^2$.

in the following verify the map is one-to-one and describe the range of u and for each u, the possible values of v and for every u, v the possible values of w.

(j) $\Omega = (0, \infty) \times (0, \infty) \times (0, \infty)$;
$u = x + y + z$, $v = x + y$, $w = x$.

(k) $\Omega = (0, \infty) \times (0, \infty) \times (0, \infty)$;
$u = x + y + z$, $v = x/(x + y + z)$, $w = z/(x + y + z)$.

(l) $\Omega = (0,\infty) \times (0,\infty) \times (0,\infty)$;
$u = x^2 + y^2 + z^2$, $v = x/\sqrt{x^2+y^2+z^2}$, $w = y/\sqrt{x^2+y^2+z^2}$

Discuss the following map (what does this mean?).

(m) $\Omega = [0,\infty) \times [0,\pi) \times [0,2\pi)$
points here are denoted by (r,θ,ϕ).
$x = r\cos\theta$, $y = r\sin\theta\cos\phi$, $z = r\sin\theta\sin\phi$.

(n) $\Omega = [0,\infty) \times [0,\pi) \times [0,2\pi) \times [0,2\pi)$
points here are denoted by (r,θ,ϕ,ψ).
$x = r\cos\theta, y = r\sin\theta\cos\phi, z = r\sin\theta\sin\phi\cos\psi., w = r\sin\theta\sin\phi\sin\psi.$

55. If you have a bounded region $\Omega \subset R^2$ with small boundary, then centroid of Ω is the point $(\overline{x},\overline{y})$ defined by

$$\overline{x} = \int\int_{\Omega} x dx dy; \quad \overline{y} = \int\int_{\Omega} y dx dy.$$

calculate centroids of the regions shown (?) below.

(a) unit disc

(b) the triangle with vertices $(-1,0)$, $(0,-1)$, $(1,1)$.

(c) the unit square $[0,1] \times [0,1]$

(d) region bounded by $y = x^2; x + y = 2$

(e) region bounded by $y = \sin^2 x$, $y = 0$, $0 \le x \le \pi$.

(f) region bounded by $y = \sin x$, $y = \cos x$, $0 \le x \le \pi/4$.

(g) region bounded by $x > 0$, $y > 0$, $\sqrt{x} + \sqrt{y} = 1$.

56. Let A be a symmetric 2×2 matrix with strictly positive eigen values. This is same as saying that A is symmetric positive definite matrix. Show that there is a symmetric positive definite matrix B such that

$$B^2 = B^T B = A$$

Generalize to higher dimensions (why should we care?)

57. Calculate the Taylor expansion around the origin upto third order.

    (a) $f(x, y) = \exp\{\sin y\}$.

    (b) $f(x, y) = \cos(xy)$.

    (c) $f(x, y, z) = \sin\{e^x + y^2 = z^3\}$.

58. In the following calculate the integrals over the regions indicated.

    (a) $Q = [-1, 1] \times [0, 2]$. $\int\int_Q \sqrt{|y - x^2|}$.

    (b) $Q = [0, \pi/2] \times [0, \pi/2]$. $\int\int_Q \sin(x + y)$.

    (c) $Q = [0, \pi] \times [0, \pi]$. $\int\int_Q \cos(x + y)$.

    (d) $Q = [0, 3] \times [0, 2]$. $\int\int_Q [x + y]$;

    where $[a]$ is the greatest integer not exceeding a.

    (e) $Q = [0, 1] \times [0, 1]$. $\int\int_Q f$;

    where f is defined on Q as follows. $f(x, y) = x+y$ if $x^2 \leq y \leq 2x^2$; and zero otherwise.

    (f) $\Omega = \{4x^2 + 9y^2 \leq 36; x > 0; y > 0\}$, $\int\int_S (3x + y)$.

    (g) $\Omega = \{x^2 + y^2 \leq 16\}$, $\int\int_S (20 + 2x + y)$.

    (h) Ω is the region bounded by $xy = 1$; $xy = 2$; $y = x$ and $y = 4x$, $\int\int_S (x^2y^2)$.

    (i) Let $D = \{x \in R^2 : 0 < ||x|| < 1\}$. Show that the integral $\int\int_D \log ||x||$ exists and calculate it.

    Same problem in other dimensions.

    (j) Let D be as above. When is the integral $\int\int_D ||x||^{-p}$ convergent $(p > 0)$.

    Same problem in other dimensions.

    (k) Let $\Omega = \{||x|| > 1\}$. When is the integral $\int\int_\Omega ||x||^p$ convergent. Here $p > 0$. Calculate the integral.

59. For the function $f(x, y) = xy(1-x^2-y^2)$ on $[0,1]\times[0,1]$ calculate local maxima, local minima, global maxima, global minima, saddle points.

60. Do the same problem for the function $f(x, y, z) = x^4 + y^4 + z^4 - 4xyz$.

61. Suppose that $f(x, y, z)$ is a C^2 function with a stationary point P. Suppose that $f''(P)$, that is, the second derivative matrix, has two diagonal entries with opposite sign.

    Show that the point P is a saddle point.

62. For the function $f(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + h$ $(a, b, \cdots$ are constants) with $a > 0$ and $b^2 < ac$ show that there is a global minimum. Calculate it.

63. Which straight line is close to $f(x) = x^2$ on $[0, 1]$. What about on $[1, 2]$. (What does the question mean?)

64. Suppose $a^1, a^2, \cdots, a^{100}$ are points in R^{121}. Show the centroid minimizes $f(x) = \sum\limits_i ||x - a^i||^2$.

65. Let A and B be $n \times n$ symmetric matrices and B is positive definite. Solve: $\quad \max x'Ax$ subject to $x'Bx = 1$.

    First show that the later set is compact and so the problem has a solution.

    Show that if x and λ are obtained by Lagrange method, then $Ax = \lambda Bx$ and λ is the max.

66. Given k points $\{(x_i, y_i) : 1 \le i \le k\}$ in R^2 (x_i are distinct), fit the best straight-line by least squares method. Give formulae for the parameters of the line.

    Find the best quadratic function by least square method. Find formulae for the parameters.

67. I have a die with 9 faces, the i-th face having probability $p_i > 0$ in a throw. I rolled the die 100 times and got the observation (?)

    $$(8, 12, 4, 11, 20, 6, 14, 9, 16)$$

    What do you think is the most likely values of the (p_i)? (I believe that what I observed must be having maximum probability).

68. Are the following functions continuous? if not describe the set of points where they are discontinuous?

(a)

$$f(x,y) = \sqrt{1-x^2+y^2}; \quad \text{if} \quad x^2+y^2 \le 1$$
$$f(x,y) = (1-x^2+y^2)^5 \quad \text{if} \quad x^2+y^2 > 1$$

(b)

$$f(x,y) = \sqrt[86]{xy} \quad \text{if} \quad (x \ge 0, y \ge 0)\text{or}(x \le 0, y \le 0)$$
$$f(x,y) = \sqrt[59]{xy} \quad \text{otherwise.}$$

(c)

$$f(x,y) = \frac{\sin(xy)}{x} \quad x \ne 0$$

and $f(x,y) = 1$ when $x = 0$.

(d)

$$f(x,y) = \begin{cases} \frac{1}{2y} & \text{if} \quad |x| < |y|; y \ne 0 \\ 0 & \text{if} \quad |x| > |y|; y \ne 0 \\ +1 & \text{if} \quad x = 0, y = 0 \\ 0 & \text{if} \quad x \ne 0; y = 0. \end{cases}$$

69. Describe the set where the functions are defined (called domain of definition) and compute ∇f at the interior points of the set.

(a)

$$f(x,y) = \exp\left\{\frac{x}{y} + \frac{y}{x}\right\}$$

(b)

$$f(x,y) = \sin^{-1}(x+y)$$

(c)

$$f(x,y) = \tan^{-1}\sqrt{\frac{x^2-y^2}{x^2+y^2}}.$$

(d)

$$f(x,y) = e^x \log y + \sin y \log x.$$

(e)

$$f(x,y) = \log \frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2=y^2}+x}$$

(f)

$$f(x,y) = \frac{x}{y}.$$

70. Find the directional derivatives.

(a) $f(x,y) = 2x^2 - y^2$ at the point $(1,2)$ in the direction of the line joining $(1,2)$ to $(4,0)$.

(b) $f(x,y) = x^3+3xy+4y^2$ at $(0,0)$ in the direction of the line making 60^o with the x-axis.

(c) $f(x,y) = x^2+y^2-3xy$ at the point $(1,2)$ in the direction of the tangent to the curve $y = x^2$ at $(0,0)$.

(d) $f(x,y) = 5x^2-3x-y-1$ at $(2,1)$ in the direction of the line from this line to $(5,5)$.

(e) $f(x,y) = x^3+3x^2+4xy+y^2$ at the point $(2/3, \quad -4/3)$ in all directions.

71. Define a function $u(x,t)$ for $0 < t < \infty$ and $-\infty < x < \infty$ by

$$u(x,t) = \frac{1}{t^{171}} \int_{-t}^{t} e^{-(x+y)}(t^2-y^2)^{85} dy.$$

Show that

$$u_{xx} = \frac{172}{t} u_t + u_{tt}.$$

72. For the curves find their length. parametric so that arc length is the parameter.

(a) Consider the circular helix

$$x_1 = \cos t; \quad x_2 = \sin t; \quad x_3 = t$$

for $0 \le t \le 100$.

(b)

$$x_1(t) = \frac{\sin t}{\sqrt{2}}; \quad x_2 = \frac{\sin t}{\sqrt{2}}; \quad x_3 = \cos t.$$

(c)

$$x_1 = 6t; \quad x_2 = 3t^2 \quad x_3 = t^3$$

for $0 \le t \le 1$.

(d)

$$x_1 = e^t; \quad x_2 = e^{-t} \quad x_3 = \sqrt{2}t.$$

(e)

$$x_1 = \frac{\sqrt{t^2+4}+t}{2} \quad x_2 = \frac{\sqrt{t^2+4}-t}{2}$$

$$x_3 = \sqrt{2}\log\frac{\sqrt{t^2+4}+t}{2}.$$

73. Show that the plane passing through the points $\{(x_i, y_i, z_,) : i = 1, 2, 3\}$ has equation

$$\begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0.$$

74. suppose f, f^2, g, g^2 are continuous or just bounded integrable functions on $[a,b]$. Show

$$\frac{1}{2}\int_a^b \left[\int_a^b \left| \begin{array}{cc} f(x) & g(x) \\ f(y) & g(y) \end{array} \right|^2 dy\right] dx \;=\; \int_a^b f^2 \int_a^b g^2 - \left(\int_a^b fg\right)^2.$$

Deduce Cauchy-Schwarz inequality for integrals (what is it?)

75. Show

$$\begin{aligned}\frac{1}{2}\int_a^b \left[\int_a^b [f(y)-f(x)][g(y)-g(x)]dy\right] dx \\ = \;(b-a)\int_a^b fg - \int_a^b f \int_a^b g.\end{aligned}$$

76. Let f be a non-negative continuous function on $[a,b]$ with $\sup f = M$. Show

$$\lim_n \left\{\int_a^b [f(x)]^n dx\right\}^{1/n} = M.$$

The following is known as Laplace principle. This will locate minimum of a function.

Let $h : [0,1] \to R$ be a continuous function. Then

$$\lim_n \frac{1}{n}\log \int_0^1 e^{-nh(x)}dx = -\min h.$$

77. Circular Helix is the curve in R^3 defined by

$$x - \rho\cos t; \quad y = \rho \sin t; \quad z = kt; \quad 0 \le t, \infty.$$

Here $k > 0$. Calculate its length from 0 to $t = 28$.

78. Twisted cubic is the curve given by

$$x = at; \quad y = bt^2; \quad z = ct^3; \quad 0 \le t < \infty.$$

Here the product $abc \neq 0$. Calculate the tangent vector at each point of the curve.

79. Find length of the curve

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),\ z = 4a\sin(\theta/2), \quad 0 \le \theta \le 2\pi.$$

80. Design a cylindrical can with lid to contain one litre (=1000 cm^3) of water using minimum amount of metal. Assume that there is a minimum.

    (diameter = length = $20/\sqrt[3]{2\pi}$)

81. A parcel delivery service requires that size of rectangular box be such that length + 2 width =2 height be no more than 108 inches. What is the volume of the largest box the company delivers?

    (11,664 cubic inches)

82. The well known Cobb-Douglas model of economy, simplest one, takes the production function as

$$Q(K, L) = A\ K^{\alpha}\ L^{(1-\alpha)}$$

    where $A > 0$ and $0 < \alpha < 1$ are constants. K denotes units of capital and L denotes units of labour. Assume that the price of one unit of capital is q rupees and price of one unit of labour is p rupees. If the total cash available is B rupees, maximize the production.

    $(\alpha B/q; (1 - \alpha)B/p)$

83. The following is apparently known as Lagrange's identity. Prove it.

$$(r \times s) \cdot (t \times u) = (r \cdot t)(s \cdot u) - (r \cdot u)(s \cdot t).$$

84. Show that the function

$$f(x + iy) = \sqrt{|x||y|}$$

    satisfies Cauchy Riemann equations at $(0, 0)$. Is it (complex) differentiable at this point?

85. Describe the following sets of points in the complex plane. (draw the picture).

    (a) $Re(z) = 3. \quad Im(z) = -1.$

    (b) $|z - c| = |z - d|$ where c and d are complex numbers.

    (c) $Re(z) + Im(z) = 0$

    (d) $|z| = Re(z) + Im(z).$

86. Show that the function $f(z) = 1/z$ is differentiable on $(|z| \neq 0)$.

87. Considering the function

$$f(x, y) = \sin x \cos y$$

show that there is a number θ between zero and one such that

$$\frac{3}{4} = \frac{\pi}{3} \cos \frac{\pi\theta}{3} \cos \frac{\pi\theta}{6} - \frac{\pi}{6} \sin \frac{\pi\theta}{3} \sin \frac{\pi\theta}{6}.$$

88. Calculate $\nabla \times F$ for the following.

(a)

$$F(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right)$$

(b)

$$F(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, 0 \right)$$

(c)

$$F(x, y, z) = (\sin x,\ \sin y,\ \sin z)$$

89. Using

$$\int\limits_0^\infty \frac{\sin x}{x} dx = \pi/2$$

evaluate

$$f(x, y) = \int_0^\infty \frac{\sin xt \quad \cos yt}{t} dt.$$

for each (x, y).

90. Suppose that f is a bounded continuous function on $(-\infty, +\infty)$. Define a function

$$u(y, t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-(y-x)^2/4t} f(x) dx$$

for

$$0 < t < \infty; \quad 0 - \infty < y, \infty.$$

show that

$$u_{yy} = u_t.$$

Do not need to justify under the integral sign.

91. Show that the function

$$\varphi(s) = \int_0^\infty e^{-st} \frac{1}{1+t^2} dt; \quad s \geq 0$$

is a continuous function.

92. Let

$$\varphi(y) = \int_0^\infty e^{-yt} \frac{\sin t}{t} dt; \quad y > 0$$

is a continuous function. Show that for $y > 0$

$$\varphi'(y) = -\frac{1}{1+y^2}$$

Show that

$$\varphi(y) = \frac{\pi}{2} - \tan^{-1} y \quad y > 0.$$

(What happens if you let $y \to 0$).