unbounded intervals/functions:

We discussed functions of one variable obtained from two variable functions by performing integration etc. We also showed that derivative of integral equals integral of derivative (loosely speaking). But the entire action took place over bounded intervals. Most of the integrals that we need in practice are over unbounded intervals. Even if it is over a bounded interval, the function is usually unbounded. These are what are (unfortunately) called improper integrals.

Instead of discussing general theory we illustrate with an example. Consider

$$f(x,t) = e^{-t}t^{x-1}, \qquad (x,t) \in (0,\infty) \times (0,\infty).$$

$$\Gamma(x) = \int_0^\infty f(x,t)dt, \qquad x \in (0,\infty).$$

We have seen last semester that this integral is finite. We shall now show that this is a continuous function of x on the interval $(0,\infty)$. For this it suffices to show that the two functions

$$\varphi(x) = \int_1^\infty f(x,t)dt; \qquad \psi(x) = \int_0^1 f(x,t)dt$$

are continuous functions.

<u>continuity of φ:</u>

Fix $0 < a < b < \infty$. Enough to show that φ is continuous on the interval $[a,b]$. Indeed to show continuity at a point α, use the fact that it is continuous on the interval $[\alpha/2,\ 2\alpha]$.

If you fix any number $c > 1$ then the function

$$\varphi_c(x) = \int_1^c f(x,t)dt$$

is a continuous from what we have discussed earlier. We show that $\varphi_c \to \varphi$ uniformly over $[a,b]$ as $c \to \infty$.

Let $\epsilon > 0$ be given. Choose $C > 1$ so that

$$\int_C^\infty f(b,t)dt < \epsilon.$$

This is possible because $\varphi_c(b) \to \varphi(b)$. Now for $x \in [a, b]$, the fact $t \geq 1$ implies $f(x,t) \leq f(b,t)$. Thus for any $c > C$ we have

$$|\varphi_c(x) - \varphi(x)| \leq \int_C^\infty f(x,t)dt \leq \int_C^\infty f(b,t)dt < \epsilon.$$

continuity of ψ:

Again we fix $0 < a < b < \infty$ and show ψ is continuous on $[a, b]$.
If you fix any number $0 < c < 1$ then the function

$$\psi_c(x) = \int_c^1 f(x,t)dt$$

is a continuous from what we have discussed earlier. We show that $\psi_c \to \psi$ uniformly over $[a, b]$ as $c \to 0$.

Let $\epsilon > 0$ be given. Choose C so that

$$\int_C^\infty f(a,t)dt < \epsilon.$$

This is possible because $\psi_c(a) \to \psi(a)$. Now if we take any $x \in [a, b]$ note that $t \leq 1$ tells us that $f(x,t) \leq f(a,t)$. Thus for any $0 < c < C$ we have

$$|\psi_c(x) - \psi(x)| \leq \int_0^C f(x,t)dt \leq \int_0^C f(a,t)dt < \epsilon.$$

This completes the proof that the Gamma function is a continuous function.

The function $\Gamma(x)$ is a differentiable and

$$\Gamma'(x) = \int_0^\infty e^{-t}\, t^{x-1}\ \log t\ dt.$$

In other words, you can differentiate under the integral sign.

First we show that this function is a continuous function and the argument is similar to the above by getting bounds. One needs to go a little below a or a little above b to compensate for the log factor. Then one shows differentiability under the integral sign by imitating argument similar to the one we used in power series: tail difference quotients are small and over bounded interval you can differentiate under integral sign.

I will not execute this. It is trivial if you have understood the idea and it needs maturity and clarity in thought. It appears highly non-trivial otherwise. So let us wait for some time. First you assimilate what is done (and understand why it is done!). These are important matters.

Inverse function theorem:

Recall that the inverse function theorem in one variable is the following.

Let $\Omega \subset R$ be an open interval, $f : \Omega \to R$ be C^1, and $a \in \Omega$. Suppose that $f'(a) \neq 0$. Then there is an open interval $U \subset \Omega$ and an open interval V such that the following hold:
(i) $a \in U$ and $b = f(a) \in V$.
(ii) f is one-one on U onto V.
(iii) The inverse map $g : V \to U$ is differentiable and for $y = f(x) \in V$ we have $g'(y) = 1/[f'(x)]$.

What is its analogue for functions of two variables? Interestingly, the exact same theorem is true. here it is.

Theorem: Let $\Omega \subset R^2$ be open, $f : \Omega \to R^2$ be C^1, and $a \in \Omega$. Suppose that $f'(a)$ (which is a 2×2 matrix) is non-singular. Then there is an open set $U \subset \Omega$ and an open set V such that the following hold:
(i) $a \in U$ and $b = f(a) \in V$.
(ii) f is one-one on U onto V.
(iii) The inverse map $g : V \to U$ is differentiable and for $y = f(x) \in V$ we have $g'(y) = [f'(x)]^{-1}$.

Let us understand the theorem. For every $x \in \Omega$, f associates a point of R^2, let its coordinates be denoted by $f_1(x)$ and $f_2(x)$. Thus $f(x) = (f_1(x), f_2(x))$. Then function f is C^1 means the two real valued functions f_1 and f_2 have continuous partial derivatives. The derivative at a point x is the linear transformation whose matrix representation is

$$f'(x) = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix} = \begin{pmatrix} D_1 f_1(x) & D_2 f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) \end{pmatrix}.$$

Why is this theorem non-trivial in R^2. After all, in R we said the following. Assume without loss of generality $f'(a) > 0$. Then in a small interval $I = (a - \delta, a + \delta)$ around a, we have $f'(x) > 0$ and $[a - \delta, a + \delta] \subset \Omega$. denote $y_1 = f(a - \delta)$ and $y_2 = f(a + \delta)$. Thus f is strictly increasing on $[a - \delta, a + \delta]$

and transforms it to $[y_1, y_2]$. It is then clear that f transforms the interval I onto $J = (y_1, y_2)$. The fact that it is increasing makes matters simple. Unfortunately in R^2 such an argument is no longer possible.

On R, if $f' \neq 0$ then the fact that f' is continuous tells us that either $f' > 0$ always or $f' < 0$ always. Thus on all of the interval Ω the function is one-one. Again this depends on the fact that f is strictly increasing or strictly decreasing. However such a conclusion can not be drawn in R^2.

Consider

$$f : R^2 \to R^2; \quad f(x, y) = (e^x \cos y, e^x \sin y).$$

Just be careful, though we denote points, in general, by $x = (x_1, x_2)$; in specific examples we do not follow this. This is done not to confuse you, but to make you comfortable. You are used to (x, y) for points of R^2 and it is better to keep it that way.

For the function above

$$f'(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

which is non-singular, in fact its determinant is e^{2x}. Thus at every point in R^2 the derivative is an invertible linear transformation. Yet this function is not one-one.

Its range is all of R^2 except the point $(0, 0)$. Each value in the range is assumed infinitely many times. Each horizontal strip $R \times [k2\pi, (k + 1)2\pi)$ is mapped to $R^2 - \{(0, 0)\}$. Any horizontal strip of width 2π does this. For each fixed x the values of f trace the circle of radius e^x around the origin. Of course, in this case given any point you can clearly plot an disc around that point on which f is one-one and invertible.

Before starting proof of the theorem, we make three observations about the landscape in R^2. These are true in all R^n, but that is for later.

1. Let us use the word 'compact' to denote closed bounded sets. here is a fact.

If $K \subset R^2$ is compact and $f : K \to R^2$, then its range $f(K)$ is again compact.

This is easy to see because from what we have already proved about real valued functions, the first coordinates of points in $f(K)$ form a bounded set and so is the set of second coordinates of points in $f(K)$. This is enough to conclude that $f(K)$ is a bounded set. to show that it is closed, let $y_n \to y$ and $f(x_n) = y_n$. We exhibit x so that $f(x) = y$. This will show that the limit y is in $f(K)$ showing that $f(K)$ is closed. Of course if x_n converges to x, then K being closed we se $x \in K$ and continuity of f tells us that

$$f(x) = \lim f(x_n) = \lim y_n = y.$$

If x_n does not converge, take a subsequence of (x_n) that converges and argue with its limit. Note that K being compact every sequence in K has a limit point and every sequence has a convergent subsequence.

2. Let K be a compact set and $z \notin K$. Then there is an $r > 0$ such that the ball $B(z, r) \subset K^c$. recall ball $B(z, r)$ means the set of points x such that $||x - z|| < r$.

This is easy because our earlier characterisation: a set is closed iff its complement is open. Thus z is in the open set K^c and hence a ball around z is contained in this open set.

3. Let $U \subset R^2$ be an open set and $f : U \to R^2$ be continuous. then for any open set $V \subset R^2$ we have $f^{-1}(V)$ is an open subset of U.

This again clear because if $x \in f^{-1}(V)$, then $f(x) \in V$. Since V is open there is $\epsilon > 0$ so that $B(f(x), \epsilon) \subset V$. Continuity of f gives a $\delta > 0$ so that $B(x, \delta) \subset f^{-1}(V)$.

We shall prove the theorem assuming that $f'(a)$ is the identity matrix. That is,

$$f'(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (\spadesuit)$$

This will make things neat, otherwise we need to hang on to maximum of its entries etc making some of the estimates ugly. the general case then follows easily as we see later. We start drawing some simple consequences of out hypotheses.

4. Non-singularity is expressed in terms of the determinant, namely

$$D_1 f_1(x) D_2 f_2(x) - D_2 f_1(x) D_1 f_2(x)$$

which is a continuous function of x we see that if it is non-zero at a, then it is non-zero in a ball around a.

5. There is a ball around a such that for x in that ball

$$|D_j f_i(x) - D_j f_i(a)| < 1/4.$$

This is follows from the continuity of the partial derivatives.

6. There is a ball around a such that for all points $x \neq a$ in that ball, $f(x) \neq f(a)$.

If this is false, we get $x_n \neq a$, $x_n \in B(a, 1/n)$, $f(x_n) = f(a)$. Since $f'(a) = I$ we conclude

$$\frac{||f(x_n) - f(a) - I(x_n - a)||}{||x_n - a||} \to 0.$$

But $f(x_n) = f(a)$ tells that this ratio is always one, leading to a contradiction.

7. Take an open ball $B(a, r)$ such that the closed ball

$$\overline{B(a,r)} = \{x : ||x - a|| \leq r\} \subset \Omega$$

and **4,5,6** hold in this ball. we claim that

$$x, \tilde{x} \in B(a, r) \Rightarrow ||f(x) - f(\tilde{x})|| \geq \frac{1}{2}||x - \tilde{x}||. \qquad (\clubsuit)$$

Note that this will in particular shows that f is one-to-one in this ball. In fact norm being continuous function,the inequality remains true in the closed ball and that f is actually one-to-one in the closed ball.

This inequality is shown as follows. Consider the function $g(x) = f(x) - x$ in the closed ball. The mean value theorem applied to the first and second coordinate functions

$$g_1(x) = f_1(x) - x_1; \qquad g_2(x) = f_2(x) - x_2$$

of g will give the following (use **5.**)

$$|[f_1(x) - x_1] - [f(\tilde{x}_1) - \tilde{x}_1]| \leq \sqrt{\frac{2}{16}}\ ||x - \tilde{x}||$$

and

$$|[f_2(x) - x_2] - [f(\tilde{x}_2) - \tilde{x}_2]| \leq \sqrt{\frac{2}{16}}\ ||x - \tilde{x}||$$

so that

$$||[f(x) - x] - [f(\tilde{x}) - \tilde{x}]|| \leq \sqrt{\frac{4}{16}}\ ||x - \tilde{x}|| = \frac{1}{2}||x - \tilde{x}||.$$

Thus using triangle inequality,

$$||x - \tilde{x}|| \leq ||[f(\tilde{x}) - \tilde{x}] - [f(x) - x]|| \quad + \quad ||f(x) - f(\tilde{x})||$$
$$\leq \frac{1}{2}||x - \tilde{x}|| + ||f(x) - f(\tilde{x})||$$

proving (♣).

8. Let us denote $f(a) = b$.

The set $K = \{x : ||x - a|| = r\}$ is clearly closed (norm is continuous) and bounded so that $f(K)$ is compact by **1.** and $b \notin f(K)$ by **7.** Hence by **2.** there is an $\eta > 0$ so $B(b, 2\eta) \cap f(K) = \emptyset$.

Let $V = B(b, \eta)$. Note that

$$y \in V \Rightarrow ||y - f(x)|| > ||y - f(a)|| \quad \forall x \in K \qquad (\bullet).$$

This is because $||y - b|| < \eta$ whereas $||y - f(x)|| \geq \eta$ for $x \in K$.

We now claim that for $y \in V$ there is a unique $x \in B(a, r)$ such that $f(x) = y$. That there can not be two points x and $\tilde{x}$ satisfying the condition follows from (♣). we only need to show the existence of a point.

As you realise, in the one dimensional case the strictly increasing nature and the intermediate value theorem for continuous functions settled the matter. Here we do not have it. But what you can not see easily will be shown by linear algebra as follows.

So take one $y \in V$ and define the function

$$\varphi(x) = ||f(x) - y||^2 = [f_1(x) - y_1]^2 + [f_2(x) - y_2]^2$$

on the compact set $\overline{B(a, r)}$. This real valued function assumes a minimum at some point. Also this is assumed in the open ball $B(a, r)$ because ($\bullet$) shows that $\varphi(x) > \varphi(a)$ for all $x \in K$. Since the minimum is attained at a point, say x^* in an open set (not on the boundary) we conclude that $\nabla\varphi(x^*) = 0$. That is,

$$[f_1(x^*) - y_1]D_1 f_1(x^*) + [f_2(x^*) - y_2]D_1 f_2(x^*) = 0$$
$$[f_1(x) - y_1]D_2 f_1(x^*) + [f_2(x) - y_2]D_2 f_2(x^*) = 0.$$

If we denote

$$f_1(x^*) - y_1 = v_1; \quad f_2(x^*) - y_2 = v_2; \quad v = (v_1, v_2)$$

$$A = \begin{pmatrix} D_1 f_1(x^*) & D_2 f_1(x^*) \\ D_1 f_2(x^*) & D_2 f_2(x^*) \end{pmatrix}$$

then in matrix notation, we have

$$vA = 0.$$

But A is non-singular by **4.** Thus $v = 0$ which means $f(x^*) = y$.

9. Let $U = f^{-1}(V) \cap B(a, r)$. Then U is open by **3.** From **8.** we see that $f : U \to V$ is one-one and onto. Thus it has inverse $g : V \to U$.

A restatement of (♣) is

$$||g(y) - g(\tilde{y})|| \leq 2||y - \tilde{y}||; \qquad y, \tilde{y} \in V.$$

This shows that g is continuous.

10. Fix $y \in V$ and let $g(y) = x$, that is, $f(x) = y$. We show that g is differentiable at y and $g'(y) = [f'(x)]^{-1} = A^{-1}$, say. Take $y_n \in V$, $y_n \neq y$ for all n. Need to show

$$\frac{g(y_n) - g(y) - A^{-1}(y_n - y)}{||y_n - y||} \to 0. \qquad (*)$$

Since A is an invertible matrix, in order to show that $v_n \to 0$ for a sequence of vectors (v_n) one could as well show $Av_n \to 0$. Denoting $g(y_n) = x_n$, that is, $f(x_n) = y_n$ we need to show

$$\frac{A(x_n - x) - (f(x_n) - f(x))}{||f(x_n) - f(x)||} \to 0.$$

That is, need to show

$$\frac{(f(x_n) - f(x)) - A(x_n - x)}{x_n - x} \; \frac{x_n - x}{||f(x_n) - f(x)||} \to 0.$$

Because of (♣) the second term is bounded by 2, the first term converges to zero because $A = f'(x)$.

This completes proof of the inverse function theorem.