We have learnt the method of substitution for integrating functions of one variable. We are now in the process of getting an analogue of that method for functions of two variables. This goes by the name of change of variable formula or the Jacobian rule. We start with the simplest case.

1.

We shall prove the following:

Theorem (simple Jacobian rule):

Let T be a non-singular linear transformation of R^2 to itself. Let $|T|$ denote the modulus of the determinant of the matrix of the linear transformation. T. Then the following are true:

(i) For any rectangle $Q = [a, b] \times [c, d]$; TQ has small boundary and $|TQ| = |T||Q|$.

(ii) (when (i) holds for a linear transformation T) For any bounded open set V with small boundary; TV is an open set with small boundary and $|TV| = |T||V|$.

(iii) (whenever (i) and (ii) hold for a linear transformation T) For any bounded continuous function f on V,

$$\int_{TV} f = \int_V foT\ |T|$$

that is,

$$\int_{TV} f(u, v)dudv = \int_V f(T(x, y))|T|dxdy.$$

In other words, if you make the substitution. $T(x, y) = (u, v)$ on the right side then $|T|dxdy = dudv$. You get left side. This is like, putting $T(x) = u$ in one dimensions and saying $T'(x)dx = du$. Here observe that the derivative of the map $(x, y) \mapsto (u, v)$ is indeed T.

Having said in part (i) that something is true for every linear transform T; we started part (ii) with the phrase 'whenever (i) is true'. It appears puzzling. The reason is the following.

It is *not* that we prove (i) for every T and then prove (ii) for every T etc. We prove (i) for simple transforms; then (ii) is available for that transform, we use this (ii) to prove (i) for more general transforms — finally covering all linear transforms. That is why it is stated in that fashion.

to put it differently, we consider all linear transformations T for which (i), (ii) and (iii) hold. We show that for certain basic transformations this is true. Then we show that if it is true for two transformations, then it is true for their composition. Then we realize that every linear transformation is a composition of the basic ones. This is executed in a different order below.

2.

If T is any one of the following linear transformations we showed that $|TQ| = |T||Q|$ for any rectangle Q (with sides parallel to the axes). Recall that $|A|$ denotes area of A.

(i) interchange of coordinates:

$T(x, y) = (y, x)$ or T is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) multiply first coordinate :

$T(x, y) = (ax, y)$ or T is given by the matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ where $a \neq 0$.

(iii) Add the second coordinate to the first;

$T(x, y) = (x + y, y)$ or T is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Keep in mind that R^2 consists of column vectors, for typographical reasons we are showing it rows; we do not want to tax our reading by putting transpose symbol all the time.

3.

We now claim that every non-singular transformation is a composition of the above three transformations. Here is how. Instead of giving a general proof that works for every R^n, I give a hands-on proof that works just for R^2. Such proofs are considered bad because you can not generalize quickly. In what follows T stands for non-singular linear transformation on R^2.

Suppose that T is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, that is

$$T(x, y) = (ax + by, cx + dy)$$

case (i): All a, b, c, d are non-zero.

$$(x, y) \to (ax, y) \to (y, ax) \to (by, ax) \to (ax, by) \to (ax + by, by)$$

$$\rightarrow (cx + c\frac{b}{a}y, by) \rightarrow (by, cx + \frac{cb}{a}y) \rightarrow ((d - \frac{cb}{a})y, cx + \frac{cb}{a}y)$$

$$\rightarrow (cx + dy, cx + \frac{cb}{a}y) \rightarrow (cx + \frac{cb}{a}y, cx + dy) \rightarrow (ax + by, cx + dy).$$

case (ii) suppose that exactly one of a, b, c, d is zero. say $d = 0$ and others are non-zero. Thus $T(x, y) = (ax + by, cx)$.

$$(x, y) \rightarrow (ax, y) \rightarrow (y, ax) \rightarrow (by, ax)$$

$$\rightarrow (ax + by, ax) \rightarrow (ax, ax + by) \rightarrow (cx, ax + by) \rightarrow (ax + by, cx).$$

Similarly, the other cases when exactly one of them is zero is done.

case(iii) Exactly two of them zero. Since T is non singular the only possibilities are $a = d = 0$ or $b = c = 0$. Let us consider the case $a = d = 0$. Thus the transformation is $T(x, y) = (by, cx)$.

$$(x, y) \rightarrow (cx, y) \rightarrow (y, cx) \rightarrow (by, cx).$$

Since there can not be three zeros this completes all the cases. thus every non-singular transformation is a composition of the three basic ones.

if you look at the proof closely, you can organise carefully and prove for linear transformations on R^n with the help of induction.

4.

If U is a bounded open set, then so is its image $T(U)$ and moreover, $T(\partial U) = \partial(TU)$.

Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $k = \max\{|a|, |b|, |c|, |d|\}$. If $(x, y) \in U$ then by Cauchy-Schwarz,

$$|ax + by| \leq \sqrt{a^2 + b^2}\sqrt{x^2 + y^2} \leq \sqrt{2} \;\; k \;\; ||(x, y)||.$$

Exactly the same inequality holds for $cx + dy$ so that

$$||T(x, y)|| \leq 2k \;\; ||(x, y)||.$$

Thus if U is bounded then so is TU.

Since T^{-1} is also linear transformation, it is continuous and thus inverse image of open sets are open (definition of continuity). In other words TU is open.

To prove the last sentence, let $p \in \partial U$. If you take any open set V containing Tp, then $T^{-1}V$ is an open set containing p and hence has points from U as well as from U^c (definition of boundary point). By looking at their images and remembering that T is one-one, we conclude that V contains points from TU as well as from $T(U^c) = (TU)^c$. This proves that if $p \in \partial U$ then $Tp \in \partial(TU)$.

similarly one argues that if $p \notin \partial U$ then $Tp \notin \partial(TU)$. This proves $T(\partial U) = \partial(TU)$.

5.

Let us say that a transformation T is good if $|TQ| = |T||Q|$ for every rectangle Q with sides parallel to the axes. Let us also abbreviate **rspa** to say 'rectangle with sides parallel to the axes'.

Let U be a bounded open set with small boundary. Then so is its image $T(U)$ under any good transformation.

The only thing to be proved, in view of the above, is that TU has small boundary. Since U is bounded, its boundary ∂U is a compact set. since it is small we can cover it by finitely many rspa $\{Q_i\}$ with total area smaller than ϵ for any pre-assigned $\epsilon > 0$. We know that their images $\{TQ_i\}$ (need not be rspa) have total area at most $|T|\epsilon$ because T is good. Also these images cover ∂U. Thus ∂U can be covered by sets whose total area is smaller than any desired quantity. This shows it is small.

We covered $\partial(TU)$ with images of rspa's. If you are uncomfortable because we did not cover $\partial(TU)$ with rspa of small total area, you can proceed as follows. Cover each TQ_i with rspa having total area at most $2|TQ_i| = 2|T||Q_i|$. (Can you do that? Yes, recall definition of area as integral of the function 1 and recall integral as inf of upper sums etc. You need to remember the full story). Consider the collection so obtained and argue.

6.

For any bounded open set V with small boundary, $|TV| = |T||V|$ under any good T.

the above discussion says that TV has small boundary too and hence both V and TV have areas. So the conclusion makes sense. We prove its truth as follows.

Recall that area is integral of the function 1. Take $\epsilon > 0$. Put V in a rspa and get a partition $\Pi = \{Q_i\}$ with rspa so that

$$U - L < \epsilon; \quad L < |V| < U. \qquad (*)$$

Since we are integrating the function 1, L is nothing but the sum of areas of rectangles in Π which are contained in V, denote these by $\mathcal{L}$. On the other hand, U is sum of areas of rectangles in Π which intersect V, denote these rectangles by $\mathcal{U}$.

Clearly,

$$\bigcup\{TQ : Q \in \mathcal{L}\} \subset TV \subset \bigcup\{TQ : Q \in \mathcal{U}\}.$$

Use the fact that T is good, area is monotone, area is additive for disjoint sets (we proved this earlier using integral is additive). Since all sets that we are considering have areas, we conclude

$$|T|L < |TV| < |T|U, \qquad |T|U - |T|L < |T|\epsilon. \qquad (**)$$

These inequalities $(*)$, $(**)$ and the fact that $\epsilon > 0$ is arbitrary will prove the result.

7.

Every non-singular transformation is good.

We knew that the three basic linear transforms are good and every non-singular transformation is a composition of these.

We shall now complete the proof of the statement by showing that composition of good transforms is again good. so let T_1 and T_2 be good. Take any rspa Q. Interior of Q, say V, is a bounded open set with small boundary. From the results above you conclude that T_1V is an bounded open set with small boundary and so, in turn, T_2 being good we conclude that $T_2(T_1V)$ is a bounded open set with small boundary.

Also $|T_2T_1(V)| = |T_2||T_1||V|$. Since determinants multiply under composition of transforms we have $|T_2 o T_1| = |T_2||T_1|$. Denoting $T = T_2 o T_1$, we thus have $|TV| = |T||V|$. Clearly $\partial TV = T(\partial V)$ and is small. We conclude that

$$|TQ| = |T||Q|.$$

In other words, composition of good transforms is again good. This completes proof that every non-singular transform is good.

We know that T_2 being good, it transforms rspa nicely. Since T_1Q need not be a rspa (though a quadrilateral), we are forced to use that T_2 transforms such sets also nicely. This is precisely, the content of part (ii) of the there. As soon as you are told T_2 is good you can deduce it transforms T_1Q also appropriately. Of course, once we know that the composition T is good, it transforms not only rspa nicely, but also bounded open sets with small boundary nicely and the formula for areas holds.

8.

Thus so far we have proved parts (i) and (ii) (**7, 6** above) of the theorem for every non-singular transform. To complete the proof of the theorem, we need to prove part (iii); integral formula for continuous functions.

Let us first assume that *the function is defined on* $\overline{TV}$. This will allow us to use uniform continuity etc. Since both V and TV have small boundaries we know the functions are integrable. That is, f is integrable on TV and foT is integrable on V. We only need to show the equality of integrals. Let $\epsilon > 0$ Let f be bounded by M.

By an earlier theorem, there is an $\delta_1 > 0$ such that if we take any good partition of TV with norm smaller than δ_1, then upper and lower sums are close to $\int\limits_{TV} f$ upto $\epsilon/4$.

Denote $g = foT$ on $\overline{V}$. This is sensible because, $T(\overline{V}) = \overline{TV}$. Choose $\delta > 0$ so that the following two hold: (i) $|Tp - Tq| < \delta_1$ whenever $|p - q| < \delta$. (ii) For any good partition of V with norm smaller than δ_1, the upper and lower sums differ from $\int\limits_V g$ by at most $\epsilon/4$.

Now let us take a partition Π of V with $||\Pi|| < \delta$. For example you can take grid of rspa. Then from what had been done so far, we conclude the following. Firstly, $T\Pi$ is a good partition of TV. Secondly norm of this partition $T\Pi$ is at most δ_1. Observe that for any set $S \in \Pi$ the sup of g over S is same as sup of f over $TS \in T\Pi$. similarly for inf. Denote by L, U the upper and lower sums of g on V for the partition Π. Similarly L_1 and U_1 for the similar sums for f on TV for the partition $T\Pi$ on TV. Then we have

$$L \leq \int_V g \leq U; \quad U - L < \epsilon/4; \quad L_1 \leq \int_{TV} f \leq U_1; \quad U_1 - L_1 < \epsilon/4.$$

$$U_1 = |T|U; \quad L_1 = |T|L.$$

These inequalities are good enough to conclude

$$|T|\int_V foT = \int_{TV} f.$$

Since $|T|$ is a number, you can put it inside integral too.

This completes proof of the theorem.

Thus the theorem is proved when f is defined on $\overline{TV}$, in fact, when f is uniformly continuous on TV.

There are several ways of deducing the result for bounded continuous functions from the above. Here is a way. Fix any $\epsilon > 0$; get finitely many rspa $\{Q_i\}$ whose interiors cover boundary of TV and have total area smaller than ϵ/M (Recall, M is the bound for f). Set

$$W_1 = TV - \cup\{Q_i\}; \qquad V_1 = T^{-1}W_1$$

Then, V_1 is a bounded open set with small boundary; $TV_1 = W_1$; Restrict f to $\overline{W_1}$; apply the earlier result. Convince yourself that the difference between this and original integrals are small.

9

We shall prove the following which generalizes the above result. This goes by the name of Jacobian rule. This is very useful in evaluating integrals.

Theorem (Jacobian rule):

Let $\Omega \subset R^2$ be a bounded open set. Let $T : \Omega \to R^2$ be a one-to-one C^1 map with non-singular derivative at every point of Ω.

At every point $(x, y) \in \Omega$ let $|T'(x, y)|$ denote the modulus of the determinant of the derivative matrix.

Then the following are true:

(i) For any rectangle $Q = [a, b] \times [c, d] \subset \Omega$; TQ has small boundary and

$$|TQ| = \int_Q |T'|.$$

that is

$$|TQ| = \int\int_Q |T'(x, y)|dxdy.$$

(ii) (when (i) holds for a T) For any open set V with $\overline{V} \subset \Omega$ with small boundary; TV is an open set with small boundary and

$$|TV| = \int_V |T'|.$$

that is,

$$|TV| = \int\int_V |T'(x,y)|dxdy.$$

(iii) (whenever (i) and (ii) hold for a T) Let V be as above. For any continuous function f on $T\overline{V} = \overline{TV}$,

$$\int_{TV} f = \int_V foT \ \ |T'|$$

that is,

$$\int_{TV} f(u,v)dudv = \int_V f(T(x,y))|T'(x,y)|dxdy.$$

If you feel uncomfortable with function given on closure $\overline{TV}$ but integrals on open set TV, you can take integral too over $\overline{TV}$. It makes no difference as we saw earlier once.

Note that when T is a linear transform defined on all of R^2, this theorem reduces to the earlier one because the derivative is now the same matrix at every point, namely, T.

10

We first deal with two special cases. Then we argue that the statement remains true under compositions. Finally show any map is composition of maps of the special kind. This last step is true only 'locally'. Then a 'patching' will prove general case without much work.

Here is a special case, y-coordinate is not changed by T.

Suppose $Q = [a,b] \times [c,d]$ is an rspa. Let $g(x,y)$ be a real C^1 function on Q such that

$$\varphi(x,y) = (g(x,y), y)$$

is a one one function on Q with non-singular derivative.

$$\varphi'(x,y) = \begin{pmatrix} g_1 & g_2 \\ 0 & 1 \end{pmatrix}$$

Thus non-singular derivative, simply means that g_1 is not zero.

We show

$$|\varphi Q| = \int_c^d \int_a^b |\varphi'|.$$

There is just one subtle point. We always talked about derivatives at points in an open set. Here we started with a closed rectangle. Since any way the present theorem is not our main focus, we want to apply this for rectangles contained in Ω, you can as well pretend that g is defined on an open set containing this rectangle. You can make other interpretations too (that do not look outside the rectangle), but let us not get distracted by this point.

Note that $|\varphi'| = |g_1|$. Since g_1 is continuous and Q is convex, mean value theorem implies that g_1 keeps same sign through out Q. We shall assume that $g_1 > 0$. Thus $|g_1| = g_1$ and we need to show

$$|\varphi Q| = \int_c^d \int_a^b g_1.$$

We can actually identify $\varphi(Q)$. You should draw the picture.

The line $[a, b] \times \{c\}$ is mapped in an increasing manner (because $g_1 > 0$) to the line segment $\{(g(x, c), c) : a \le x \le b\}$. Note that continuity forces that the image is a line segment and since y-coordinate is unchanged, the image is part of the horizontal line $y = c$.

The line $[a, b] \times \{d\}$ is mapped to the line segment
$\{(g(x, d), d) : a \le x \le b\}$.

The line $\{a\} \times [c, d]$ is mapped to the curve $\{(g(a, y), y) : c \le y \le d\}$ and the line $\{b\} \times [c, d]$ is mapped to the curve $\{(g(b, y), y) : c \le y \le d\}$

By the the theorem on iterated integration,

$$\int_{\varphi(Q)} 1 = \int_c^d \int_{g(a,y)}^{g(b,y)} dxdy = \int_c^d [g(b, y) - g(a, y)]dy.$$

$$\int_Q g_1 = \int_c^d \left(\int_a^b g_1 dx \right) dy = \int_c^d [g(b, y) - g(a, y)]dy.$$

and hence they are equal. for the second integral we used the fundamental theorem of Calculus. For each fixed y, g as a function of x is a primitive for g_1.

11.

Let $Q = [a, b] \times [c, d]$ be as above. Let $h(x, y)$ be a real C^1 function on Q such that

$$\varphi(x, y) = (x, h(x, y))$$

is a one one function on Q with non-singular derivative.

$$\varphi'(x, y) = \begin{pmatrix} 1 & 0 \\ h_1 & h_2 \end{pmatrix}$$

Thus non-singular derivative, simply means that h_2 is not zero.

We can show exactly as above

$$|\varphi Q| = \int_c^d \int_a^b |h_2|.$$

12.

Let now Ω be an bounded open set and

$$\varphi(x,y) = (\xi(x,y), \eta(x,y))$$

be a one-to-one C^1 function on Ω with derivative non-singular at every point.

$$\varphi'(x,y) = \begin{pmatrix} \xi_1(x,y) & \xi_2(x,y) \\ \eta_1(x,y) & \eta_2(x,y) \end{pmatrix}$$

$|\varphi'|$ is the modulus of the determinant of the above derivative matrix. The matrix above is referred to as Jacobian. We show that for any rectangle $Q = [a,b] \times [c,d] \subset \Omega$

$$|\varphi Q| = \int_Q |\varphi'|. \qquad (\spadesuit)$$

The main idea is that φ is 'locally' a composition of the above two kinds. Here is the precise statement.

Step 1: For every point x there is an open set V containing x such that ($\spadesuit$) holds for rectangles contained in V.

Step 2: Step 1 implies the full ($\spadesuit$).

First let us argue step 2. Let $Q = [a,b] \times [c,d] \subset V$ be any rectangle. for each $p \in Q$ there is a V_p as above. We can take the V_p to be a rspa with p at its centre. Let W_p be the rectangle with sides consisting of the middle halfs of sides of V_x. Take finitely many of these, say $\{W_i\}$ such that they cover Q; Remember Q is compact. Note that these $\{W_i\}$ and the Q determine a partition of Q into rspa. Since each of these sets of the partition is contained in one single V_i we can apply the result coming from step 1 and add up. Remember areas add up over disjoint regions and so do integrals.

We shall execute step 1.

Take a point $(x_0, y_0) \in \Omega$. Since $\varphi'(x_0, y_0)$ is non-singular, its first column can not be zero. We must have either $\xi_1(x_0, y_0) \neq 0$ or $\eta_1(x_0, y_0) \neq 0$. There is no loss to assume that $\xi_1(x_0, y_0) \neq 0$. Otherwise you need to apply interchange of coordinates in the range space. If you think about it you will see. But first just assume this and proceed.

Since ξ_1 is a continuous function, there is no loss to assume that in a ball around (x_0, y_0) it is strictly positive. If it is negative, similar proof applies. consider the map

$$\Psi(x, y) = (\xi(x, y), y)$$

Then

$$\Psi' = \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & 1 \end{pmatrix}$$

is clearly non-singular in this ball and hence there is an open set, say V, containing (x_0, y_0) such that Ψ maps this onto an open set W and the inverse map Ψ^* is differentiable etc. This is by the inverse function theorem. Note that Ψ changes only one coordinate, namely, the first coordinate. Also observe

$$|\Psi'| = |\xi_1| = \xi_1. \qquad (\bullet)$$

Now define on the open set W^* the following map;

$$\zeta(a, b) = (a, \eta(\Psi^*(a, b))); \qquad (a, b) \in W$$

Thus ζ changes only one coordinate, namely the second coordinate.

Let us see what is the composition $\zeta(\Psi(x, y))$ on V. Observe $\Psi(x, y) = (\xi(x, y), y)$. Since ζ does not change the first coordinate we see that first coordinate of the composition, namely $\zeta(\Psi(x, y))$ is $\xi(x, y)$.

Regarding second coordinate of the composition, to calculate ζ of this point (a, b) we need to apply Ψ^* which is inverse of Ψ and so you get back (x, y), that is, (x, y) with $\Psi(x, y) = (a, b)$ and then you should look at eta of this, you get $\eta(x, y)$. Thus second coordinate of the composition $\zeta o \Psi$ is η. Thus

$$\zeta(\Psi(x, y)) = (\xi(x, y), \eta(x, y)) = \varphi(x, y).$$

Let us also note that

$$\zeta'(a, b) = \begin{pmatrix} 1 & 0 \\ ? & ?? \end{pmatrix} \qquad (\dagger)$$

where the second row is $\nabla \eta o \Psi^*$ and so by chain rule

$$(?,\ ??) = \eta'(\Psi^*(a,b))(\Psi^*)'(a,b)$$

$$= (\eta_1(\Psi^*(a,b)), \eta_2(\Psi^*(a,b))) \quad [\Psi'(\Psi^*(a,b))]^{-1}$$

But

$$\Psi'(P) = \begin{pmatrix} \xi_1(P) & \xi_2(P) \\ 0 & 1 \end{pmatrix}$$

$$[\Psi'(P)]^{-1} = \begin{pmatrix} 1/\xi_1(P) & -\xi_2(P)/\xi_1(P) \\ 0 & 1 \end{pmatrix}$$

So

$$(?,\ ??) \; =$$

$$(\eta_1(\Psi^*(a,b)), \eta_2(\Psi^*(a,b))) \begin{pmatrix} 1/\xi_1(\Psi^*(a,b)) & -\xi_2(\Psi^*(a,b))/\xi_1(\Psi^*(a,b)) \\ 0 & 1 \end{pmatrix} \tag{††}$$

Since both Ψ and ζ change only one coordinate, they are good, that is we can use theorem for them. Note that (i) of the theorem holds for each of them and hence the other parts too. Thus if you take any rspa $Q \subset V$ then $\Psi(Q) \subset W$. We can no longer say that this later set is a rectangle. However (ii) and (iii) of the theorem can be used. Thus

$$|\varphi(Q)| = |\zeta[\Psi Q]| = \int_{\Psi Q} |\zeta'(u,v)| dudv$$

by part (ii) of the theorem applied to the transform ζ and the set ΨQ. We now apply part (iii) to the transform Ψ and the integrand $|\zeta'|$ to see

$$|\varphi(Q)| = \int_Q |\zeta'(\Psi(x,y))||\Psi'(x,y)| dxdy.$$

To calculate the first term in the integrand, we use (†) and (††). Remember Ψ^* is inverse of Ψ and evaluate (††) at $(a,b) = \Psi(x,y)$ substitute in (†) to get

$$|\zeta'(\Psi(x,y))| = \left| \frac{\eta_1(x,y)\xi_2(x,y)}{\xi_1(x,y)} - \eta_2(x,y) \right|$$

From (•)

$$|\Psi'(x,y)| = \xi_1(x,y)$$

Substituting these two in the above integral we get

$$|\varphi Q| = \int_Q |\eta_1\xi_2 - \eta_2\xi_1| = \int_Q |\varphi'|.$$

This completes the proof of (♠). What the above shows is that (♠) holds for all rectangles contained in the open set V that we have exhibited. Thus around every point there is a open set such that this holds for all rspa contained in that open set. As argued in step 2, this is enough.

Remember the above proof used the fact that (a) the result is true for transformations which change only one coordinate and (b) when part (i) of the theorem is proved for a transformation then you can use other parts for that transformation. The result (a) has already been proved earlier.

13.

Thus we should now prove parts (ii) and (iii) of the theorem assuming (i) holds.

Let V be an open set with small boundary. The inverse function theorem already assures that the image TV is an open set. To show it has small boundary, we repeat the same argument that we had when T is a linear transformation (item (5) above). We cover ∂V with finitely many rspa of small total area and use the fact that $|T'|$ is bounded on compact sets to get an estimate for the area of the image.

14.

We shall now prove part (iii) of the theorem assuming part (ii) for T.

I have avoided Riemann sums so far in two dimensions. It is better to have them handy now, to avoid a nuisance.

Let f be a bounded function defined on $S = \overline{V}$. In the following; partition Π of S is either with rspa or a good partition. We defined upper and lower sums. We now define Riemann sums. A selection ξ for Π is a selection of one point p_A from each set A of the partition. Given a partition and selection we define the Riemann sum

$$R(\Pi, \xi) = \sum_{A \in \Pi} f(p_A)|A|.$$

Thus instead of taking inf or sup over a set we evaluate the function at a point in that set. These Riemann sums also converge to the integral. More precisely, Given $\epsilon > 0$ there is a $\delta > 0$ such that the following holds:

$$||\Pi|| < \delta \text{ good; } \xi \text{ selection for } \Pi \Rightarrow \left| R(\Pi, \xi) - \int f \right| < \epsilon. \quad (\dagger)$$

Integral is over V. There is nothing new in this. We already knew of a δ so that the upper sum, lower sum, integral are close as soon as $||\Pi|| < \delta$. It is obvious that the Riemann sum is in between the lower and upper sums and hence the above conclusion holds for it, no matter what the selection is.

We shall slightly generalise the notion of above Riemann sum when we have a product function. Suppose we have two continuous functions f and g on $\overline{V}$. We want to integrate the product fg. Let Π be a good partition of V and a selection ξ for it. We make an interesting Riemann sum. For a set A of the partition, instead of taking the term $fg(p_A)|A| = f(p_A)g(p_A)|A|$ we take $f(p_A) \int_A g$. After all, if the set A is small, value of g at any point of A is close to $g(p_A)$ and so the integral is indeed close to $g(p_A)|A|$. Thus we define

$$R_1(\Pi, \xi) = \sum_{A \in \Pi} f(p_A) \int_A g.$$

This makes sense because our partition is good and g is continuous bounded. We claim that, as $||\Pi|| \to 0$, these sums also converge to the integral $\int fg$. More precise statement is the following.

Given $\epsilon > 0$ there is a $\delta > 0$ such that

$$||\Pi|| < \delta \text{ good; } \xi \text{ selection for } \Pi \Rightarrow \left| R_1(\Pi, \xi) - \int fg \right| < \epsilon. \qquad (\dagger\dagger)$$

This is proved as follows. fix $\epsilon > 0$. Let M be a bound for $|f|$ on $\overline{V}$. Choose $\delta > 0$ such that

$$||\Pi|| < \delta \text{ good; } \xi \text{ selection;} \Rightarrow \left| R(fg, \Pi, \xi) - \int_V fg \right| < \epsilon/4.$$

and

$$p, q \in \overline{V}; \quad ||p - q|| < \delta \Rightarrow |g(p) - g(q)| < \epsilon/(4M|V|).$$

Now take any good partition Π; with $||\Pi|| < \delta$ and any selection ξ. Let $A \in \Pi$.

$$p, q \in A \Rightarrow ||p - q|| < \delta \Rightarrow |g(p) - g(q)| < \epsilon/c; \quad c = 4M|V|.$$

So

$$\left| g(p_A)|A| - \int_A g \right| = \left| \int_A [g - g(p_A)] \right| \leq \frac{\epsilon|A|}{c}.$$

hence

$$\left|f(p_A)g(p_A)|A| - f(p_A)\int_A g\right| \le \frac{\epsilon M|A|}{c} = \frac{\epsilon M}{4M|V|}|A|$$

so that

$$\left|\sum_A f(p_A)g(p_A)|A| - \sum_A f(p_A)\int_A g\right| \le \sum_A \frac{\epsilon}{4|V|}|A| = \frac{\epsilon}{4}.$$

Thus

$$|R - R_1| < \epsilon/4.$$

But by choice of δ

$$|R - \int fg| < \epsilon/4.$$

combining these last two inequalities we have

$$\left|R_1 - \int fg\right| < \epsilon/2$$

as required. this completes the proof of the claim.

Let us now return to our problem. We want to show

$$\int_{TV} f = \int_V foT\ |T'|. \qquad (**)$$

Let $\epsilon > 0$ be fixed. We show that the difference between the two quantities above is at most ϵ.

Use (†) with the set $\overline{TV}$ and function f and $\epsilon/4$. Get $\delta > 0$ so that

$$||\Pi|| < \delta \ \text{ good for } \ \overline{TV}; \ \ \xi \ \text{ selection for } \ \Pi \Rightarrow \left|R(\Pi,\xi) - \int_{TV} f\right| < \epsilon/4 \ \ (\bullet)$$

Use (††) with the set $\overline{V}$; function f as foT and function g as $|T'|$ and $\epsilon/4$. Get $\delta_1 > 0$ so that

$$||\Pi|| < \delta_1 \ \text{ good for } \ \overline{V}; \ \ \xi \ \text{ selection for } \ \Pi \Rightarrow \left|R_1(\Pi,\xi) - \int_V foT\ |T'|\right| < \epsilon/4 \ \ (\bullet\bullet)$$

Take δ_1 smaller if necessary so that the following holds

$$p, q \in \overline{V}; \ \ ||p - q|| < \delta_1 \Rightarrow ||Tp - Tq|| < \delta/2. \ \ (\bullet\bullet\bullet)$$

Now let us take a good partition Π of $\overline{V}$ with $||\Pi|| < \delta_1$ and a selection ξ for it. Let $T\Pi$ be the partition of $\overline{TV}$ given by $\{TA : A \in \Pi\}$. Let $T\xi$ be the selection for it given by $p_{TA} = Tp_A$.

It is not difficult to show that $T\Pi$ is a good partition of $\overline{TV}$. By ($\bullet\bullet\bullet$) we see $||T\Pi|| < \delta$ and so by ($\bullet$) we have

$$\left| R(T\Pi,\ \ T\xi,\ \ f) \ \ - \ \ \int_{TV} f \right| \ \ < \ \ \epsilon/4.$$

Also

$$\left| R_1(\Pi,\ \ \xi,\ \ foT\ |T'|) \ \ - \ \ \int_V foT\ \ |T'| \right| \ \ < \ \ \epsilon/4$$

For any set $A \in \Pi$, we already know from part (ii) of the theorem that $|TA| = \int_A |T'|$ so that it is easy to see that

$$R(T\Pi,\ \ T\xi,\ \ f) \ \ = \ \ R_1(\Pi,\ \ \xi,\ \ foT\ |T'|)$$

Hence the two inequalities show that

$$\left| \int_V foT\ \ |T'| \ \ - \ \ \int_{TV} f \right| < \epsilon/2.$$

This completes proof of ($**$) and thus proof of part (iii) of the theorem.

This completes proof of all the three parts of the theorem.

15.

We have completed the proof the Jacobian formula as stated.

it appears unsatisfactory. Would have been nice if part (iii) is stated for any bounded continuous function on Ω rather than for open sets V with $\overline{V} \subset \Omega$. Actually this is how I stated in the classes starting with Ω which is bounded open set with small boundary. But I proved only this much.

Yes, it is also true if you started with Ω bounded open set with small boundary. Proof is simple. You 'approximate' U with sets V as above. This can be made precise as follows. Cover $\partial\Omega$ with finitely many open rspa, take their closures and take V to be the part of Ω outside these finitely many closed rectangles. Since you can make total area of these rectangles as small as you please you ken get the result for Ω too.

However once you understand the main ideas, certain beautifications you can do yourself. At this stage you need not bother (unless you wish to).

16.

Improper integrals are dealt with as in the one dimensional case. This is a long but routine route and would appear boring if spelt out with all details. You may have a bounded region but the function is unbounded like $1/||x||$. Or the function may be bounded but the region is unbounded, like $\exp\{-||x||^2\}$ on R^2.

Just to give you a feel let us discuss the case of a function f given on R^2. We say that the integral

$$\int_{R^2} f$$

exists if whenever you take a sequence of rspa $Q_n \uparrow R^2$ the numbers

$$\int_{Q_n} f$$

converge to a limit and the limit is independent of the sequence of rectangles taken. Then this common value is called the value of the integral.

For example, if

$$\sup_n \int_{[-n,n]\times[-n,n]} |f(x,y)|dxdy < \infty$$

Then f is integrable. In other words there is a number c such that no matter what rectangles (or regions) you take which increase to R^2, your integrals converge to c.

We shall work out some examples rather than developing the theory of improper integrals.