We shall continue our discussion of properties of integrals and how to calculate integrals. We are only considering bounded functions on a closed bounded interval. We showed that every continuous function is integrable.This we did by showing that for every $\epsilon > 0$, there is a partition P such that $U(P, f) - L(P, f) < \epsilon$. Actually we observed a more precise property.

8. Let f be a continuous function on $[a, b]$. Given $\epsilon > 0$, there is a $\delta > 0$ with the following property: whenever we take any partition with difference between successive points smaller than δ, then $U(P, f) - L(P, f) < \epsilon$.

Let us introduce a word that reduces our writing. Given a partition $P = \{a = a_0 < a_1 < a_2 < \cdots < a_k = b\}$, the maximum distance between successive points, $\max_i(a_{i+1} - a_i)$ is denoted by $||P||$, *norm* of P. A *selection* for a partition is simply a selection of points from each partition interval; more precisely, a (finite) sequence of points $s = \{x_0 \le x_1 \le x_2 \le \cdots x_{k-1}\}$ such that $x_i \in [a_i, a_{i+1}]$ for $i = 0, 1, 2, \cdots, k-1$. Given any interval, there are several possible partitions of the interval. Given one partition, there are several possible selections for the partition; we can pick any one point from each interval of the partition.

Given a partition P and a selection s for the partition, we define

$$R(P, f, s) = \sum_{i=0}^{k-1} f(x_i)[a_{i+1} - a_i].$$

This is called the Riemann sum for the partition and selection. Recall that, instead of value of the function at the selected point, if we used the infimum and supremum in each partition interval we get the lower and upper Riemann sums.

We denote integral of f over the interval $[a, b]$ by $\int_a^b f(x)dx$ or $\int_a^b f$ or simply $\int f$ if the interval is clear from the context.

9. Let f be a continuous function on $[a, b]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $|R(P, f, s) - \int f| < \epsilon$ for any partition P with $||P|| < \delta$ and for any selection s for the partition.

We only need to observe that both $\int f$ and $R(P, f, s)$ are between $L(P, f)$ and $U(P, f)$. So the same δ as above would do.

10. Let $\{P_n\}$ be a sequence of partitions of $[a, b]$ with $||P_n|| \to 0$ and for each n, let s_n be a selection for the partition P_n. Then for any continuous function f on $[a, b]$ we have:

$$U(P_n, f) \to \int f; \qquad L(P_n, f) \to \int f.; \qquad R(P_n, f, s_n) \to \int f$$

This is clear from the previous statement. Thus even though the appearance of selection appears an extra complication, you should keep in mind that it is one more choice at our disposal and some times some one (like mean value theorem) may already make a choice for us. You will see this in the fundamental theorem of integral calculus.

Though we are concentrating on continuous functions now, one naturally wonders whether there are functions which are not continuous but integrable. the answer is yes. The evidence is also easy to get. Just consider the function $f(x) = 1$ for $0 < x < 1$; and $f(0) = 54, f(1) = 1/2$. This function is integrable. In fact lower sums and upper sums are easily calculable and they yield $U(f) = 1 = L(f)$.

But what is not easy to answer is the following question: what precisely are the functions which are integrable? The answer roughly is that f is integrable when and only when its set of discontinuities is 'small'. This is an important issue but but shall not enter this discussion now. It is more important and basic to see how to calculate integrals and how to use integrals to our benefit. However, some of the observations we made above are true without assuming that we have a continuous function. Here is an example whose proof is easy.

11. Let f be bounded function on $[a, b]$. Then f is integrable iff for every $\epsilon > 0$, there is a partition P such that $U(p, f) - L(P, f) < \epsilon$; or equivalently, for every $\epsilon > 0$ there is a partition P such that $|R(P, f, s) - R(P, f, s')| < \epsilon$ for any two selections s and s'.

This is immediate from the following simple fact. Suppose we have two sets S_1 and S_2. Suppose that $a \leq b$ for every $a \in S_1$ and every $b \in S_2$. Then $\sup S_1 = \inf S_2$ iff for every $\epsilon > 0$, there are $a \in S_1$ and $b \in S_2$ with $b - a < \epsilon$.

For some of the statements below, continuity of the functions is not needed, integrability is enough; of course proofs have to be done with more care. But, as mentioned earlier, we shall not complicate life now.

11. if f and g are continuous, then

$$\int (f+g) = \int f + \int g; \qquad \int (39f) = 39 \int f.$$

if $f \equiv 28$ on $[a,b]$, then $\int\limits_a^b f = 28(b-a)$.

12. If f is continuous on $[a,b]$ and $a < c < b$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Just take a sequence of partitions P_n such that $||P_n|| \to 0$ and each P_n includes the point c. Then the set $Q_n = \{x \in P_n : x \leq c\}$ and the set $R_n = \{x \in P_n : x \geq c\}$ will constitute partitions of $[a,c]$ and $[c,b]$ respectively. Clearly

$$||Q_n|| \to 0; \qquad ||R_n|| \to 0; \qquad U(P_n f) = U(Q_n, f) + U(R_n, f)$$

and proof is completed by noting

$$U(P_n, f) \to \int_a^b f; \quad U(Q_n, f) \to \int_a^c f; \quad U(R_n, f) \to \int_c^b f.$$

13. for any continuous function $|\int f| \leq \int |f|$.
Note that $|f|$ is again a continuous function and hence its integral makes sense. proof is simple because for any partition

$$|U(P,f)| \leq U(P,|f|).$$

14. **Fundamental Theorem of Integral Calculus:**
let f be a continuous function on $[a,b]$.
(i) Define $F(a) = 0$ and $F(x) = \int\limits_a^x f$ for $a < x \leq b$.
Then F is continuous on $[a,b]$; it is differentiable; $F' = f$.

(ii) Let G be any continuous function on $[a,b]$ which is differentiable on (a,b) and $G'(x) = f(x)$ for $a < x < b$. Then $\int\limits_a^b f = G(b) - G(a)$.

(iii) If G_1 and G_2 are two such functions as in (ii), then there is a number α such that $G_1(x) = \alpha + G_2(x)$ for every $x \in [a, b]$.

This is an extremely useful and powerful theorem. Firstly, it relates integration to derivatives. Secondly, it reduces our job of calculating integrals to finding functions whose derivative is the given function. This is easier than struggling with partitions, sups, infs, and limits.

Any function G as in (ii) is called a *primitive* for f. Part (i) says F is always a primitive, unfortunately, its definition depends on integration again. So it can not be used as a way to evaluate integrals. It is important and assures us of the existence of a primitive.

Sometimes only part (ii) is called fundamental theorem. Usually it is stated for integrable functions rather than only for continuous functions as we did above. For example if you take the function f which is one on $(0, 1)$ and our choice of numbers at $x = 0$ and $x = 1$; then f need not be continuous on $[0, 1]$ but the function $G(x) = x$ fits the bill.

Before proving this theorem, let us see two useful applications. The fundamental theorem makes it possible to translate theorems on derivatives to theorems on integrals, which help in calcualting integrals. We 'translate' the chain rule' and 'product rule' of differentiation.

15. Let φ be a strictly increasing continuously differentiable function on $[a, b]$ onto $[c, d]$. Let f be a continuous function on $[c, d]$. then

$$\int_a^b f(\varphi(x))\varphi'(x)dx = \int_c^d f(y)dy.$$

since φ' is assumed to be continuous, the integrand on the left side is continuous and hence integral makes sense.

Thus integrating f on the interval $[c, d]$ is not simply integrating the composed function on $[a, b]$, but you need to multiply this composed function with φ'. the reason is that when you calculate Riemann sums on $[a, b]$ you multiply with length of partition interval. If $x_1 < x_2$ then length of the interval $[x_1, x_2]$ is $(x_2 - x_1)$. If $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$ then length of the image interval $[y_1, y_2]$ is $\varphi(x_2) - \varphi(x_1)$ which is $\varphi'(\theta)[x_2 - x_1]$ for some number θ between x_1 and x_2. Thus when an interval in domain is shifted to the range, the length gets magnified by a factor (the factor could be less than one).

In practice, the method above is implemented as follows. You need to evaluate an integral which you recognize as the left side. you say put $y = \varphi(x)$ so that $dy = \varphi'(x)dx$ and the left side becomes the right side after noting that $y = c$ when $x = a$ and $y = d$ when $x = b$.

You can state a similar theorem when φ is decreasing. We need to multiply with $|\varphi'(x)| = -\varphi'(x)$ instead of simply $\varphi'(x)$.

Proof of the formula is simple. Let F be a primitive for f on $[c, d]$. Thus it is differentiable and $F'(y) = f(y)$ for $c < y < d$. Set $G(x) = F(\varphi(x))$ on $[a, b]$. Clearly, G is continuous on $[a, b]$, differentiable on (a, b), and by chain rule,

$$G'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x).$$

Thus by fundamental theorem

$$\int_a^b f(\varphi(x))\varphi'(x)dx = G(b) - G(a) = F(d) - F(c) = \int_c^d f(y)dy.$$

We have taken a simple route by using the extra concept of primitive assured by the fundamental theorem. You should remember that, such a method of taking easy way out will not help us when life does get complicated. But can we prove it starting from definition of integral? Yes. Here is another proof, you may ignore if you wish.

For any partition P of $[a, b]$, let $\varphi(P)$ denote the partition of $[c, d]$ obtained by taking images of points in P. First observation is this: If $||P_n|| \to 0$, then $||\varphi(P_n)|| \to 0$. This is a simple consequence of uniform continuity of φ.

Let us temporarily name $g(x) = f(\varphi(x))\varphi'(x)$ defined on $[a, b]$. The second observation is the following. Given any partition P of $[a, b]$, there is a selection s for the partition P of $[a, b]$ so that

$$R(P, g, s) = R(\varphi(P), f, \varphi(s)).$$

First you should note that image of a selector is a selector for the image partition. To show the stated selector, if $[\alpha, \beta]$ is an partition interval for P, then by MVT, there is a γ in this interval so that

$$[\varphi(\beta) - \varphi(\alpha)] = \varphi'(\gamma)[\beta - \alpha].$$

Let s be the selection for the partition P so that from any partition interval $[\alpha, \beta]$ it picks a point γ satisfying the above equation from this interval. MVT assures there is at least one such point. Then

$$g(\gamma)[\beta - \alpha] = f(\varphi(\gamma))[\varphi(\beta) - \varphi(\alpha)]$$

It is clear that this selector will do the job.

Now to complete the proof is simple. Take a sequence of partitions P_n with $||P_n|| \to 0$. Then $||\varphi(P_n)|| \to 0$. For each n select s_n as described above. Then

$$R(P_n, g, s_n) = R(\varphi(P_n), f, \varphi(s_n)).$$

Here the left side converges to $\int g$ and right side to $\int f$ over the appropriate intervals.

This completes the alternative proof.

This is the most commonly used form of method of substitution. However, the fundamenttal theorem tells us better. Do not assume φ is strictly increasing. Just assume that it is a continuously differentiable on $[a, b]$ onto $[c, d]$ with $\varphi(a) = c$ and $\varphi(b) = d$. Then the proof using the fundamental theorem still holds good and hence the formula is still true.

Here is translation of product rule.

16. Let F and G be two continuously differentiable functions on $[a, b]$ with derivatives f and g. Then

$$\int_a^b F(x)g(x)dx = \{F(b)G(b) - F(a)G(a)\} - \int_a^b f(x)G(x)dx.$$

Proof is trivial. The function FG is continuous and its derivative equals $Fg + fG$ and so is a primitive for the later. By Fundamental theorem

$$F(b)G(b) - F(a)G(a) = \int (Fg + gF) = \int Fg + \int fG.$$

Let us now return to the proof of the fundamental theorem. Interestingly, the proof is straight forward.

(i) Shall show uniform continuity of F on $[a, b]$. Fix $\epsilon > 0$. Since f is continuous, it is bounded, say $|f(x)| < M$ for all x. Take $\delta = \epsilon/M$ Now take x, y with $|x - y| < \delta$. No loss to assume $x < y$, If $x = a$ then $F(a) = 0$ and so

$$|F(y) - F(a)| = |F(y)| = |\int_a^y f| \leq \int_a^y |f| \leq M(y - a) < \epsilon.$$

If $a < x < y$ our earlier observation tells

$$\int_a^y f = \int_a^x f + \int_x^y f; \quad i.e, \quad F(y) = F(x) + \int_x^y f;$$

so that

$$|F(y)-F(x)| = |\int_x^y f| \le \int |f| \le M(y-x) \le \epsilon.$$

We show that F is differentiable and $F'(x) = f(x)$. Fix x and let us denote $f(x) = \alpha$. In what follows when we integrate α over an interval, it is understood that we are talking about the constant function identically equal to α on that interval. Also all the points appearing below are in the interval $[a.b]$

Let $\epsilon > 0$. We exhibit $\delta > 0$ so that

$$0 < y - x < \delta \Rightarrow |\frac{F(y)-F(x)}{y-x} - \alpha| < \epsilon.$$

and

$$0 < x - y < \delta \Rightarrow |\frac{F(y)-F(x)}{y-x} - \alpha| < \epsilon.$$

This will prove the stated result. Do not worry, if your point $x = b$ then first implication can not arise and when $x = a$ the second can not. We take $\delta > 0$ so that

$$y \in [a,b]; |y-x| < \delta \Rightarrow |f(y) - \alpha| < \epsilon.$$

Note that x is fixed and $f(x)$ is named α. So the above is possible by continuity of f. For any $y > x$, we have

$$F(y) - F(x) = \int_x^y f(t)dt; \qquad \int_x^y \alpha dt = \alpha(y-x).$$

$$|\frac{F(y)-F(x)}{y-x} - \alpha| = |\int \frac{1}{y-x}\int_x^y [f(t)-\alpha]dt| \le \frac{1}{y-x}\int_x^y |f(t)-\alpha|dt.$$

If $|y-x| < \delta$ then for every $t \in [x,y]$ we have $|t-x| < \delta$ so that the integrand above is at most ϵ and so the integral is at most $\epsilon(y-x)$ showing what we wanted.

Similar computation yields the result for $0 < x - y < \delta$.

This completes proof of (i).

Proof of (ii). First we observe the following. Given any partition P, there is a selection s so that $G(b) - G(a) = R(P, f, s)$. This will complete proof as follows. Take a sequence of partitions P_n with $||P_n|| \to 0$. For each n, get selector s_n for P_n as claimed above. Then proof is completed by noting

$$R(P_n, f, s_n) \to \int f; \qquad R(P_n, f, s_n) = F(b) - F(a) \quad \text{for all } n.$$

So let

$$P = \{a = a_0 < a_1 < a_2 < \cdots < a_k = b\}$$

and let us get selection s as claimed. For each i let $x_i \in (a_i, a_{i+1})$ be given by the MVT to satisfy

$$G(a_{i+1}) - G(a_i) = f(x_i)(a_{i+1} - a_i).$$

This is the selection s. then

$$G(b) - G(a) = \sum_{i=0}^{k-1} [G(a_{i+1}) - G(a_i)] = \sum f(x_i)(a_{i+1} - a_i) = R(P, f, s).$$

This completes proof of (ii)

Proof of (iii): suppose that there are two functions G_1 and G_2 having the same derivative f. Then $G_1 - G_2$ has zero derivative and hence is a constant in the interval (a, b). Since $G_1 - G_2$ is continuous on the interval $[a, b]$ and is a constant in the interval (a, b) it must be constant in the interval $[a, b]$.

This completes proof of the fundamental theorem.

We have excellent tools before us to evaluate integrals and also put them to use. sometimes we use a notation as follows: $\int f = F$ without mentioning any interval $[a.b]$. We use the same variable x for both f and F. This is to be interpreted as saying that F is a primitive for f. It simply means that over a 'meaningful' interval $F' = f$. Thus if you have two numbers $a < b$ in this interval then $\int\limits_a^b f(x)dx = F(b) - F(a)$.

I assume that you have come across the following in high school and so go over them fast to reach new and interesting things. If you did not go through them in high school or if you do not remember, then you should convince yourself about their truth. Do not take anything for granted.

$$\int x^n = \frac{x^{n+1}}{n+1}. \qquad \int x^a dx = \frac{x^{a+1}}{a+1} \quad \text{if} \quad a \neq -1.$$

We have explained the calculation of Archimedes for calculating $\int\limits_0^1 x^2 dx$. It follows exactly the upper and lower sums. It would be a nice exercise to calculate $\int x^n$ for positive integers n, following the same idea.

$$\int e^x dx = e^x, \quad x \in R. \qquad \int \frac{1}{x} dx = \log x, \quad x > 0.$$

$$\int \cos x dx = \sin x. \qquad \int \sin x dx = -\cos x.$$

$$\int \sinh x dx = \cosh x; \qquad \int \cosh x dx = \sinh x.$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \log(x + \sqrt{1+x^2}).$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \log(x + \sqrt{x^2-1}) \qquad |x| > 1.$$

$$\int \frac{1}{-\sqrt{x^2-1}} dx = \log(x - \sqrt{x^2-1}) \qquad |x| > 1.$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \log \frac{1+x}{1-x} \qquad |x| < 1.$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \log \frac{x+1}{x-1} \qquad |x| > 1.$$

If we write $\int f = \varphi - \int g$ it simply means

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a) - \int_a^b g(x)dx.$$

Walli's product:

$$\int_0^{\pi/2} \sin^0 x dx = \frac{\pi}{2}; \qquad \int_0^{\pi/2} \sin x dx = \cos 0 - \cos(\pi/2) = 1.$$

If $m > 1$, then integration by parts gives

$$\int_0^{\pi/2} \sin^m x dx = \int_0^{\pi/2} \sin^{m-1} x (-\cos x)' dx$$

$$= \int_0^{\pi/2} \cos x \ \ (m-1) \sin^{m-2} x \cos x dx$$

$$= (m-1) \int_0^{\pi/2} \sin^{m-2} x dx - (m-1) \int_0^{\pi/2} \sin^m x dx$$

so that

$$\int_0^{\pi/2} \sin^m x dx = \frac{m-1}{m} \int_0^{\pi/2} \sin^{m-2} x dx.$$

Thus

$$\int_0^{\pi/2} \sin^{2m} x dx = \frac{2m-1}{2m} \frac{2m-3}{2m-2} \frac{2m-5}{2m-4} \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2}.$$

$$\int_0^{\pi/2} \sin^{2m+1} x dx = \frac{2m}{2m+1} \frac{2m-2}{2m-1} \frac{2m-4}{2m-3} \cdots \frac{4}{5} \frac{2}{3} 1.$$

dividing the first equation by the second

$$\frac{\int_0^{\pi/2}\sin^{2m}xdx}{\int_0^{\pi/2}\sin^{2m+1}xdx} = \frac{(2m-1)(2m+1)}{(2m)^2}\ \frac{(2m-3)(2m-1)}{(2m-2)^2}\ \cdots$$

$$\cdots\frac{3\times 5}{4^2}\ \frac{1\times 3}{2^2}\ \frac{\pi}{2}.$$

$$\frac{\pi}{2} = \frac{2^2}{1\cdot 3}\ \frac{4^2}{3\cdot 5}\ \frac{6^2}{5\cdot 7}\cdots\frac{(2m-2)^2}{(2m-3)(2m-1)}\ \frac{(2m)^2}{(2m-1)(2m+1)}$$

$$\times\frac{\int_0^{\pi/2}\sin^{2m}xdx}{\int_0^{\pi/2}\sin^{2m+1}xdx}.$$

We shall now show that as $m\to\infty$;

$$\frac{\int_0^{\pi/2}\sin^{2m}xdx}{\int_0^{\pi/2}\sin^{2m+1}xdx}\to 1. \qquad (\spadesuit)$$

It will then follow that

$$\frac{\pi}{2} = \lim_{m\to\infty}\frac{2^2}{1\cdot 3}\ \frac{4^2}{3\cdot 5}\ \frac{6^2}{5\cdot 7}\cdots\frac{(2m-2)^2}{(2m-3)(2m-1)}\ \frac{(2m)^2}{(2m-1)(2m+1)}.$$

This is called wall's product.

$$\frac{\pi}{2} = \lim_{m\to\infty}\ \frac{2^{2m}(m!)^2}{3^2\cdot 5^2\cdots(2m-1)^2(2m+1)} = \lim_{m\to\infty}\ \frac{2^{4m}(m!)^4}{[(2m)!]^2(2m+1)}.$$

Or

$$\sqrt{\frac{\pi}{2}} = \lim_{m\to\infty}\frac{2^{2m}(m!)^2}{(2m)!\sqrt{(2m+1)}}$$

Or

$$\sqrt{\pi} = \lim_{m\to\infty}\ \frac{2^{2m}(m!)^2}{(2m)!\sqrt{(m+1/2)}}$$

Since $\sqrt{m}/\sqrt{m+1/2}\to 1.$ we get

$$\sqrt{\pi} = \lim_{m\to\infty}\ \frac{2^{2m}(m!)^2}{(2m)!\sqrt{m}}$$

This is called Walli's formula for $\sqrt{\pi}$.

Euler's constant:

We know that
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\frac{1}{n}$$
increases to infinity as n becomes large. But how large is the above quantity, in other words, how fast is the above sequence increasing towards infinity?. Knowledge of integration helps to answer this question. The above quantity is like $\log n$. We shall show this now. Let

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\frac{1}{n}-\log n$$

Observe,
$$\log(k+1)-\log k = \int_k^{k+1}\frac{1}{x}dx$$
so that
$$\frac{1}{k+1} \leq \log(k+1)-\log k \leq \frac{1}{k}$$
So that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\frac{1}{n} \geq \sum_1^n[\log(k+1)-\log k] = \log(n+1).$$

$$a_n \geq \log(n+1)-\log n \geq 0.$$

Also the same inequality above shows

$$a_n - a_{n+1} = \log(n+1)-\log n - \frac{1}{n+1} \geq 0.$$

Thus (a_n) is a decreasing sequence of non-begative numbers and hence converges. The limit is usually denoted by γ, called Euler's constant.

There is another (essentially same argument as above) argument to show convergence of (a_n) and also to see that it is strictly positive. The above inequalities show

$$1 \geq [\log 2 - \log 1] \geq \frac{1}{2} \geq [\log 3 - \log 2] \geq \frac{1}{3} \geq [\log 4 - \log 3] \geq \frac{1}{4}\cdots\cdots.$$

Leibnitz's theorem on alternating series tells that the alternating series with above terms converges. The sequence a_n we have is nothing but its partial sums (not all, a subsequence) and hence converges and this also shows that the sum is at least $1-\log 2$.

Incidentally, no nice alternate description seems to be known to decide whether γ is rational or not.

some more friends:

There are certain functions which are important and we have not yet met them. We saw the exponential function, e^x and some trigonometric functions; $\sin x$ and $\cos x$. We have also calculated their derivatives.

$(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ and hence $(\cos^2 x + \sin^2 x)' = 0$. Since this sum of squares equals one at $x = 0$ we see $\sin^2 x + \cos^x = 1$. This also shows

$$-1 \leq \sin x \leq 1; \qquad -1 \leq \cos x \leq +1.$$

Not to interrupt our plan of meeting more friends, we assume the following fact and prove it later. This shows that the functions sin and cos are just as you know in high school.

These functions, $\sin x$ and $\cos x$, are periodic of period 2π,

where π is the area of the circle of radius one. $(\spadesuit)$.

and also $\pi/2$ is the least positive number where $\cos x = 0$.

Thus the function $\sin x$ is one-to one strictly increasing function on the interval $[-\pi/2, \ \ +\pi/2]$ onto $[-1, +1]$. general theory regarding continuous functions tells us that the inverse function g is continuous on the interval $[-1, +1]$ onto $[-\pi/2, \ \ +\pi/2]$ Also it is differentiable at every point in $(-1, +1)$ with derivative given by the general formula as follows. temporarily denote by f the sin function on the interval $[-\pi/2 \ \ + \pi/2]$ we see $f(g(y)) = y$ for each $y \in [-1, +1]$. Since the derivative of f is non zero at every point in $(-1, 1)$ general theory tells us g is differentiable in $(-1, 1)$ and

$$f'(g(y))g'(y) = 1;$$

so that

$$g'(y) = \frac{1}{\cos g(y)} = \frac{1}{\sqrt{1 - \sin^2(g(y))}} = \frac{1}{\sqrt{1 - y^2}}.$$

where in the last equality we used that g is inverse of sine function.

Usually $g(y)$ is denoted by $\sin^{-1}(y)$ (inverse of the sine function) or $\arcsin y$ (arc whose sine is y, here arc refers to the angle subtended at the centre by the arc). Since cos is positive in the interval $[-\pi/2, +\pi/2]$ we have taken positive root in the second equality above.

Of course, the sine function is on-to-one in the interval $[+\pi/2, \quad 3\pi/2]$ onto $[-1,+1]$. we could have defined the inverse sine function so that it takes values in this interval. Nothing wrong with it, it would also be differentiable on $(-1,+1)$. Of course it would then be decreasing and derivative will be negative — cosine function is negative in the interval $[+\pi/2, \ 3\pi/2]$.

Thus several 'branches' are possible for the inverse function. We settled on one branch, that is all. This is the branch usually one takes.

The cosine function is not one-to-one on the interval $[-\pi/2, \ \ +\pi/2]$. But it is one to one, strictly decreasing on $[0,\pi]$ onto $[-1,1]$. Its inverse h is defined on the interval $[-1,+1]$; takes values in $[0,\pi]$,; strictly decreasing and continuous; differentiable on $(-1,1)$ with derivative

$$h'(y) = \frac{1}{-\sin(h(y))} = \frac{-1}{\sqrt{1-\cos^2 h(y)}} = \frac{-1}{\sqrt{1-y^2}}.$$

Usually, $h(y)$ is denoted by $\cos^{-1}(y)$ (inverse of the cosine function) or $\arccos x$ (arc whose cosine is y).

Again as in the case with sine function, several branches of the inverse function are possible for the cosine function too.

The function $\tan x$ is defined as $\sin x/\cos x$. Of course, this is not defined on all of R. This is not defined precisely at those points where $\cos x = 0$. It is defined at all other points. It is one-to-one on $(-\pi/2, \quad +\pi/2)$ onto $(-\infty,+\infty)$, strictly increasing and differentiable. If $-\pi/2 < x < 0$, sine is negative and since $\cos x$ approaches zero as x approaches $-\pi/2$ we see that tan approaches $-\infty$ as x approaches $-\pi/2$. similarly it approaches $+\infty$ as x approaches $+\pi/2$.

$$(\tan x)' = \frac{1}{\cos^2 x}.$$

Thus inverse g of the tangent function is defined on all of R, continuous, increasing, takes values in $(-\pi/2, \ \ +\pi/2)$, it is differentiable and

$$g'(y) = \frac{1}{\cos^2(h(y))} = \frac{1}{1+\tan^2(h(y))} = \frac{1}{1+y^2}.$$

Usually $g(y)$ is denoted $\tan^{-1}(y)$ or $\arctan y$.

We can define $\cot x$, $\sec x$ and cosec x. Since there is nothing we can add to what you know from high school, we shall not continue in this direction. But you should recollect them, use composition rule for differentiation to calculate their derivatives.

The trigonometric functions are also called circular functions because $(\cos x, \sin x)$ form points on circle. we shall now introduce hyperbolic functions,

$$\cosh x = \frac{e^x + e^{-x}}{2}. \qquad \sinh x = \frac{e^x - e^{-x}}{2} \quad x \in R.$$

These are called hyperbolic cosine and hyperbolic sine respectively, because the points $(\cosh x, \sinh x)$ lie on the hyperbola $y^2 - x^2 = 1$.

$$\cosh 0 = 1; \qquad \cosh x \geq 1;$$

$$\cosh(-x) = \cosh x; \qquad \lim_{x \to \pm\infty} \cosh x = \infty.$$

$$\sinh 0 = 0; \qquad \sinh(-x) = -\sinh x;$$

$$\lim_{x \to -\infty} \sinh x = -\infty.; \qquad \lim_{x \to \infty} \sinh x = \infty.$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

$$(\sinh x)' = \cosh x; \qquad (\cosh x)' = \sinh x.$$

The hyperbolic sine is strictly increasing and has range all of R. Hence its inverse map is defined on all of R, is differentiable. Here we can calculate explicitly the inverse map.

Similarly, hyperbolic cosine is strictly increasing on $[0, \infty)$ and has range $[1, \infty)$. Thus its inverse is defined on $[1, \infty)$ with values in $[0, \infty)$ and is continuous. It is differentiable on $(1, \infty)$. We can explicitly solve for the inverse. Of course, cosh is one to one on $(-\infty, 0]$ onto $[1, \infty)$ and so we can think of another branch of its inverse. We shall not take it for inverse.

$$\cosh^{-1}(y) = \log\left(y + \sqrt{y^2 - 1}\right); \qquad \sinh^{-1}(y) = \log\left(y + \sqrt{y^2 + 1}\right).$$

By chain rule you can calculate their derivatives too.

Fine tuning of integral:

(i) After defining upper sums; lower sums; integrability and after showing that very continuous function on a closed bounded interval is integrable, we have been specializing to continuous functions. However, one does come across functions which are not continuous or functions which are bounded continuous but defined only on an open interval.

For example the function $\sin(1/x)$ defined on the open interval $(0,1)$, continuous and bounded. We, at this moment, are unable to talk about integrability because we did everything on a closed bounded interval. This was only done to fix ideas and have concrete picture in mind. Discussing open intervals poses no serious problems. This we do first.

(ii) so let us take a bounded interval (a,b) and a bounded function f on this interval. As earlier, partition is a finite sequence of points

$$P = \{a = a_0 < a_1 < a_2 < \cdots < a_k = b\}.$$

Given a partition P we define $U(P,f)$ and $L(P,f)$ the upper and lower sums as earlier; just that for the first and last interval we take sup and inf only over $(a, a_1]$ and $[a_{k-1}, b)$. As earlier, we say that f is Riemann integrable if Sup of all lower sums equals inf of all upper sums and in that case, the common value is called integral of f and is denoted

$$\int\limits_a^b f; \qquad \int_a^b f(x)dx.$$

The fact that every lower sum is smaller than every upper sum is obvious and also the fact that upper sums decrease whereas lower sums increase as the partition becomes finer. It is also easy to show that f is integrable iff for any given $\epsilon > 0$, we can get a partition P so that $U(P,f) - L(P,f) < \epsilon$..

(iii) We can show that a bounded continuous function is integrable. Earlier we used uniform continuity, but now this is no longer immediately possible because continuous function on an open interval need not be uniformly continuous. However, we can take advantage of the fact that the function is bounded.

Let $|f(x)| \leq M$ for all $x \in (a,b)$. Let $\epsilon > 0$. Let us choose $\delta > 0$ so that $2M\delta < \epsilon/4$. Note that on any subinterval, sup minus inf of f is at most $2M$. Thus in particular, if you consider the interval $(a, a+\delta]$ or the interval $(b-\delta, b]$ you see that sup minus inf over that interval times delta is smaller than $\epsilon/4$.

The function being uniformly continuous on $[a+\delta, b-\delta]$, get a partition P_1 of this $[a+\delta, a-\delta]$ so that $U(P_1, f) - L(P_1, f) < \epsilon/4$. The partition P for (a, b) is simply the points in P_1 along with a at the beginning and b at the end. Thus the first interval of this partition is $(a, a+\delta]$. and the last interval of this partition is $(b-\delta, b]$. It is easy to see that

$$U(P, f) - L(P, f) < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.$$

(iv) it is interesting to see that in the notation for integral, $\int\limits_a^b$ there is nothing to show wether we have open interval or closed interval. Since area of a line is zero, it makes no difference whether you include the lines at the end points or not, in calculating the area under the curve.

More precisely, suppose f were actually continuous on $[a, b]$ and you ignore it and consider the function only on (a, b) and calculate the integral. You get the same answer as you would get when you calculated for the closed interval. The proof is very simple, for each partition whether you calculate the sums over (a, b) or $[a, b]$ you get the same answer. After all, the only difference is the end intervals and there continuity of the function tells you whether you include end point or not you get the same value for the sup or inf.

(v) You can also show that selections (given a partition, select points from each partition interval) and Riemann sums also lead to the value of the integral, even on an open interval. In fact, whenever $||P_n|| \to 0$ the sums $R(P_n, f, s_n)$ converge to the integral whatever be the selection s_n for P_n.

(vi) The idea in the above argument has something more to offer. suppose that f is a bounded function on $[a, b]$ but only known to be continuous on (a, b). Is it integrable? Yes, the argument above show that it is integrable and equals the integral of f over the open interval (a, b).

(vii) In fact one can go further. Suppose f is a bounded function defined on (a, b) and is continuous at all but finitely many points. then f is integrable and the integral equals integral of f on the complement of this finite set. Note that complement of this finite set is made up of finitely many disjoint open intervals, so from the above, f is integrable on each of these intervals, so calculate them and add them up. This is the meaning of integral of f over the complement of the finite set.

Carefully understand the statement above. Firstly, we said that f is integrable. Secondly, we said to calculate the integral, you do on the complement of the finite set. the first statement is easy to see by improvising the above idea, enclose each of the finitely many discontinuity points in small intervals and get partitions of the remaining parts carefully and put all together to get a partition P of (a, b) to see $U(P, f) - L(P, f) < \epsilon$. The second statement also follows from this, but its importance is that it gives us a method of calculating the integral.

$(viii)$ Actually all this is a reflection of the fact that finitely many lines have area zero. of course, countably many lines also have area zero and even if f has countably many discontinuity points the result must be true. Yes, it is indeed so. But we shall not discuss. But are they the only functions which are integrable.? No, there are more.

(ix) The properties of integrals that we verified for continuous functions hold good for integrable functions. For example, if f and g are bounded integrable functions on (a, b) (or $[a, b]$) then so is their sum $f + g$ and $\int(f + g) = \int f + \int g$. Also $7f$ is integrable and $\int(7f) = 7 \int f$.

In other words a fine tuning is possible and would make the theory better and complete. But if you understand the story of functions with finitely many discontinuities and how to calculate integrals, ti would suffice for a first course.

You must keep in mind that all this story we developed is for *bounded functions* on *bounded intervals.* if the function is unbounded, you can immediately see that this procedure is useless. There one interval of the partition where the sup is $+\infty$ or there is a partition interval where the inf is $-\infty$. In the first two cases we get $\pm\infty$ for each partial sum. However when the last case occurs, we can not even define Riemann sum we will be involved in $\infty - \infty$ for which we have not given any meaning.

When the interval is infinite, then also we enter a similar situation.

Most of the integrals are of this kind, that is, either the function is unbounded or the interval of integration is unbounded. We deal with such situations next.