discussion of HA1.

Q2. For each $n \geq 1$, let A_n denote the set $\{1, 2, \cdots, n\}$.

We have a bijection $f : S \to A_{11}$ and a bijection $g : S \to A_{13}$. We want to show that this is not possible. What should we do?

The first step is to eliminate the unknown set S from the problem.

Step 1: We define a function $h : A_{11} \to A_{13}$ as follows. Take $x \in A_{11}$. There is one element $s \in S$ such that $g(s) = x$, because g is onto A_{11}. Also such a point is unique, because g is one-to-one. This s is denoted by $g^{-1}(x)$. We take $f(s)$ and this is our $h(x)$. Thus

$$h(x) = f(g^{-1}(x))$$

Let $y \in A_{13}$. There is one $s \in S$ with $f(s) = y$, because f is onto. Let $g(s) \in A_{11}$ be denoted by x. Then it is easy to see $h(x) = y$. In other words h is onto. Thus h is a function on A_{11} onto A_{13}. Next step shows there can not be such a function. [Actually this h is one-one too, though we do not need: If $x, y \in A_{11}$ and $x \neq y$, then $g^{-1}(x) \neq g^{-1}(y)$ and now use f is one-to-one to conclude $h(x) \neq h(y)$.]

Now we have a function from A_{11} ONTO A_{13}. Shall show this is not possible.

If we start associating with 1,2 etc of A_{11} an element of A_{13} we will run out of elements from A_{11} whereas elements of A_{13} are still left out.

Why? how do you know this happens?

Because A_{11} has 11 elements and A_{13} has 13 elements.

Firstly, I do not know 'number of elements'. Even if I agree with you on a suitable definition, why should there be no such function? Actually, this was our main problem. We did, definitely, believe that A_{11} has 11 elements. So if we can establish a bijection between A_{11} with S then we felt eligible to say that S also has 11 elements, we decided to write $|S| = 11$. You are absolutely right. But to see that this is a sensible definition, we must make sure that there can not be bijection from S to both A_{11} and A_{13}. If this happens we do not know whether S has 11 elements or 13 elements.

But is it not obvious that A_{11} has 11 elements and A_{13} has 13 elements.

Hold on, I think we are going in circles. You are again using 'number of elements' and also the word obvious. Remember, to use an adjective like obvious, easy, trivial; you must first verfiy the truth of the statement; then depending on the nature of the argument needed to verify its truth (may be length, may be depth of facts used), you use these adjectives. We have not even verified the truth of the statement.

To make you appreciate the issue involved, let me rephrase the problem. Someone walks into this class room and tells us that he has a function from A_{11} onto A_{13}. If he shows you his function, you are sure, you can find an error. You might even think it is absolutely trivial for you to find his error. I believe you. But we have a road-block. He refuses to show us his function. Should we leave matters like this or are we capable of telling him that he is wrong, we can prove it to him without seeing his function? Yes, we accept the challenge of showing that he is wrong. Here is how we achieve it.

Step 2: For $m = 1, 2, \cdots$ let P_m be the following statement:
Whatever be $n > m$, we can not find a function on A_m *onto* A_n.
This is proved by induction.

$\underline{m=1}$. Take $n > 1$. Let f be any map of A_1 to A_n. If $f(1) \neq 1$, then 1 is not in the range of f. If $f(1) = 1$, then 2 is not in the range of f; just remember $n \geq 2$ and $A_n = \{1, 2, \cdots, n\}$. Thus f can not be onto.

done for $m = 1, 2, \cdots, k-1$. Shall do for $\underline{m = k.}$ Let $n > k$ be fixed and if possible a function f on A_k onto A_n. We shall produce a function h on A_{k-1} onto A_{n-1}. But $n - 1 > k - 1$ and the statement P_{k-1} is true, leading to a contradiction.
Let $f(k) = a \in A_n$. We define h as follows. Take x with $1 \leq x \leq k-1$. If $f(x) < a$, then declare $h(x) = f(x)$; if $f(x) \geq a$ declare $h(x) = f(x) - 1$. It is clear that $h(x) \leq n - 1$, so that h takes values in A_{n-1}. Is it onto A_{n-1}? Yes, to see this take y with $1 \leq y \leq n-1$. In case $y < a$, using the fact that f is onto A_n, we get x with $f(x) = y$. But then, $h(x) = y$ and of course $x \neq k$, so that $x \in A_{k-1}$. If $y \geq a$, then using the fact that $1 \leq y \leq n-1$ first conclude that $y + 1 \leq n$ and hence there is a $x \in A_k$ with $f(x) = y + 1$. Observe $x \neq k$ because $f(k) = a < y + 1$. Thus $x \in A_{k-1}$ and $h(x) = y$. Thus whatever be $y \in A_{n-1}$ there is $x \in A_{k-1}$ with $h(x) = y$.

Q3 . The reason we are going through this is the following. Many-a-times we 'think' we know certain things while we really do not know them. From school we know,

$$0.1 = \frac{1}{10}; \quad 0.12 = \frac{12}{100}; \quad 0.121 = \frac{121}{1000} \cdots\cdots.$$

Such an understanding is correct and enough in school. But then what is the meaning of $0.12121212 \cdots\cdots\cdots\cdots.$?

$$\frac{12121212 \cdots\cdots\cdots\cdots}{100000000 \cdots\cdots\cdots\cdots}$$

does not make sense. A perfectly equivalent, but probably looking complicated, way of putting the earlier understanding is the following.

$$0.1 = \frac{1}{10}; \quad 0.12 = \frac{1}{10} + \frac{2}{100}; \quad 0.121 = \frac{1}{10} + \frac{2}{100} + \frac{1}{1000}, \cdots\cdots.$$

With such an equivalent way of putting things, we can easily understand the infinite decimal expansion.

$$0.12121212 \cdots\cdots = \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \cdots\cdots.$$

Let us return to the problem. When $x = 0$, we see the required expansion by taking each $\epsilon_i = 0$. So let us now consider $0 < x \leq 1$.

How shall we get the decimal expansion?

??

You can give a try.

We take gif of 10x.

What is gif, I do not know.

greatest integer function.

Oh, let us denote it as $[x]$: the largest integer which is less than or equal to x. With this notation you are saying that $[10x]$ is the first decimal digit. What is the second digit?

We take $[10\{x - [10x]\}]$. This is second decimal digit.

Yes, you are right. But how do we show that the decimal expansion with these digits gives us the number x?

??

I shall put the exact same thing in a more picturesque way, you can see convergence too.

Given a number x with $0 < x \leq 1$, here is an algorithm for obtaining its decimal expansion. Denote by I_k the interval

$$I_k \quad = \quad \left(\frac{k}{10}, \ \frac{k+1}{10}\right]; \qquad k = 0, 1, \cdots, 9.$$

These intervals are disjoint, each having length 1/10. They make up all of (0,1]. Thus x must be in exactly one of these intervals. Let that interval be I_k and put $\epsilon_1 = k$. Since

$$\frac{\epsilon_1}{10} \quad < \quad x \quad \leq \quad \frac{\epsilon_1 + 1}{10},$$

we see that

$$0 \quad < \quad x - \frac{\epsilon_1}{10} \quad \leq \quad \frac{1}{10}.$$

Now divide this interval I_k (just remember now $k = \epsilon_1$) into ten parts by conisdering

$$I_{kl} \quad = \quad \left(\frac{k}{10} + \frac{l}{10^2}, \ \frac{k}{10} + \frac{l+1}{10^2}\right]; \qquad l = 0, 1, \cdots, 9.$$

Since $x \in I_k$ and the intervals $\{I_{kl} : 0 \leq l \leq 9\}$ are disjoint making up all of I_k, there is exactly one l such that $x \in I_{kl}$. Put $\epsilon_2 = l$ and immediately observe, as earlier, (recalling that $k = \epsilon_1$) that

$$0 \quad < \quad x \ - \ \frac{\epsilon_1}{10} \ - \ \frac{\epsilon_2}{10^2} \quad \leq \quad \frac{1}{10^2}.$$

By induction now one can obtain the required digits and show the stated properties too by using the inequalities deduced at each stage.

Suppose a number x has two different expansions

$$\cdot\epsilon_1\epsilon_2\epsilon_3\cdots = x = \cdot\eta_1\eta_2\eta_3\cdots.$$

Remember this means,

$$\frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2} + \frac{\epsilon_3}{10^3} + \cdots = x = \frac{\eta_1}{10} + \frac{\eta_2}{10^2} + \frac{\eta_3}{10^3} + \cdots.$$

Let $\epsilon_i = \eta_i$ for $i = 1, 2, \cdots k-1$ and $\epsilon_k < \eta_k$. (of course, η_k may be smaller. Either you can repeat the same argument, or, to start with itself, you can say the one having smaller value at the first digit where they differ, is denoted with epsilons). Thus, we see

$$\frac{\epsilon_k}{10^k} + \frac{\epsilon_{k+1}}{10^{k+1}} + \frac{\epsilon_{k+2}}{10^{k+2}} + \cdots = \frac{\eta_k}{10^k} + \frac{\eta_{k+1}}{10^{k+1}} + \frac{\eta_{k+2}}{10^{k+2}} + \cdots . (\spadesuit)$$

Observe that the left side of ($\spadesuit$) is at most, (use sum of geometric series)

$$\text{LHS of } (\spadesuit) \leq \frac{\epsilon_k + 1}{10^k}$$

and strict inequality holds if $\epsilon_m < 9$ for at least one $m > k$. Also

$$\text{RHS of } (\spadesuit) \geq \frac{\eta_k}{10^k}$$

and strict inequality holds if $\eta_m > 0$ for at least one $m > k$.

But LHS and RHS of ($\spadesuit$) are equal and $\epsilon_k < \eta_k$; we must have
$\epsilon_k + 1 = \eta_k$; and
$\epsilon_m = 9$ for each $m > k$; and
$\eta_m = 0$ for each $m > k$.

Q6. For the time being do not try the part about algebraic numbers. It is simple, but needs a new idea. Interestingly, if a and b are algebraic, we can not *simply* show that $a + b$ is algebraic; we need to show (at the same time) that several things are algebraic and $a + b$ is one of them.

exponentiation. (continued).

x^n for $x \neq 0$ and $n \in N$.

This is defined by induction: $x^1 = x$ and if we have defined x^n for $n = 1, 2, \cdots k$ then we put $x^{k+1} = x^k \cdot x$. Do you know how to prove the law of indices: $x^{n+m} = x^n \cdot x^m$ and $(xy)^n = x^n y^n$. If you never saw a proof, now is the time to write a proof of this fact.

x^n for $x \neq 0$ and $n \in Z$.

For $n \in N$ it is defined above. For $n = 0$, we put $x^0 = 1$. For $n < 0$ it is defined as $x^n = (1/x)^{-n}$. Prove the law of indices.

<u>$x^{1/n}$ for $x > 0$ and $n \in N$.</u>
In the last class we proved the existence of exactly one number $y > 0$ such that $y^n = x$. We define this y as $x^{1/n}$, also denoted as $\sqrt[n]{x}$.

If $0 < x < y$ then $x^{1/n} < y^{1/n}$. Also for any $x, y > 0$, $(xy)^{1/n} = x^{1/n}y^{1/n}$.

<u>x^r for $x > 0$ and $r \in Q$</u>
Let $r = m/n$ where m, n are integers and $n \geq 1$. we put $x^r = (x^m)^{1/n}$. This makes sense because $x^m > 0$ whatever be $m \in Z$. Is this well defined? Someone expreses the same rational m/n as $(km)/(kn)$ (here $k \in N$), and calculates. Will he get the same answer? In other words

$$(x^m)^{1/n} = (x^{km})^{1/(kn)}?$$

To see that this is indeed true, first observe that the right side is positive. If we show that its n-th power equals x^m, then the right side equals the left side (by uniqueness of n-th roots).

$$(x^{km})^{1/(kn)}(x^{km})^{1/(kn)} \cdots (n - \text{times}) = (x^{km}x^{km} \cdots (n - \text{times})^{1/kn}$$

$$= (x^{kmn})^{1/(kn)} = [(x^m)^{kn}]^{1/(kn)} = x^m.$$

Here the last equality is by uniqueness of (kn)-th root.

More generally, suppose a rational number is expressed as m/n and also p/q where all m, n, p, q are integers and $n, q \geq 1$. — for example $(4/6)$ and $(6/9)$. The question is whether

$$(x^m)^{1/n} = (x^p)^{1/q}?$$

This follows from

$$(x^m)^{1/n} = (x^{mq})^{1/(nq)}; \qquad (x^p)^{1/q} = (x^{np})^{1/(nq)}; \qquad mq = np.$$

Thus x^r is well defined for every $x > 0$ and rational number r. Verify the laws of indices:

$$x^{r+s} = x^r x^s; \qquad (xy)^r = x^r y^r; \qquad x^r = (1/x)^{-r}.$$

$$x > 1, r < s \Rightarrow x^r < x^s; \qquad x < 1, r < s \Rightarrow x^r > x^s.$$

$\underline{x^a \text{ for } x > 1 \text{ and } a \in R}$
For $x > 1$, we define $x^a = lub\{x^r : r \in Q, r \leq x\}$. If we take any rational t, with $a - 1 < t < a$, then x^t is in the above set; if we take any rational s with $a < s < a + 1$ then x^s is an upper bound for that set. Thus lub is sensible. Also if a happens to be rational then this definition gives the same answer as the definition in the previous clause. Show this by using the last property (monotonicity) stated above.

$\underline{x^a \text{ for } x > 0 \text{ and } a \in R}$
If $x > 1$ the above clause defines x^a. If $x = 1$, we put $x^a = 1$ whatever be a. If $0 < x < 1$, we put $x^a = (1/x)^{-a}$. This makes sense because, $1/x > 1$ and above clause applies.

The laws of indices still hold. We postpone this study. We need to develop some other stories of importance.

modulus.

For a real number x, we define $|x|$, modulus of x, as follows: if $0 \leq x$ then $|x| = x$ while we define $|x| = -x$ in case $x < 0$.

Fact: (i) $|x| \geq 0$; $|x| = 0$ iff $x = 0$. (ii) $|x| = |-x|$. (iii) $|x+y| \leq |x|+|y|$.

(i) If $x \neq 0$ then $-x \neq 0$ as well and hence if $x \neq 0$ then $|x| \neq 0$.
If $x > 0$, then $|x| = x > 0$. If $x < 0$, then we know $|x| = -x > 0$.
(ii) If $x > 0$, then $-x < 0$ so that $|-x| = -(-x) = x = |x|$.
If $x < 0$, then $-x > 0$ so that $|-x| = -x = |x|$.
If $x = 0$, then $-x = 0$ so that $|-x| = 0 = |x|$.
(iii) case 1: $x \geq 0$, $y \geq 0$. Then $x + y \geq 0$ so that

$$|x + y| = x + y = |x| + |y|.$$

case (ii): $x < 0$, $y < 0$. Then $x + y < 0$ so that

$$|x + y| = -(x + y) = (-x) + (-y) = |x| + |y|.$$

case (iii): $x > 0$ and $y < 0$. Then $x + y \geq 0$ and $x + y < 0$ are both possible. In case $x + y \geq 0$, use $y < -y$ to see

$$|x + y| = x + y \leq x + (-y) = |x| + |y|.$$

In case $x + y < 0$, use $-x < x$ to see

$$|x + y| = -(x + y) = (-x) + (-y) < x + (-y) = |x| + |y|.$$

case (iv): $x < 0$ and $y > 0$. argue as above.

cardinality.

We shall now return to cardinality of sets: Recall the definitions of finite, countable, uncountable sets. Generally, $|A|$ is called the cardinality of the set A. Of course, as of now this is defined for finite sets. For countably infinite set $\aleph_0$ denotes its cardinality. For R, the symbol is $\mathbf{c}$. Even among uncountable sets, there are several kinds; we shall not enter that topic. Here are some simple rules.

Fact:

(i) Let $n \geq 1$ be an integer. If $|A| = n$ and $f : A \to B$ a bijection, then $|B| = n$.

(ii) If A is countably infinite, $f : A \to B$ is a bijection, then B is also countably infinite.

(iii) The set $N = \{1, 2, 3, \cdots\}$ is countably infinite.

(iv) Any infinite subset of N is countably infinite. Any subset of a countable set is countable.

(v) The set of integers, Z is countable.

(vi) The set of pairs $S = \{(m, n) : m, n \in N\}$ is countable.

(vii) The set of positive rational numbers is countably infinite.

(viii) The set of rational numbers is countable.

(ix) If for each $n = 1, 2, 3, \cdots$ we have countable sets, then their union $\cup A_n$ is also countable set.

Proof: (i) Since $|A| = n$, fix a bijection $g : \{1, 2, \cdots, n\} \to A$. Then $h(x) = f(g(x))$ gives a bijection of $\{1, 2, \cdots, n\}$ to B.

(ii) Similar proof as above holds.

(iii) Identity map $f(x) = x$ shows this.

(iv) Let A be an infinite subset of N. Define, by induction, for each $n \in N$, an element $a_n \in A$ as follows:

$$a_1 = \min A; \qquad a_{n+1} = \min\left(A \cap \{a_1, \cdots, a_n\}^c\right); \quad n > 1.$$

Recall that if $A \subset N$ is not empty, then it has a first element; that is, $a \in A$ such that $x > a$ for all $x \in A; x \neq a$. This is denoted as min above. Also the set being infinite, this process can be continued for ever. Define

$$f : \{1, 2, \cdots\} \to A; \qquad f(n) = a_n.$$

This is one to one. Indeed, if $m < n$, then by definition of a_n we see that $a_n \neq a_m$. To see that this is onto, we proceed as follows. Observe that $a_1 \geq 1$. This, in turn, implies

$$x \in A, \ x \neq a_1 \Rightarrow x > 1.$$

Using induction, we can prove that $a_n \geq n$ for every n. This, in turn, gives

$$x \in A, \ x \neq a_1, a_2, \cdots, a_n \Rightarrow x > n.$$

Thus,

$$x \in A \Rightarrow x = a_n \quad \text{for some } n.$$

This shows that f is onto A.
The last part is easy now.

(v) Define $f : Z \to N$ by $f(n) = 2^n$ if $n \geq 1$; $f(n) = 3^{-n}$ if $n \leq -1$ and $f(0) = 1$. Then f is a one-to-one function. But of course it is *not onto* N. However it is onto its range, namely, the set $A = \{f(x) : x \in Z\}$. Since A is countably infinite, so is Z.

(vi) The function $f(m, n) = 2^m 3^n$ establishes a on-one function on S onto its range contained in N.

(vii) The set of strictly rational numbers is identified with the set of pairs (m, n) wher $m \geq 1$, $n \geq 1$ are integers and have no common factors. But then this is a subset of the earlier set.

(viii) Proof similar to (v). Fix a bijection f from strictly positive rationals to N. For $x \in Q$, put

$$g(x) = 2^{f(x)} \text{ if } x > 0; \quad g(x) = 3^{-f(-x)} \text{ if } x < 0; \quad g(0) = 1.$$

(ix) We can safely assume that $A_n \neq \emptyset$ for each n (why?). For each $n = 1, 2, \cdots$, fix a function $f_n : A_n \to N$ a one-one function. We are not saying 'onto' because in case A_n is finite it would be impossible. Define a function $f : \cup A_n \to N$ as follows. Take $x \in \cup A_n$. Let i be the first integer such that $x \in A_i$. Put

$$f(x) = 2^i 3^{f_i(x)}.$$

This establishes a one-one function onto its range contained in N.

Fact: The interval $(0, 1)$ is uncountable. The set of real numbers is uncountable. The set of irrational numbers is uncountable. The set of transcendental numbers is uncountable.

Proof: <u>the set $(0, 1)$ is not countable.</u>

If possible, let $f : N \to (0, 1)$ be any function. We show that there is an element of $(0, 1)$ which is not in the range of f; showing, in particular, that there can not be a bijection between the two sets. The method of proof is called Cantor's diagonal argument, which is simple, yet powerful.

For each n, fix a decimal expansion of the number $f(n)$. Let us define numbers ϵ_n for $n = 1, 2, \cdots$ as follows. $\epsilon_n = 7$ if the n-th digit in the decimal expansion of $f(n)$ is different from 7, while $\epsilon_n = 8$ if the n-th digit in the decimal expansion of $f(n)$ equals 7.

Let

$$a = \frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2} + \frac{\epsilon_3}{10^3} + \cdots.$$

The series converges and defines a number.
Is it between zero and one? Yes, actually it is between 7/10 and 9/10.
Does it differ from every number in the range of f? Yes, from the number $f(n)$ it differs at the n-th decimal digit.
Since some numbers have more than one expansion, how can we say that this number a is different from each $f(n)$ just because there is a difference in one decimal place? Since the expansion of a does not end either with zeros or with

nines, a has exactly one expansion and it does not agree with any of the $f(n)$.

R is not countable.
If R were countable its subset $(0,1)$ would be countable too.

Set of transcendental numbers is not countable.

The set of algebraic numbers is known to be countable; if the set of transcendental numbers is also countable then the union of these two sets would be countable set too.