CMI(BSc I) Calculus/Analysis HA-1 07-08-2013

While "pleasure" and "enjoyment" are often used to characterize one's efforts in science, failures, frustrations, and disappointments are equally, if not the more, common ingradeints of scientific experience. Overcoming difficulties, undoubtedly, contributes to one's final enjoyment of success.

S. Chandrasekhar.

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

John von Neumann

Following is expected from you.

Reach the class room on time.

If you have to be absent, find out what was done — before you
come to the next class. Also let me know why you were absent.

Every week there will be home assignment, please go through, think,
work out, pen solutions and Finally read what you have written.

If you have trouble understanding an exercise, keep reading
again and again. You are sure to succeed.

If an exercise asks you to do something complicated, see
if you can do a simpler thing and build on your success.

Regarding discipline and hard work, stick to your school routine.
Regarding Math, breathe an air of freedom,
start thinking and questioning.
What is a proof? How to communicate your proof to others?

Make a habit to consult books in the library, for example,
Tom Apostol; Robert Bartle; Bartle and Donald Sherbert
Richard Courant and David Hilbert; Courant and Fritz John;
Walter Rudin; George Simmons etc etc, unlimited!

* * * * * *

1. (a) Show

$$xy \le \frac{1}{2}(x^2+y^2) \qquad x, y \in R.$$
$$2\sqrt{xy} \le x+y \qquad x>0, y>0.$$
$$|x_1+x_2+\cdots+x_{97}| \le |x_1|+|x_2|+\cdots+|x_{97}|.$$

(b) If $x_1 \ge x_2 \ge \cdots \ge x_n \ge 0$ and $y_1 \ge y_2 \ge \cdots \ge y_n \ge 0$, show that

$$n\sum x_i y_i \ge (\sum x_i)(\sum y_i).$$

(c) If x, y, z are non-negative numbers, show that

$$x^2+y^2+z^2 \ge xy+yz+zx,$$

$$(x+y)(y+z)(z+x) \ge 8xyz,$$
$$x^2y^2+y^2z^2+z^2x^2 \ge xyz(x+y+z).$$

2. I have a finite set S. Anand tells me that S has 13 elements, that is, he has a function $f : S \to \{1, 2, \cdots, 13\}$; one-to-one and onto. Bhakta tells me that S has 11 elements, that is, he has a function $g : S \to \{1, 2, \cdots, 11\}$; one-to-one and onto. We all believe that one of them (at least) must be wrong. They refuse to show us their functions. How do you convince them that one of them must be wrong?

3. Given a real number $x \in [0, 1]$ show that there exists a sequence of integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ each ϵ_i being one of $0, 1, \cdots, 9$ such that

$$x = \frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2} + \cdots + \frac{\epsilon_n}{10^n} + \cdots.$$

This sequence $(\epsilon_1, \epsilon_2, \cdots)$ is called the decimal expansion of x. It is usually written as $\cdot\epsilon_1\epsilon_2\ldots$.

Show that if a number x has two expansions, then one expansion must 'end' with all zeros (called terminating expansion) and the other must end with all 9 (called non-terminating expansion). Further, a number can not have more than two expansions.

4. We have described decimal expansions of numbers in the interval $[0, 1]$. To complete the picture, show that every non-negative integer can be expressed as a finite sum $\eta_k(10)^k + \eta_{k-1}(10)^{k-1} + \cdots + \eta_1(10) + \eta_0$ where each η_i is one of the numbers $\{0, 1, 2\cdots, 9\}$. Moreover such an expression is unique subject to $\eta_k \ne 0$, in case $k \ge 1$. We consider only such expressions.

Conclude that if $x \in R$ and $x \geq 0$, then we can express

$$x = \eta_k(10)^k + \eta_{k-1}(10)^{k-1} + \ldots + \eta_0 + \epsilon_1(\frac{1}{10}) + \ldots + \epsilon_j(\frac{1}{10})^j + \ldots,$$

where each of the numbers ϵ and η are among $\{0, 1, \cdots, 9\}$ (with $\eta_k \neq 0$ in case $k \geq 1$). If we were to *denote* $1/10$ *by* z, and η_i by ϵ_{-i} then this expression takes the pleasing form

$$x = \frac{\epsilon_{-k}}{z^k} + \cdots + \frac{\epsilon_{-1}}{z} + \epsilon_0 + \epsilon_1\, z + \epsilon_2 z^2 + \cdots.$$

$$\text{symbolically,} \quad x = \epsilon_{-k}\epsilon_{-k+1}\ldots\epsilon_0 \;\cdot\; \epsilon_1\epsilon_2\ldots\epsilon_j\ldots.$$

5. To conclude this circle of ideas about decimal representation, let us consider once again decimal expansion of numbers $x \in (0, 1)$. Say that a decimal expansion $\cdot\epsilon_1\epsilon_2\ldots$ is recurring if (a block repeats after some stage) there are integers $k \geq 0$ and $l \geq 1$ such that

$$(\epsilon_{k+1}\cdots\epsilon_{k+l}) \;=\; (\epsilon_{k+l+1}\cdots\epsilon_{k+2l}) \;=\; (\epsilon_{k+2l+1}\cdots\epsilon_{k+3l}) \;=\; \cdots.$$

Show that x is rational iff it has a recurring expansion.

6. Recall the axioms for real number system $(R, +, \cdot, <)$ which we have adapted.

Axiom set I: for addition $(+)$. Additive identity is denoted 0 and additive inverse of x is denoted $-x$.

Axiom set II: for multiplication $(\cdot)$. Multiplicative identity is denoted $1 (\neq 0)$, multiplicative inverse for $x \neq 0$ is denoted $1/x$.

Axiom set III: says $(+, \cdot)$ are friendly. $x \cdot (y + z) = x \cdot y + x \cdot z$.

Axiom set IV: $(<)$. For any x, y exactly one of $x < y$, $x = y$, $y < x$ holds. If $x < y$ and $y < z$ then $x < z$.
We use $x \leq y$ as abbreviation for '$x < y$ or $x = y$'.

Axiom set V: says $(<)$ is friendly with $(+, \cdot)$, namely,
($y < z$ implies $x + y < x + z$) and ($0 < x,\ 0 < y \Rightarrow 0 < x \cdot y$) .

Final Axiom set VI: is the least upper bound axiom. Let $S \subset R$ be non-empty. Suppose S has an upper bound — $(\exists y)(\forall x \in S)(x \leq y)$. Then S has a least upper bound —
$(\exists z)\ \{[\forall x \in S,\ x \leq z]\ \&\ [(\forall x \in S,\ x \leq y) \Rightarrow z \leq y]\ \}$.
Sometimes this axiom is also called 'continuity axiom' or 'completemess axiom', because it tells us that our geometric picture of real numbers

as a line without breaks/gaps is justified.

Fix such a system $(R, +, \cdot, <)$ once and for all. Elements of R are called real numbers.

Here are some questions that need to be answered *immediately*. (1) Did we not know real numbers already? (2) What is this business of axioms? (3) Is there such a system at all? (4) How many such systems are there? (5) What is the relation of such a system to the real numbers we have been using all along, in particular, in school?

In a nutshell here are the answers. (1) Yes, we have working knowledge of real numbers, afterall we have been working with real numbers. We know real numbers just as we know 'colour' or 'green colour' and so on. Think about it. (2) The axioms are the rules we accept once and for all about numbers. You can be assured that when we have a question (about real numbers) to be answered, we only use these few rules and would not make new rules. (3) Yes, such a system exists. We postpone construction to the end of the course, not because it is difficult,but because it is boring/dull. We need to collect some bricks, arrange them, throw in some cement, water and so on. (4) Such a system is unique, in the sense, if there are two such systems then there is an isomorphism between them that preserves the operations $(+)$ and $(\cdot)$ as well as the relation $\leq$. (5) What we have been using all along is just such a system, no more and no less. You need not panic. To justfy this, we need to show that everything we used so far about real numbers can be *deduced* from the few axioms listed above. This is interesting, though tiresome at times. We see some examples in the class, just to convince ourselves.

Show sum and product of rational numbers is rational. Show that sum and product of two algebraic numbers is algebraic. Do you think this will be true if algebraic (through out the sentence) is replaced by irrational?

Is there a rational number whose square is 42? Write a rigorous argument for your answer so that others are convinced after reading it. (You will not be there to explain *what you meant*, they only read *what you have written*).

Let me remind you that having square roots and cube roots (for positive numbers) is not our birth-right. Think of the set of rational numbers. It does not have a number whose square is 3. The set of reals R can provide a number whose square is 3, but is unable to provide a number whose square is (-3). Later, you will know that the set of complex numbers provides this also. Think about these matters *till* you are convinced that what we are doing in class *needs* to be done.

I would also like you to appreciate the subtle point in showing that *there is no one-to-one map on* $\{1, 2, \cdots, 11\}$ *onto* $\{1, 2, \cdots, 13\}$. You may start thinking: if I associate with 1, then with 2 etc, I run out of numbers in the domain but numbers remain in the range etc etc. It is fine, but does not solve our problem. It is important and necessary to feel what goes wrong, but you have to support the feeling with argument that no-one can refute. For example, some one says, I have such a function, but I will not show you. How do you convince him: you can not fool me; I know, for sure, you are wrong?

I am calling your attention to tricky points in arguments so that you get a hang of 'what is a proof' and 'how to write a proof.'.

You have to keep two things separate: abstraction and clarity. We are not trying to be abstract. We want to be clear about what we are saying. The purpose of saying something is to get across that thing to another person. If he/she did not understand what you said, the purpose is lost. That is why clarity is important. For this, you should first know what you are saying.

One important thing to keep in mind is that, in the middle of all this, we should not loose track of the inherent beauty of ideas and arguments.

7. Given a real number $x \in [0, 1]$ show that there exists a sequence of integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ each ϵ_i is either zero or one such that

$$x = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots + \frac{\epsilon_n}{2^n} + \cdots .$$

This sequence $(\epsilon_1, \epsilon_2, \cdots)$ is called binary expansion of x. It is usually written as $\cdot\epsilon_1\epsilon_2\ldots$.

Show that if a number x has two expansions, then one expansion must 'end' with all zeros (called terminating expansion) and the other must end with all ones (called non-terminating expansion). Further, a number can not have more than two expansions.

8. Let us now fix an integer $r \geq 2$. Let me repeat that this integer is fixed. Given a real number $x \in [0, 1]$ show that there exists a sequence of integers $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ each ϵ_i being one of $0, 1, \cdots, r-1$ such that

$$x = \frac{\epsilon_1}{r} + \frac{\epsilon_2}{r^2} + \cdots + \frac{\epsilon_n}{r^n} + \cdots.$$

This sequence $(\epsilon_1, \epsilon_2, \cdots)$ is called expansion of x to the base r. It is usually written as $\cdot\epsilon_1\epsilon_2\ldots$.

Show that if a number x has two expansions, then one expansion must 'end' with all digits being zero after some stage (called terminating expansion) and the other must end with all digits being $r-1$ after some stage (called non-terminating expansion). Further, a number can not have more than two expansions.

When $r = 2$ you get binary expansion; $r = 10$ gives decimal expansion; $r = 3$ gives ternary expansion for the number x.

Why are you boring us with all these expansions? Well, if there is a main center in a town, it is good /important to know several roads that lead to the place — later you can choose whatever road is convenient for you.

9. Let S be a non-empty subset of R.

Say that S is bounded below if there is an $a \in R$ such that $a \leq x$ holds for every $x \in S$. Any such number is called a lower bound for the set S. Greatest lower bound (glb), if exists, is a number l such that it is a lower bound for S and $m \leq l$ holds for every lower bound m of S. glb of S is also called infimum of the set S.

Say that S is bounded above if there is an $b \in R$ such that $x \leq b$ holds for every $x \in S$. Any such number is called an upper bound for the set S. lub of S is also called supremum of the set S.

Say that S is bounded if there is an $a \in R$ and an $b \in R$ such that $a \leq x \leq b$ holds for every $x \in S$.

Show that the lub axiom for R, which we have assumed, implies the following:
(i) Every non-empty subset of R which is bounded below has a glb.
(ii) Every non-empty subset of R which is bounded has a glb
(iii) Every non-empty subset of R which is bounded has a lub.
Show conversely the following: If you did not assume the lub axiom for R, but instead assumed any one of the above three statements then you can prove lub axiom as a theorem.

Moral: We did not show any partiality in assuming lub axiom, ti is same as glb axiom.

10. It will be good to have a criterion to recognize lub of a set. Let S be a non-empty set bounded above. Show the following. A number s is lub of S iff the following two conditions hold: (i) If $x \in S$ then $x \leq s$ and (ii) if $\epsilon > 0$ then there is at least one element $x \in S$ such that $x > s - \epsilon$.

    Let S be a non-empty subset of R which is bounded below. A number m is glb of S iff following two conditions hold: (i) If $x \in S$ then $m \leq x$ and (ii) if $\epsilon > 0$, there is at least one element $x \in S$ such that $x < m + \epsilon$.

11. I have an interval $[a, b]$. Can there be two numbers x, y in this interval such that $x - y > b - a$?. Suppose I have two intervals $[a, b]$ and $[c, d]$. Suppose every element of $[a, b]$ belongs to $[c, d]$. In other words $[a, b] \subset [c, d]$. Show that $c \leq a \leq b \leq d$.

12. Let S be a non-empty subset of R. Define a new set, $T = \{-x : x \in S\}$. If S is bounded above then show that T is bounded below. Also show that, if s is lub of S, then $-s$ is glb of T.

13. Let A and B be two non-empty sets of positive real numbers. Let us make a new set, $C = \{xy : x \in A, y \in B\}$ If s is lub of A and t is lub of B, show that st is lub of C.
    Do you think this will be true if A and B are arbitrary (not necessarily positive) non-empty subsets?

14. Let A and B be two non-empty sets of real numbers. Let us make a new set, $C = \{x + y : x \in A, y \in B\}$ If s is lub of A and t is lub of B, show that $s + t$ is lub of C.

15. Let $x \in R, x \neq 0, x \neq 1$. Consider the sequence $a_n = x^n$ for $n = 1, 2, 3, \cdots$. Show that this is an increasing sequence iff $x > 1$. Show

that this is a decreasing sequence iff $0 < x < 1$. Show that this is a monotone sequence iff $x > 0$.

16. Let $x \neq 0$. Find conditions on x so that the set $\{x^n : n \in Z\}$ is bounded. Find conditions on x so that the set $\{x^n : n \in N\}$ is bounded.

17. Show that the set P, of polynomials in one variable x with integer coefficients, is a countable set. Show that the set of algebraic numbers is a countable set.

    Numbers which are not algebraic are said to be transcendental.

18. Show that every interval (a, b) with $a < b$ is uncountable. In fact show that every such interval has the same number of elements as the interval $(0, 1)$.

    Show that between any two distinct real numbers there is a transcendental number.

We shall discuss some of these problems next week.

I hope you are paying attention to the first page of the first assignment. In particular, solve and write the solution and equally important, read what you have written. This last instruction is to be executed sincerely.

Sometimes I do not understand what I have written. Obviously, I can not expect others to understand.

Sometimes I understand, but it is incorrect.

Sometimes I understand, it is correct, but there are some steps for which I have not provided justification. If I did not justify, then the proof is incomplete (and also the reader might think that I am bluffing my way through).

Remember, you will not be sitting next to the reader explaining what you meant! Writing a proof needs practice and you have plenty of time (unless, you want to convert exam as practice session!).

If you have any doubts, you are free to discuss with other students or meet me.

19. Draw the number line and plot the following set.

    (i) $\{x : x^2 - 5x + 6 > 0\}$ (ii) $\{x : x^2 - 5x + 6 \leq 0\}$

    (iii) $\{x : -5 < x^4 < 16\}$.

20. I am sure you all know the following. Prove them.

$$\sum_1^n i = \frac{n^2}{2} + \frac{n}{2}; \qquad \sum_1^n i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

$$\sum_1^n i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}; \qquad \sum_1^n i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

21. For each of the intervals I: $(-23, +85)$ $[-23, +85)$ $(-23, +85]$ $[-23, +85]$ show that $\sup A = +85$ and $\inf A = -23$.

22. For a number $x > 0$ we defined $\sqrt[n]{x} = \sup\{z > 0 : z^n < x\}$. Show that $\sqrt[n]{x} = \inf\{z > 0 : z^n > x\}$.

23. Solve: $x > 0$, $x^2 - x - 1 = 0$.
    We are taught in school how to solve quadratic equations. We get $x = \frac{1\pm\sqrt{5}}{2}$. Since we want positive solution we take $(1 + \sqrt{5})/2$. Solving quadratics with the formula $[-b \pm \sqrt{b^2 - 4ac}]/2a$ is very important.

It is such an excellent answer, we stopped thinking further about the problem. Let us think afresh. First of all x can not be zero.

Want $x^2 = x+1$. That is, $x = 1 + \dfrac{1}{x}$. This is very interesting equation. It does not give us the value of x, but explains x in terms of x. We can use this information for 'self improvement'. Use this value of x on right side,

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{x}}; \quad x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{x}}}$$

It is natural to believe that the solution is

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Non-sense or meaningful? We are the masters to give a meaning.

Put $x_0 = 1$, put inductively for $n \geq 1$, $x_n = 1 + \dfrac{1}{x_{n-1}}$. Shall show the sequence (x_n), so defined, converges to the solution.

If $\frac{3}{2} \leq x_{n-1} \leq 2$ then show that the same holds for x_n too. Show x_1 satisfies these inequalities and hence all x_n for $n \geq 1$, satisfy. Show

$$|x_{n+1} - x_n| \leq (\frac{2}{3})^2 |x_n - x_{n-1}|; \qquad n \geq 3.$$

Use this to show (x_n) converges. If the limit is a, show $\frac{3}{2} \leq a \leq 2$ and $a = 1 + \frac{1}{a}$ and hence $a = (1+\sqrt{5})/2$.

24. We have seen that a real number has several dresses — binary, decimal and so on — in which it can appear.

    For example, consider a, the multiplicative inverse of $1 + 1 + 1$, you can write as $1/3$ or $\frac{1}{3}$ or 3^{-1}. When it wears decimal dress it appears to you as $0.33333333\cdots\cdots$. When it wears binary dress, it appears as $0.0101010101\cdots\cdots$. In ternary dress it has two different styles of appearance $0.100000\cdots\cdots$ and $0.0222222222222\cdots\cdots$..

    We discuss one more colourful dress possessed by numbers.

    For any non-negative number a, let $\langle a\rangle$ and (a) denote the largest integer not exceeding a and the fractional part of a respectively. Thus, $(a) = a - \langle a\rangle$. It is customary to denote $\langle x\rangle$ by $[x]$. Unfortunately, in the present context, the brackets [ ] are reserved for something else.

Fix $x \in (0,1)$, we define a sequence (finite or infinite) of integers $(n_1, n_2, \cdots,)$, where each $n_i \geq 1$, as follows. Set

$$n_1 = \langle 1/x \rangle; \quad r_1 = (\frac{1}{x}) = \frac{1}{x} - n_1.$$

In general, having defined n_i and r_i for $1 \leq i \leq k$; put

$$n_{k+1} = \langle 1/r_k \rangle; \qquad r_{k+1} = (\frac{1}{r_k}) = \frac{1}{r_k} - n_{k+1}.$$

If at some stage we find $r_k = 0$, we stop and say that $[n_1, n_2, \ldots, n_k]$ is the *continued fraction expansion* of x. If this process continues for ever, then we say that the infinite sequence $[n_1, n_2, \ldots]$ is the continued fraction expansion of x. In the first case, we say that x has a terminating expansion and in the second case, we say it has a non-terminating expansion.

We write $x = [n_1, n_2, n_3, \cdots]$ or also as (which occupies more space)

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \frac{1}{\ddots}}}}.$$

What is the meaning of left side above? If each $a_i > 0$, we define $[a_1, \cdots, a_k]$ by induction on k as follows. (pause/think)

$$[a_1] = 1/a_1; \quad \text{for } k > 1, \quad [a_1, \cdots, a_k] = [a_1, \cdots, a_{k-2}, a_{k-1} + \frac{1}{a_k}].$$

We define $[n_1, n_2, n_3, \cdots] = \lim_k [n_1, \cdots, n_k]$. Does this limit exist? Yes, and equals x. We postpone these matters, but do the following.

Show that $x \in (0,1)$ is rational iff the expansion is terminating. What is the expansion for $4/5$?, $144/89$? What number is $[1, 2, 3, 4]$?

If $x > 1$ and not an integer, then its continued fraction expansion is given by $[n_0; n_1, n_2, \cdots]$ where $n_0 = \langle x \rangle$ and $[n_1, n_2, \cdots]$ is the expansion of (x). Notice that n_0 is separated from the rest by a semi-colon, so there is no confusion. If $x \geq 1$ is an integer, we simply say $[x;]$ is its continued fraction expansion (funny, semicolon followed by bracket?).

25. If $\{x_n\}$ converges, show that $\{|x_n|\}$ converges. Is the converse true?

26. For which real numbers x does the sequence $\{x^n\}$ converge. In such a case, what is the limit?

27. Show

$$\frac{(n+47)^{589}}{2^n} \to 0; \qquad \sqrt[n]{n^{43}} \to 1; \qquad \lim \sqrt[2n+1]{n^2+n} = 1.$$

28. If $\sum_1^{59} a_i = 0$, show that $\lim \sum_1^{59} a_i \sqrt{n+i} = 0$.

29. Show $\sqrt{2}, \quad \sqrt{2+\sqrt{2}}, \quad \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots \to 2$.

30. If $a_n > 0$ and $\lim(a_{n+1}/a_n) = L > 0$, then show that $\sqrt[n]{a_n} \to L$. Use this to evaluate limits of $\sqrt[n]{n}; \quad \sqrt[n]{n^5+n^4}, \quad \sqrt[n]{n!/n^n}$

    Use the last limit to show that $n! = n^n e^{-n} a_n$ where $\sqrt[n]{a_n} \to 1$.

    (Much later, you will learn something called Stirling's formula which gives a better understanding of $n!$).

31. Purba defines: a sequence (x_n) converges to a number x iff *given* any integer $m = 1, 2, \cdots$ *there is* an integer n_0 (possibly depending on m) such that $|x_n - x| < 9^{-m}$ for all $n \geq n_0$.

    Uma defines: a sequence (x_n) converges to a number x iff *given* any integer $m = 1, 2, \cdots$ *there is* an integer n_0 (possibly depending on m) such that $|x_n - x| < 2^{-m}$ for all $n \geq n_0$.

    Do you think they are related to our definition? What if they replace $<$ by $\leq$?

32. Let $f : \{1, 2, 3, \cdots\} \to Q \cap (0, 1)$ be a bijection. Let $x_n = f(n)$. Calculate $\liminf x_n$ and $\limsup x_n$. Find the set of all limit points of the sequence.

****

12345678910111213141516171819202122232425

****

For R we have adopted six axiom sets. A system $(R, +, \cdot)$ satisfying the first three sets is called an *algebraic field* or simply, a field. A system $(R, +, \cdot, <)$ satisfying the first five sets of axioms is called an *ordered field*. A system satisfying all the six sets is called a *complete ordered field* or *complete Archimedean ordered field*. Thus Real number system is nothing but a complete ordered field.

These terms or useful when you communicate with others. You need not pay much attention to these technical terms, especially because all your teachers would probably be introducing many new words and it becomes hard for you. As far as our course is concerned, it is important to remember what the rule or axiom says and what can be done with those; rather than remembering just these technical names.

I am not yet sure if you have started writing solutions for exercises. If you could not solve an exercise and we did it in class, then you should, after the class, spend a few minutes analyzing why you could not solve and what is it that you missed. This is important.

33. Suppose that a sequence (x_n) converges. show that the following sequence also converges.

$$x_1, x_1, x_2, x_2, x_3, x_3, x_4, x_4, x_5, x_5, \cdots\cdots.$$

What is the n-th term of this sequence?

What if I repeated each term ten times.

What if I repeated n-th term n times, like

$$x_1, x_2, x_2, x_3, x_3, x_3, x_4, x_4, x_4, x_4, x_5, \cdots\cdots.$$

You may, at first sight, think that these are trivial and I am showing you the same sequence again and again. But please do recall the definition of sequence and convince yourself that these are all different sequences.

What if I deleted all odd terms, that is, consider the sequence,

$$x_2, x_4, x_6, x_8, x_{10}, x_{12} \cdots\cdots.$$

What if I take integers $1 \leq n_1 < n_2 < n_3 < n_4 < \cdots$ and defined a sequence $y_k = x_{n_k}$ for $k = 1, 2, 3, 4, \cdots$.

34. I have a sequence (x_n). I know that the sequence $y_n = x_{2n}$ $(n \geq 1)$ converges. I know that the sequence $z_n = x_{2n-1}$ $(n \geq 1)$ converges. That is, the sequence of even terms converges and the sequence of odd terms converges.

    Do you think the sequence (x_n) converges. Under what conditions on the limits of these two sequences can you conclude that the sequence (x_n) converges?

    Can you think of a generalization of the above.

35. Suppose a sequence (a_n) of real numbers converges to a number a. I have a polynomial of one variable $P(x)$. Show that the sequence of numbers $\{P(a_n)\}$ converges to the number $P(a)$.

36. Later we shall define the sine function rigorously. But starting with your knowledge, discuss the following. For which real numbers x does the sequence $\{(\sin x)^n\}$ converge. In case the sequence does not converge, explain what are all its limit points.

37. (not easy) If your sequence of numbers are getting close to a then successive averages of your numbers also get close to a.

    If a sequence $\{x_n\}$ converges, then show that the sequence $\{a_n\}$ where $a_n = \frac{1}{n}\sum_{1}^{n} x_k$ also converges. Actually, these averages also converge to the same limit as the original sequence.

    If a seqeunce $\{x_n\}$ is such that the sequence of successive averages converges, then the sequence $\{x_n\}$ is said to converge in the sense of *Cesaro*. Thus any convergent sequence is also convergent in the sense of Cesaro.

    Show that the converse is not true by considering the ± 1 sequence.

38. Show, without using the fact that Cauchy sequences converge, that sum and product of Cauchy sequences is again Cauchy. That is, if (x_n) and (y_n) are Cauchy sequences, then so are the sequences $(x_n + y_n)$ and $(x_n y_n)$. should not use the difficult fact that Cauchy sequences converge.

    Suppose that (x_n) is a Cauchy sequence and $x_n \neq 0$ for every n. do you think $(1/x_n)$ is a Cauchy sequence?

The reason, I did not want you to use convergence of Cauchy sequences is the following. Afterall, if your world consists of the set of rationals (and no more) then also the definition of Cauchy sequence makes sense; but of course in this world now there are Cauchy sequences that do not converge.

39. Let (x_n) be a sequence and $a \leq x_n \leq b$ for each n. If $x_n \to x$, then show that $a \leq x \leq b$.

    More generally, show that the above inequality holds for any limit point of the sequence.

40. Write complete proof of the fact explained in class: The limsup and liminf of a bounded sequence are indeed limit points of the sequence. Remember these are defined as the supremum and infimum of the set of limit points. (We showed that this set is non-empty).

41. Let (x_n) be a bounded sequence.

    A number s is limsup of the sequence if and only if the following two conditions hold:

    (i) for any $\epsilon > 0$, there are only finitely many n such that $x_n > s + \epsilon$.

    (ii) for any $\epsilon > 0$, there are infinitely many n such that $x_n > s - \epsilon$.

    A number l is liminf of the sequence if and only if the following two conditions hold:

    (i) for any $\epsilon > 0$, there are only finitely many n such that $x_n < l - \epsilon$.

    (ii) for any $\epsilon > 0$, there are infinitely many n such that $x_n < l + \epsilon$.

42. Let (x_n) and (y_n) be bounded sequences. Show that

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

$$\liminf x_n = -\limsup(-x_n).$$

$$\limsup x_n = -\liminf(-x_n).$$

$$\limsup(29x_n) = 29 \limsup x_n.$$

$$\liminf(29x_n) = 29 \liminf x_n.$$

43. Let (x_n) be a bounded sequence.

Define $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \cdots\}$. Show that (y_n) is decreasing and bounded below and hence converges to a limit. Show that this limit is indeed $\limsup x_n$. Thus limsup is actually limit of the supremums of the 'tails'(?) of the sequence (not only supremum of limit points).

Define $z_n = \inf\{x_n, x_{n+1}, x_{n+2}, \cdots\}$. Show that (y_n) is increasing and bounded above and hence converges to a limit. Show that this limit is indeed $\liminf x_n$. Thus liminf is actually limit of the infimums of the 'tails' of the sequence (not only infimum of limit points).

44. A sequence (x_n) converges if and only if it is bounded and $\limsup x_n \le \liminf x_n$.

This last inequality is same as saying that the bounded sequence has excatly one limit point.

****

23571113171923293137414347535961677173798397101

****

45. Fix a number h. Let us define $(x)_n$ for $n \geq 0$ by

$$(x)_0 = 1, ; \quad (x)_n = x(x-h)(x-2h)\cdots(x-[n-1]h).$$

Show the following for $n = 1, 2, 3\cdots$.

$$(x+y)_n = (x)_n + \binom{n}{1}(x)_{n-1}(y)_1 + \binom{n}{2}(x)_{n-2}(y)_2 + \cdots + (y)_n.$$

How does this read if $h = 0$?

46. Do you think a convergent sequence (a_n) must have a maximum, that is, an k such that $a_k \geq a_n$ for all n?

Do you think a convergent sequence (a_n) must have a minimum, that is, an k such that $a_k \leq a_n$ for all n?

Show that a convergent sequence (a_n) must have either a maximum or a minimum.

47. Suppose that $a_n > 0$ for each n and $\liminf a_n = 0$. Show that for infinitely many values of n the following happens:
a_n is strictly smaller than $a_1, a_2, a_3, \cdots, a_{n-1}$.

48. Suppose that $a_n > 0$ for each n and the sequence (a_n) converges to zero. Show that for infinitely many values of n the following happens:
a_n is strictly larger than $a_{n+1}, a_{n+2}, a_{n+3}, \cdots\cdots$.

49. Suppose $a_n \leq b_n \leq c_n$. if $\lim a_n = \alpha = \lim c_n$, then show that $\lim b_n$ exists and equals α. If you are only told that $\lim a_n$ and $\lim c_n$ exist, then can you conclude that $\lim b_n$ exists?

50. Test for convergence of $\sum a_n$ where,

$$a_n = \frac{2n}{2n+1} - \frac{2n-1}{2n}, \qquad a_n = (-1)^n \frac{n}{n+1}.$$

51. If $\sum a_n$ and $\sum b_n$ are series of strictly positive terms and $\lim \frac{a_n}{b_n} \to 1$, show that that the series $\sum a_n$ converges iff the series $\sum b_n$ converges. What if the limit were 1000 instead of one.

52. Test the following for convergence.

$$\sum \frac{2000}{\sqrt[3]{29n^4 - 35}}; \qquad \sum \frac{2000}{\sqrt[4]{29n^3 - 35}};$$

$$\sum \frac{1}{n(1.01)^n}; \qquad \sum \frac{\log n}{n^{1.0001}}; \qquad \sum \frac{n^{100000}}{n!}; \qquad \sum \frac{(100000)^n}{n!}.$$

You may need to use facts about log function, that we have not yet discussed.

53. Let $P(x)$and $Q(x)$ be two polynomials in one variable x. Assume that $Q(n) \neq 0$ for $n = 1, 2, 3, \cdots$. Discuss convergence of $\sum \frac{P(n)}{Q(n)}$.

54. If $\sum a_n^2$ converges, show that $\sum \frac{a_n}{n}$ converges.

Sometimes a more general formulation will help solve the problem, because it gives an idea. If $\sum a_n^2$ and $\sum b_n^2$ converge, then show that the series $\sum a_n b_n$ converges absolutely.

If $\sum |a_n|$ converges, show that $\sum a_n^2$ converges. If $\sum a_n$ converges, do you think that $\sum a_n^2$ converges?

55. Show that the series below converge iff $p > 1$.

$$\sum_{n \geq 100} \frac{1}{n \log n \, (\log \log n)^p} \qquad \sum_{n \geq 10000} \frac{1}{n \log n \log \log n \, (\log \log \log n)^p}$$

56. If $0 < r < 1$, you already know that $\sum r^n$ converges. Show that $\sum n^{100} r^n$ converges. More generally, if $P(x)$ is any polynomial in one variable x, show that $\sum P(n) r^n$ converges.

57. Show that for $0 < x < 1$, the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots$$

converges.

58. Show that the series

$$\sum (-1)^n \frac{\sin(19/n)}{\log \log(100 + n)}. \qquad \sum \frac{1}{n} \sin \frac{1}{n}.$$

converge. You can proceed with your knowledge of sine function, though we develop this function rigorously later. Do you think this series is absolutely convergent?

59. discuss convergence of the following two series.

$$\sum_{n\geq 2} \frac{\log(n+1) - \log n}{(\log n)^2}; \qquad \sum_{n\geq 1} \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}.$$

For the first series, you need to know a little more than the definition of log function.

60. Let $0 < a < b < c < 1$. Show that the following series converges.

$$a + b + c + a^2 + b^2 + c^2 + a^3 + b^3 + c^3 + \cdots\cdots.$$

Show that the series $\sum x_n$ also converges, where $x_n = a^n + b^n + c^n$. Note that this series is different from the series above (just ask yourself: what is the first term, what is the second term). Do these two series have the same limit? Justify your answer.

61. Sometimes the statement of the problem is long and probably frightening, but the solution is immediate. Here are two such problems.

Let $\sum a_n$ be a series of non-zero numbers.

Suppose that there is an $\epsilon > 0$ and and an integer n_0 such that for every $n \geq n_0$

$$\frac{\log \frac{1}{|a_n|}}{\log n} > 1 + \epsilon.$$

Then show that the series $\sum a_n$ converges.

Suppose that there is an $\epsilon > 0$ and and an integer n_0 such that for every $n \geq n_0$

$$\frac{\log \frac{1}{|a_n|}}{\log n} < 1 - \epsilon.$$

Then show that the series $\sum a_n$ does not converge.

62. One suggestion to show convergence of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is to write it as

$$(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots$$

and show that this series converges. This needs justification, because the two series are different (ask yourself: what is the first term etc).

Justify by doing the following. Show that the second series converges. So partial sums are Cauchy. Then argue that for the first series also partial sums are Cauchy.

63. The theorem on alternating series said the following: If $a_n \downarrow 0$, then the series
$$a_1 - a_2 + a_3 - a_4 + - \cdots$$
converges.

Here is a generalization. Let $\sum b_n$ be a series with bounded partial sums. Let $a_n \downarrow 0$. Then the series $\sum a_n b_n$ converges. Note that by taking the series $\sum b_n$ to be $\sum \pm 1$ series, you get the theorem on alternating series (remember, we are not demanding convergence of $\sum b_n$).

To prove this generalization, procees ad follows. Let (s_n) be the partial sums of the series $\sum b_n$. Show for $m > n$
$$a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + a_{n+3}b_{n+3} + \cdots + a_{m-1}b_{m-1} + a_m b_m =$$
$$-s_n a_{n+1} + s_{n+1}(a_{n+1} - a_{n+2}) + s_{n+2}(a_{n+2} - a_{n+3})$$
$$+ \cdots + s_{m-1}(a_{m-1} - a_m) + s_m a_m.$$
Use this to show convergence of the series $\sum a_n b_n$.

This generalization comes to our rescue in difiicult situations.

Using your understanding of sine and cosine functions, show that the series $\sum \sin nx$ has bounded partial sums. (Hint: multiply and divide partial sum by $\sin(x/2)$ and see.) Conclude that the series $\sum \frac{\sin nx}{n}$ converges.

64. Using Cauchy product of series, show
$\sin(x + y) = \sin x \cos y + \cos x \sin y$, and
$\cos(x + y) = \cos x \cos y - \sin x \sin y$.

****

71737983899710110310710911312713113713914915115716316 7173

****

65. Suppose that (a_n) is a Cauchy sequence. Suppose that a subsequence converges to a. This means, there are $n_1 < n_2 < \cdots\cdots$ and if $b_k = a_{n_k}$ then the sequence (b_k) converges to a. Show that $a_n \to a$.

66. Suppose $x_n \to a$. Let π be a permutation of $\{1, 2, \cdots\}$. Put $y_n = x_{\pi(n)}$. Show that $y_n \to a$. (This is just to make sure that you do not confuse between sequences and series).

67. We have shown that Cauchy product of two absolutely convergent series is convergent. Show that it is absolutely convergent.

68. $\sum a_n$ is a convergent series of strictly positive numbers, show $\sum \frac{\sqrt{a_n}}{n}$ converges.

69. $\sum a_n$ is a series of positive numbers which does not converge. Show that $\sum \frac{a_n}{1+a_n}$ does not converge.

    If $s_n = a_1 + \cdots + a_n$, show that

$$\frac{a_{n+1}}{s_{n+1}} + \cdots + \frac{a_{n+k}}{s_{n+k}} \geq 1 - \frac{s_n}{s_{n+k}}$$

    Deduce that $\sum \frac{a_n}{s_n}$ does not converge.

    Show that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

    Deduce that $\sum \frac{a_n}{s_n^2}$ converges.

    Discuss convergence of

$$\sum \frac{a_n}{1 + na_n} \qquad \sum \frac{a_n}{1 + n^2 a_n}.$$

70. $\sum a_n$ is a convergent series of strictly positive numbers. Put $r_n = \sum\limits_{m \geq n} a_m$. If $m < n$ show that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}.$$

    Deduce that $\sum \frac{a_n}{r_n}$ does not converge.

    Show that

$$\frac{a_n}{\sqrt{r_n}} \leq 2(\sqrt{r_n} - \sqrt{r_{n+1}}).$$

    Deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

71. Let $a > 0$. Choose a number $x_1 > \sqrt{a}$. Define recursively

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}), \qquad n \geq 2$$

Show $x_n \downarrow \sqrt{a}$.

If $\epsilon_n = x_n - \sqrt{a}$, show

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{x_n} \leq \frac{\epsilon_n^2}{\sqrt{a}}.$$

72. Fix $a > 1$. Choose a number $x_1 > \sqrt{a}$. Define recursively

$$x_{n+1} = \frac{a + x_n}{1 + x_n} = x_n + \frac{a - x_n^2}{1 + x_n}.$$

Show $x_1 > x_3 > x_5 \cdots\cdots$.

Show $x_2 < x_4 < x_6 < \cdots\cdots$.

Show $\lim x_n = \sqrt{a}$.

73. Let $f : R \to R$.
Verify: : f is continuous at a is same as

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)\Big\{[|x - a| \geq \delta] \bigvee [|f(x) - f(a)| < \epsilon]\Big\}.$$

understand the symbols as well as its meaning in words.
How do you understand its negation: f is not continuous at a. Remember, it is not simply putting a negation symbol $\neg$ before this formula. You need to write positive statement, that is, where negations do not appear for quantifiers.
Explain in symbols as well as words. (I repeat: expressing in words is not simply saying: f is not continuous at a. It means, first express in symbols as mentioned above without negation symbols for quantifiers and then write it in words).

Verify: f is continuous on R is same as

$$(\forall a \in R)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)\Big\{[|x - a| \geq \delta] \bigvee [|f(x) - f(a)| < \epsilon]\Big\}.$$

understand the symbols as well as its meaning in words.
How do you understand its negation: f is not continuous on R. Explain in symbols as well as words.

74. Verfiy: $x_n \to x$ is same as

$$(\forall \epsilon > 0)(\exists k)(\forall n \geq k)(|x_n - x| < \epsilon).$$

Understand the symbols as well as the meaning in words.
Express its negation, $x_n \not\to x$ both in symbols as well as words.

75. Verify: x is a limit point of (x_n) is same as

$$(\forall \epsilon > 0)(\forall k)(\exists n > k)(|x_n - x| < \epsilon).$$

Express its negation: x is not a limit point of (x_n) both in symbols as well as words.

76. x is limsup of (x_n) is same as

$$(\forall \epsilon > 0)\Big\{\Big[(\forall k)(\exists n > k)(x_n > x - \epsilon)\Big] \bigwedge \Big[(\exists k)(\forall n > k)(x_n \leq x + \epsilon)\Big]\Big\}.$$

77. Let $f : R \to R$. We say that f is uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that $f(x) - f(y) < \epsilon$ whenever $|x - y| < \delta$. Verify the following formula expresses this.

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y)\Big\{[|x - y| \geq \delta] \bigvee [|f(x) - f(y)| < \epsilon]\Big\}.$$

What is its negation (in symbols as well as words).

78. Just like limit point of a sequence, you can define limit poiint of a set. Let $A \subset R$. Say that a number a is a limit point of the set A if there are infinitely many points of A close to a, more precisely, given $\epsilon > 0$, there are infinitely many points of A in $(a - \epsilon, a + \epsilon)$. Verify that the following formula expresses the same thing.

$$(\forall \epsilon > 0)(\forall k)\left(\text{cardinality}\Big\{(a - \epsilon, a + \epsilon) \bigcap A\Big\} \geq k\right).$$

What does it mean to say that a is not a limit point of the set A.?

79. Define $f : R \to R$ by $f(x) = 0$ if x is rational and $f(x) = 1$ if x is irrational.

Describe the set of all points a such that f is continous at a.

Do the same thing for the following functions:

(a) $f(x) = 0$ if x is an integer and $f(x) = 1$ when x is not an integer.

(b) $f(x) = 1$ if $x \geq 0$ and $f(x) = 55$ if $x < 0$.

(c) $f(x) = 1$ if $x > 0$; $f(x) = 55$ if $x < 0$ and $f(0) = 44$.

(d) $f(x) = [x]$, the greatest integer not exceeding x.

(e) $f(x) = x - [x]$, the fractional part of x.

80. You know that the function $f(x) = x^2$ is continuous. If I give $a = 2$ and $\epsilon = 1$ what will be your δ? What if $a = 20$ and same ϵ. What if $a = 200$ and same ϵ?

    Let $f(x) = 1/x$ defined on the interval $(0, 33)$. Take $\epsilon = 0.1$ and $a = 5$, find δ. With the same ϵ and $a = 1$, $a = 1/5$ and $a = 1/100$ find δ.

****

179181191193197199211223227229233239241251257263269271277281283293

****

Most of you have done reasonably well in the midsem.

Some of you (those who got at least 30) have done very well, and should maintain this level. Remember maintaining current level also needs effort.

Some of you (those who got at least 20, but below 30) have done well. You can improve if you work/think slightly harder. Remember, you should not try to stay where you are.

Some of you (those who got less than 20) are not doing well at this stage. However, you are definitely capable, but you need to put in your best efforts. You can recover and do well. Remember, if you can learn to stand, it is not difficult to walk and then it is not difficult to run!

I hope you have all realized that the story of calculus is continuation of high school story with essential differences — concept of proof, clarity regarding what you can use and what you can not, understanding of the ideas, the freedom to question things (and not to blindly reproduce what teacher says), accept responsibility to what we write (and not to blame the book or someone else), developing the ability to communicate what you want to say, etc etc.

Some exercises in this set need thinking on your part, they are not routine.

81. Show that
$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$
is an integer. Here n is a natural number.

82. Let $a_1, a_2, \cdots, a_n$ be strictly positive numbers.

    Show that $\sqrt[2]{a_1 a_2} \le (a_1 + a_2)/2$.

    When n is an integer of the form $2, 2^2, 2^4, 2^8, 2^{16}, \cdots$ (the exponent of 2 is itself a power of 2), show that
$$\sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Actually this is true for all integers $n \geq 2$. You need not do.

However, let me explain how to do it for $n = 3$, if you know for $n = 2^2$. Start with a_1, a_2, a_3 and take $a_4 = \sqrt[3]{a_1 a_2 a_3}$ and try your luck.

83. For each natural number n, show that there is a natural number k (of course, depending on the given n) such that

$$(\sqrt{2} - 1)^n = \sqrt{k} - \sqrt{k-1}.$$

84. In introducing the number e we found that it is limit of a series of numbers and also it is limit of the *increasing* sequence $\left(1 + \frac{1}{n}\right)^n$.

    Show that the sequence $\left(1 + \frac{1}{n}\right)^{n+1}$ is *decreasing*. What is its limit?

85. If f is a montone function defined on an interval and has the intermediate value property, show that f is a continuous function.

    Intermediate value property means the following: If $a < b$; and u is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = u$.

    Do you think that intermediate value property without monotonicity will imply continuity?

    Remember continuous function defined on an interval has the intermediate value property.

86. Let $f(x)$ be defined on the real line as follows: if x is irrational number then $f(x) = 0$. If x is a rational number p/q in lowest terms, then $f(x) = 1/q$. Show that f is continuous at a iff a is irrational number.

    p/q in lowest terms means p and q are integers without common factor. This means: if x is a natural number that divides both p and q, then $|x| = 1$.

87. Let ABC be a triangle in the plane (A, B, C not on a line) . Suppose that a line L in the plane is given. Show that there is a line parallel to L that bisects the triangle into two parts of equal area.

88. Consider the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots.$$

    If the n-th number is p_n/q_n, then $p_{n+1} = p_n + 2q_n$ and $q_{n+1} = p_n + q_n$.

    Show that all the fractions are in lowest terms.

    Show that p_n/q_n converges to $\sqrt{2}$.

89. Let $0 < a < b$ be given. Define

$$a_1 = \sqrt{ab} \qquad b_1 = \frac{a+b}{2}.$$

$$a_2 = \sqrt{a_1 b_1} \qquad b_2 = \frac{a_1+b_1}{2}.$$

and in general,

$$a_{n+1} = \sqrt{a_n b_n} \qquad b_{n+1} = \frac{a_n+b_n}{2}.$$

Show that the sequence (a_n) converges. Show that the sequence (b_n) converges. Show that these two limits are equal.

This is called arithmetic-geometric mean or arithmetico-geometric mean of the two given numbers.

90. For $x, y > 0$ show that

$$\frac{x^n + y^n}{2} \geq \left(\frac{x+y}{2}\right)^n.$$

What does this mean geometrically?

91. Let $0 \leq x \leq 1$. Put

$$s_1 = x; \qquad s_{n+1} = \frac{1}{2}\left(s_n + \frac{x}{s_n}\right) \qquad n \geq 1.$$

Show $|x - \sqrt{x}| \leq 1/4$ and $|s_{n+1} - \sqrt{2}| \leq |s_n - \sqrt{2}|/2$ and deduce

$$|s_n - \sqrt{x}| \leq \left(\frac{1}{2}\right)^{n+1}.$$

Thus $r_n = s_n/\sqrt{x} \to 1$. How fast does it converge? Show

$$r_{n+1} - 1 = \frac{1}{2r_n}(r_n - 1)^2; \qquad s_{n+1} - \sqrt{x} = \frac{1}{2s_n}(s_n - \sqrt{x})^2$$

Deduce that

$$0 \leq r_{n+1} - 1 \leq \frac{1}{2}(r_n - 1)^2.$$

92. Suppose $f : R \to R$ is differentiable and f' is a polynomial of degree $n-1$. Show that f must be a polynomial of degree n.

93. Let $f(x) = x^2 + bx + c$. Show that f is increasing for $x > -b/2$ and decreasing for $x < -b/2$.

94. Let $f : R \to R$ be a function. Then f is said to be an even function if $f(-x) = f(x)$ for every $x \in R$. f is said to be an odd function if $f(-x) = -f(x)$.

    Give two examples of even functions and two examples of odd functions.

    If f is differentiable and is even, show that f' is an odd function.

    If f is differentiable and is odd, then show that f' is an even function.

    If f' is odd show that f is even function.

    If f' is even and $f(0) = 0$, show that f is odd.

95. Find the point of intersection of: tangent to the curve $y = x^2 - x$ at the point $(2, 0)$ and tangent to the curve $y = 1 - x^2$ at the point $(1, 0)$.

96. Show that the functions

    $f(x) = \sqrt{x^2 - 1}$ defined on the interval $(1, \infty)$ and

    $g(y) = \sqrt{y^2 + 1}$ defined on the interval $(0, \infty)$

    are inverses of each other. Verify

$$f'(g(y))g'(y) = 1; \qquad g'(f(x))f'(x) = 1.$$

97. Show that the function

    $f(x) = x^2 + 3x + 1$ defined on the interval $(1, \infty)$ is invertible.

    Find its inverse g. Verify

$$f'(g(y))g'(y) = 1; \qquad g'(f(x))f'(x) = 1.$$

98. (difficult to understand but trivial to solve) Let $r_1, r_2, \cdots$ be an enumeration of the set fo rationals on the real line. For each $x \in R$, let us put

$$f(x) = \sum_{n : r_n \leq x} \frac{1}{2^n}.$$

    That is, given a number x do the following: see if $r_1 \leq x$, if so put $1/2$ in your bag, if not do not put;see if $r_2 \leq x$, if so put $1/2^2$ in your bag, if not do not put. Continue this way. Now add all the numbers you have put in your bag. It is meaningful. What you get is declared as $f(x)$.

    Show f is continuous at a point a iff a is irrational. for every $x \in (0, 1)$

99. Continuing the previous exercise, show the following: Let $D \subset R$ be a countable set given to you. Find a monotone function on R whose set of discontinuity points is exactly the given set D.

    (The idea is not to complicate life, but to see if you understood the previous exercise.)

100. Let f be a strictly increasing continuous function on a closed bounded interval $[a, b]$. Let $c = f(a)$ and $d = f(b)$. Show that range of f is exactly $[c, d]$. Show that the inverse function g, defined on $[c, d]$ is strictly increasing and is again continuous.

    Let f be as above but is moreover differentiable at every point in (a, b). Show that the inverse function g defined on $[c, d]$ is differentiable at every point in (c, d). show that

$$g'(y) = \frac{1}{f'(g(y))}, \quad c < y < d; \quad \text{and} \quad f'(x) = \frac{1}{g'(f(x))}, \quad a < x < b.$$

    Prove a similar statement for strictly decreasing continuous function on $[0, 1]$.

    Suppose that f is a continuous function on $[0, 1]$. Assume that it is one-to-one. That is, if $x \neq y$ then $f(x) \neq f(y)$. Then show that f must either be strictly increasing or strictly decreasing.

    (Idea is to see if you can unravel the meaning of: suppose f is neither increasing nor decreasing $\cdots$.)

101. Can you define a continuous function on the interval $[0, 1]$ whose range is the set of natural numbers.

    Let S be the set of irrational numbers on the interval $[0, 1]$. Is there a continuous function defined on S having its range the set of natural numbers.

    (The idea is *not* to understand continuous functions defined on arbitrary subset S. Who cares about this in a first course of calculus. The idea is to see if you made friends with real numbers.)

102. Let $f(x) = x^{230}e^{-x}$. Show that $f(x) \to 0$ as $x \to \infty$. This means the following: given any $\epsilon > 0$, we can find x_0 so that $|f(x)| \leq \epsilon$ for all $x \geq x_0$.

(Idea is to see if you are comfortable with numbers. Do not complicate life.)

Use the above to calcuate

$$\lim_{x \to 0} \frac{1}{x} e^{-1/x^2}.$$

Let $P(x)$ be any polynomial. Show that $P(x)e^{-x} \to 0$ as $x \to \infty$.

103. We have defined two functions f and g in the class and named them as $\sin x$ and $\cos x$ — without any evidence that they are indeed sine and cosine functions, in fact you may find it confusing. By looking at the series, it is not even clear that they take values in the interval $[-1, +1]$.

In any case, $f(-x) = -f(x)$ and $g(-x) = g(x)$ follow from definition.

You saw one evidence in the class: $f' = g$ and $g' = -f$.

Here is another evidence: Show that $f^2(x) + g^2(x) = 1$ for every x. Deduce that these functions indeed take values in the interval $[-1, +1]$. Also when one of them takes the value zero, then the other one must take a value ± 1.

Hre is another evidence. Using power series, argue

$$\frac{f(x)}{x} \to 1 \quad \text{as} \quad x \to 0.$$

Here is another evidence, this is only for fun because we have no plans, now, of doing complex numbers. However let us see what can be achieved if we knew complex numbers. Write the series definition of exponential function, with ix instead of x, let us name it e^{ix}. Calculate its real part and imaginary parts. What do you see? Remember $i^2 = -1$.

Do you know something called DeMoivre's formula? do you see any glimpse of it?

104. I have a function f defined on the interval $[0,1]$. I do not know what exactly is the function, but I know the following:

$$|f(x) - f(y)| \leq 589 \sin(|x-y|^{235}); \qquad x, y \in [-1,1].$$

Show that f is a continuous function. Show that f is differentiable. Show that f is a constant function.

105. Let f be a continuous function on the interval $[0,1]$ which is differentiable at every point of $(0,1)$. Assume that $|f'(x)| \leq 33$ for every $x \in (0,1)$. Show that

$$|f(x) - f(y)| \leq 33|x-y|; \qquad x, y \in [0,1].$$

106. Consider the two power series:

$$P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots.$$

$$Q(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \cdots.$$

Show that for any $x \in R$, the first series converges iff the second series converges. Conclude that both have the same radius of convergence. Thus

$$\limsup \sqrt[n]{|\alpha_n|} = \limsup \sqrt[n]{|\alpha_{n+1}|}.$$

Let (a_n) and (b_n) be sequences of positive numbers. Suppose that the sequence $\{a_n\}$ converges to a finite non-zero number a. Then $\limsup a_n b_n = a \limsup b_n$.
Use this to show that the following power series also has the same radius of convergence as the earlier two.

$$T(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + \cdots.$$

107. Let $f(x) = \sin(1/x)$ defined for $x \neq 0$. Can you prescribe a number $f(0)$ so that f is continuous on R?

Let $f(x) = x\sin(1/x)$ defined for $x \neq 0$. Can you prescribe a number $f(0)$ so that f is continuous on R. Then will your function be differentiable at zero?

Let $f(x) = x^2\sin(1/x)$ defined for $x \neq 0$. Can you prescribe a value $f(0)$ so that f is continuous on R. Then will your function be differentiable at zero? Is the functuion $f'(x)$ continuous at zero?

Answer the same questions if $f(x) = x^3 \sin(1/x)$.

This story can go on — why not higher powers? why only integer powers of x? why not powers for $1/x$ too within the sine function?

108. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

Show f is differentiable function on R.

More generally, Let P be a polynomial in one variable.

$$f(x) = \begin{cases} P(1/x)\, e^{-1/x^2} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

Show f is differentiable function on R.

109. Since you are learning linear algebra, here is an interesting formula. suppose I have nine differentiable functions $\{f_{ij}; 1 \leq i, j \leq 3\}$ on R. Define a function φ on R by

$$\varphi(x) = \begin{vmatrix} f_{11}(x) & f_{12}(x) & f_{13}(x) \\ f_{21}(x) & f_{22}(x) & f_{23}(x) \\ f_{31}(x) & f_{32}(x) & f_{33}(x) \end{vmatrix}$$

Here $|A|$ is determinant of A. Show that φ is differentiable and

$$\varphi' = \begin{vmatrix} f'_{11} & f'_{12} & f'_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f'_{21} & f'_{22} & f'_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f'_{31} & f'_{32} & f'_{33} \end{vmatrix}$$

3073113133317331337347349353359367373379383389397

Some exercises below are computational, you need to think how large or how small are things. Some are theoretical, but they are easy. If you open your pen without understanding the problem or try random paths blindly, you will get frustrated. Please do not do so.

110. Find derivatives of the following functions.

$$e^{(x^{33})}. \qquad (e^x)^{33}. \qquad \sin(\cos x). \qquad \frac{e^{\cos x}}{1+x^2}.$$

111. Let $f(x) = -x\log x - (1-x)\log(1-x)$ $(0 < x < 1)$. Is it possible to define this function at the points zero and one so that it is continuous on the closed interval $[0,1]$? Where is the function increasing, where is it decreasing, Where is its maximum value, what is it? Sketch its graph.

112. Prove the following formula, called Leibnitz's formula.

If f and g are each differentiable n times, then so is the product fg and

$$(fg)^{(n)} = f^{(n)} + \binom{n}{1} f^{(n-1)}g^{(1)} + \binom{n}{2} f^{(n-2)}g^{(2)} + \cdots + g^{(n)}.$$

113. For any real number α let us define

$$\binom{\alpha}{0} = 1; \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\times 2\times 3\times\cdots\times n}.$$

If α is a positive integer, does this agree with what you know from high school?

Eventhough there is a formula for radius of convergence of power series, many times it is convenient to use ratio test. Let us fix any $\alpha \neq 0$. Consider the power series

$$\sum_0^\infty \binom{\alpha}{n} x^n.$$

If $\alpha \geq 1$ is an integer, this is actually a finite sum and so has radius of convergence ∞ and equals $(1+x)^n$. Prove it.

If α is not an positive integer, show that its radius of convergence is one. Show that for any $|x| < 1$, sum of the above series is $(1 + x)^\alpha$. Ask Taylor for help.

Here are special cases.

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{1 \times 3}{2!}\left(\frac{x}{2}\right)^2 + \frac{1 \times 3 \times 5}{3!}\left(\frac{x}{2}\right)^3 + \cdots$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{1 \times 3}{2!}\frac{x^4}{2^2} + \frac{1 \times 3 \times 5}{3!}\frac{x^6}{2^3} + \frac{1 \times 3 \times 5 \times 7}{4!}\frac{x^8}{2^4} + \cdots.$$

114. Find the radius of convergence of the following power series (ratio test is better than the formula). All are easy, serve to test if you made friends with numbers.

$$\sum \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}\, x^n$$

$$\sum \frac{(2n)!}{n!n!}\, x^n. \qquad \sum \frac{(n+1)! - n!}{2^n}\, x^n. \qquad \sum \frac{(n!)^2}{(n^2)!}\, x^n. \qquad \sum \frac{n!}{n^n}\, x^n.$$

$$\sum \frac{e^{n^2}}{n!}\, x^n. \qquad \sum \frac{n^3}{e^n}\, x^n. \qquad \sum \frac{e^n}{n!}\, x^n. \qquad \sum \frac{2^n}{(n!)^{\sqrt{2}}}\, x^n.$$

115. Evaluate
$\lim_{x \to 0^+} x^{1/3}(\log x)^3. \qquad \lim_{x \to 0^+} x(\log x)^2. \qquad \lim_{x \to 0}(\sin x)^x.$
Taking help of L'Hopital, evaluate
$\lim_{x \to 0^+} \frac{\log \sin x}{\log(e^x - 1)}. \qquad \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right).$

116. You know that a sequence of real numbers converges iff it is a Cauchy sequence. Similar result is true for uniform convergence of functions too. Let (f_n) be a seqence of functions on a set S. Say that the sequence is uniformly Cauchy if the following holds: given $\epsilon > 0$, there is n_0 such that $|f_n(x) - f_m(x)| < \epsilon$ for every $n, m \geq n_0$ and every $x \in S$.

Show that (f_n) converges uniformly on S to some function iff it is uniformly Cauchy sequence.

117. There are several important ideas in proof of the main theorem on power series.

Suppose, I have a series of functions $\sum f_n$ on a set S. Suppose there are numbers M_n such that $\sum M_n$ converges and for each n, the function

f_n is bounded by M_n, That is, $|f_n(x)| \le M_n$ for every n and every $x \in S$. Show that the series $\sum f_n$ converges uniformly. This is called Weierstrass M-test.

Suppose I have a sequence (f_n) of differentiable functions on $(-30, +30)$ and $f_n(0) \to 17$. Suppose that $f'_n \to g$ uniformly on $(-30, +30)$. Then show that the sequence $\{f_n\}$ converges uniformly to a function f; f is continuous, f is differentiable; $f' = g$.

118. Show that if

$$f(x) = \sum_1^\infty \frac{\cos nx}{n^{7/3}}$$

then

$$f'(x) = -\sum \frac{\sin nx}{n^{4/3}}.$$

119. find if the following sequences/series converge unformly on the intervals menioned.

(a) Sequence (f_n) where $f_n(x) = \dfrac{n^2 x}{1 + n^3 x^2}$, $[-1, 1]$? $[0, \infty)$?

(b) $\sum_1^\infty (x \log x)^n$ on $(0, 1)$.

(c) $\sum n^{77} e^{-nx}$ on $(0.001, \infty)$? on $(0, \infty)$?.

(d) The series $\sum_1^\infty [1 - \cos(x/n)]$ converges uniformly on the interval $[-10^{10}, +10^{10}]$ but does not converge uniformly on R.

120. Similar to series which are made up of powers of x, there are important class of series which are made up of powers of n.

Let $(a_n : n \ge 1)$ be real numbers. A series of the form $\sum \dfrac{a_n}{n^x}$ is called Dirichlet series. We shall not complicate our life and be content with a simple instance.

Show that the series, $f(x) = \sum \frac{1}{n^x}$, converges uniformly on the interval $[1 + \epsilon, \infty)$, whatever be $\epsilon > 0$.

Show that the series $g(x) = -\sum \frac{\log n}{n^x}$ also converges uniformly on the interval $[1 + \epsilon, \infty)$ for every $\epsilon > 0$.

Argue that f is a continuous function on $(1, \infty)$; indeed f is differentiable on $(0, \infty)$; indeed $f' = g$.

121. A man in a boat on one shore of a river wishes to reach a point on the other shore that is 4 miles down the stream. The river is 2 miles wide and has negligible current. He can row at 4 mph and run along the opposite bank at 8 mph. If he rows along a straight line path to a point on the opposite bank and then runs along the opposite bank to his destination, where should he land to minimize time taken to reach destination.

122. A window in the form of a rectangle surmounted by an isosceles triangle (with top side of the rectangle as base), the altitude of the triangle being (3/8)-th of its base. If the perimeter of the window is 30 ft, find the dimensions of the window for admiting maximum light.

123. A wire bent in the form of a circle of radius a exerts an attractive force upon a particle on the axis of the circle (that is on the line through center and perpendicular to the plane of the circle). From the theory of attraction (let us believe it), this force is proportional to $h/(a^2 + h^2)^{3/2}$ where h is the height of the particle above the plane of the circle. At what height is the attraction maximum?

{Birkbeck college once announced an evening lecture by John Buchan, with the title 'Margins of life' and I expected the speaker to talk about some scientific field, perhaps about viruses or very large molecules on the border line between inorganic matter and living organisms. Not a bit: what he spoke about, to my growing astonishment, was the importance for a student not to work too hard! A student should not devote his entire time to the study of his subject; he should leave a margin on which he could scribble notes on what went around him. I was quite amazed that such advice should be regarded as necessary; I felt that students were generally a scatter-brained lot and in my view ought to be encouraged to stick to their books. But the lecturer obviously thought that the opposite advice was necessary to prevent them from becoming narrow-minded. (Otto Frisch)}

The problems in this set are of two kinds — Profound looking statements which are trivial; not too complicated applications of the machinery we developed.

124. Show

$$\sum_1^\infty \frac{1}{(n+x)(n+x+1)(n+x+2)} = \frac{1}{2(x+1)(x+2)}.$$

Here x is a number which is not negative integer.

$$\sum_2^\infty \frac{\log[(1+\frac{1}{n})^n(1+n)]}{\log n^n \log(n+1)^{n+1}} = -\log 4.$$

$$\sum_2^\infty \frac{1}{n^2-1} = \frac{3}{4}; \qquad \sum \frac{n^2x^n}{n!} = (x^2+x)e^x.$$

$$\sum_1^\infty nx^n = \frac{x}{(1-x)^2}; \qquad \sum (n+1)x^n = \frac{1}{(1-x)^2}. \quad |x|<1.$$

$$\text{Simplify } \sum \frac{(n-1)(n+1)}{n!}.$$

125. Test the following series for convergence.

$$\sum \frac{\log n}{n\sqrt{n+1}}; \qquad \sum \frac{n!}{(n+2)!}; \qquad \sum \frac{1}{(\log n)^{1000}}; \qquad \sum \frac{1+\sqrt{n}}{(n+1)^3-1}.$$

$$\sum \frac{\sin(1/n)}{n}; \qquad \sum [1-n\sin(1/n)]; \qquad \sum \log(n\sin(1/n)).$$

$\sum a_n$ where a_n equals $1/n$ if n is a square and equals $1/n^2$ if n is not a square.

$\sum a_n$ where a_n equals $1/n^2$ if n is odd and $-1/n$ if n is even.

126. if $\sum a_n$ is a convergent series of positive terms, show that the series $\sum \sqrt{a_n a_{n+1}}$ converges.

127. If the series $\sum a_n$ is absolutely convergent, show that the following series are also convergent. For the second eeries below we assume that all the a's are different from -1.

$$\sum a_n^2; \qquad \sum \frac{a_n}{1+a_n}; \qquad \sum \frac{a_n^2}{1+a_n^2}.$$

128. Evaluate the following limits using L'Hopital when needed.

$$\lim_{x\to 0} \frac{\log(\cos ax)}{\log(\cos bx)}. \qquad \lim_{x\to 1} \frac{\sum_1^n x^k - n}{x-1}. \qquad \lim_{x\to 0+} \frac{x-\sin x}{(x\sin x)^{3/2}}.$$

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}. \qquad \lim_{x\to 1+} \frac{x^x - x}{1-x+\log x} \qquad \lim_{x\to\infty} x^{1/x}.$$

129. Taking the help of Taylor, when needed (sometimes you do not need), prove the following.

(i) For $|x| < 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

$$\log\frac{1}{1-x} = \sum \frac{x^n}{n}.$$

$$\frac{1}{2}\log\frac{1+x}{1-x} = \sum_1^\infty \frac{x^{2n-1}}{2n-1}.$$

$$(1+x)\log(1+x) = x + \sum_2^\infty (-1)^n \frac{x^n}{n(n-1)}.$$

$$\log(x+\sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \cdots$$
$$+(-1)^{n+1}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!}\frac{x^{2n+1}}{2n+1} + \cdots.$$

(ii) For $x \in R$,

$$(1+x)e^{-x} = 1 + \sum_2^\infty (-1)^{n-1}\frac{n-1}{n!}x^n$$

$$\frac{e^x-1}{x}=1+\frac{x}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{n!}+\cdots$$

$$\sin^2 x=\frac{2x^2}{2!}-\frac{2^3x^4}{4!}+\frac{2^5x^6}{6!}-\cdots+(-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\cdots.$$

$$\cos^2 x=1+\frac{1}{2}\sum_1^\infty(-1)^n\frac{(2x)^{2n}}{(2n)!}.$$

$$e^x\sin x=x+x^2+\frac{2x^3}{3!}-\frac{4x^5}{5!}+\cdots$$

I do not think the general term is $\pm(n-1)x^n/n!$, have not checked.

130. Starting from $z_0=2$ use newton's approximation for finding $\sqrt{2}$. Calculate the first ten approximations using acalculator. Square them and see.

Do the same thing for $\sqrt[3]{2}$ starting from 2 again.

131. We defined uniform convergence of a sequence of functions, uniform continuity of a function. Here is another 'uniformity' phenomenon.

Let f be a function differentiable on $[0,1]$. Thus at zero and one the left and right derivatives exist Asssume that f' is continuous. Show the following. Given $\epsilon>0$, there is a $\delta>0$ such that

$$x,a\in[0,1];\ x\neq a;\ |x-a|<\delta\ \Rightarrow\left|\frac{f(x)-f(a)}{x-a}-f'(a)\right|<\epsilon.$$

This verifies the ratio $[f(x)-f(a)]/[x-a]$ converges to $f'(a)$. However given $\epsilon>0$, we can find δ that works for all point intervals in this interval.

132. Is the function $f(x)=\sin(1/x)$ uniformly continuous on the interval $(0,1)$. What about the function $g(x)=x\sin(1/x)$ on the same interval $(0,1)$. Is the function $f(x)=\sin(1/x)$ uniformly continuous on the interval $(1,\infty)$?

133. This exercise is not difficult. However, if you do not want to do, it is fine.

(i) For $0<a\leq 1$ put

$$\zeta(s,a)=\sum_0^\infty(n+a)^{-s}.$$

Show that the series converges absolutely for $s > 1$. the function $\zeta(s) = \zeta(s, 1)$ is the Riemann zeta function. Prove

$$\sum_{h=1}^{k} \zeta\left(s, \frac{h}{k}\right) = k^s \zeta(s).$$

Prove

$$\sum_{1}^{\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s})\zeta(s); \qquad s > 1.$$

(ii) If $\sum a_n$ is a convergent series of positive numbers show that the series $\sum \sqrt{a_n}\, n^{-p}$ converges for $p > 1/2$.

(iii) Just like Cauchy product of power series, we can talk about product of Dirichlet series. Suppose that

$$A(s) = \sum_{1}^{\infty} \frac{a_n}{n^s}; \qquad B(s) = \sum_{1}^{\infty} \frac{b_n}{n^s}$$

are two absolutely convergent Dirichlet series. Show that

$$\sum_{1}^{\infty} \frac{c_n}{n^s} = A(s)B(s); \qquad c_n = \sum_{d|n} a_d b_{n|d}.$$

(iv) Show that for the Riemann zeta function $\zeta(s)$ satisfies,

$$\zeta^2(s) = \sum_{1}^{\infty} \frac{d(n)}{n^s},$$

where $d(n)$ is the number of divisors of n — including 1 and n.