discussion of HA2.

Q7: Expansions to base other than 10 (to which we are used to) are also important. The basic idea is same as for the decimal expansion. For example, here is how you get binary expansion. If $x = 0$ take all digits zero. Let us consider x, $0 < x \leq 1$.

Divide the interval $(0, 1]$ into two parts: $(0, 1/2]$ and $(1/2, 1]$. If x is in the first part (left) declare $\epsilon_1 = 0$ and if it is in the second part (right) declare it to be 1. Observe $|x - \epsilon/2| \leq 1/2$. Divide each of these two first level intervals into two halves. Declare $\epsilon_2 = 0$ or 1 according as the point x is in the left or right second level intervals within the first level interval. Observe $|x - \epsilon_1/2 - \epsilon_2/2^2| \leq 1/2^2$. continue and complete proof.

Again the division points are esentially the ones having more than one expansion.

Q9: To show that a non-empty set bounded below has greatest lower bound.

Take all lower bounds of S, denote this set by T.

Is T non-empty? Yes, by hypothesis there are lower bounds for S.

Is T bounded above? Yes, any point $x \in S$ is an upper bound, because if y is any lower bound of S then $y \leq x$ for every point x of S.

Let z be least upper bound of T.

Is z a lower bound of S? Yes, if there is a $x \in S$ such that $x < z$, then every poiny of S being an upper bound of T, we see z can not be least upper bound of T, a contradiction.

Is z greatest lower bound of S? Yes, If there is a lower bound y of S with $z < y$ then $y \in T$ and z is not an upper bound of T, a contradiction.

Q14: Given s is lub of A and t is lub of B, to show $s + t$ is the lub of $C = \{x + y : x \in A, y \in B\}$.

Is $s + t$ an upper bound of C? Yes, if $x \in A$ and $y \in B$, then $x \leq s$ and $y \leq t$ so that $x + y \leq s + t$.

Let $\epsilon > 0$. Are there points of C above $s + t - \epsilon$? (we are using the criterion already derived for lub). Yes, because there is $x \in A$ with $x > s - \epsilon/2$ and $y \in B$ with $y > t - \epsilon/2$. This $x + y \in C$ will do.

Q13: We now have $C = \{xy : x \in A; y \in B\}$. Of course we were told A and B have only positive numbers.

If A or B consists of only one point, namely, $\{0\}$ then easy to see that C also consists of one point $\{0\}$ and so the result is true. We assume that both A and B have strictly positive numbers. Hence $s > 0$ and $t > 0$.

Is st an upper bound of C? Yes everything being positive $x \leq s$ and $y \leq t$ implies that $xy \leq st$.

Let $\epsilon > 0$, Are there points of C above $st - \epsilon$. (we are using the criterion for lub derived already). We can safely assume that $st - \epsilon > 0$, otherwise any non-zero point of C would do. Since we are in the 'multiplicative set up' (an intuitive expression, just to motivate what we are doing), we shall write $st - \epsilon$ as αst with $0 < \alpha < 1$. We are just saying take $\alpha = (st - \epsilon)/st$. Express $\alpha = \alpha_1 \alpha_2$ with $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$. Can we do this? Yes. If you wish take $\sqrt{\alpha}$ as both α_1 and α_2. (Did you realize we are imitating the $\epsilon/2$ argument of the previous 'additive set up'?). Of course, you can take any α_2 with $0 < \alpha < \alpha_2 < 1$ and put $\alpha_1 = \alpha/\alpha_2$. Since $\alpha_1 s < s$ and $\alpha_2 t < t$ get $x \in A$ and $y \in B$ with $x > \alpha_1 s$ and $y > \alpha_2 t$ so that $xy > \alpha st$.

Is this result true without assuming that we have sets of positive numbers? Not necessarily, take $A = B = [-1, 0]$.

set operations.

In the last class we talked about cardinality of sets, countable union of sets etc. Since some of you are not sure about set theoretical jargon, let us briefly digress a little bit and recall certain definitions and notations.

As I said in the first class, we shall not define what is a set. We still proceed with our understanding that a set is a well-defined collection of objects. Given something, we should be able to say whether it is in the set or not. If we can not tell, then we do not know what our set is. In that case any discussion of such a set does not make sense (essentially because, we would not know what we are discussing about!)

I have used too much English, but do not worry. We can, if challenged, precisely define what a set is. But we decided not to do it now.

Suppose A is a set. To express that an object x is in the set A, we use

$x \in A$. To say x is not in the set A we use $x \notin A$.

Suppose A and B are two sets. We say $A \subset B$ (read: A is a subset of B, or A is contained in B) in case the following happens: whenever an object is in A, then it is in B also. In symbols $x \in A$ implies $x \in B$. Sometimes the same thing is expressed by writing $B \supset A$ (read: B is superset of A or B contains A or B includes A). For example

$$\frac{1}{2} \in (0,1); \quad 2 \notin (0,1); \quad (0,1) \subset [0,1]; \quad [0,1] \supset (0,1).$$

Suppose A and B are two sets. Then their union is the collection of all objects which either belong to the set A or objects which belong to the set B. This union includes all objects which are in both the sets, because such an object satisfies both the clauses. In symbols

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

The intersection of the two sets, is the collection of those objects which belong to both the sets. In symbols

$$A \cap B = \{x : x \in A \text{ and } x \in B.\}$$

The concepts of union and intersection can be defined for any collection of sets, not necessarily two. Suppose I have a collection of sets. Then their union is the set of all objects which belong to *at least one* of the sets in the collection. The intersection of the collection of sets consists of those objects which belong to *all* sets in the collection.

If we have a sequence of sets $A_1, A_2, \cdots$, one set for each $n \in N$, we denote their union by $A_1 \cup A_2 \cup A_3 \cup \cdots$ or $\cup A_n$ or $\bigcup\limits_{n \geq 1} A_n$. Similar notation is used for intersection; $A_1 \cap A_2 \cap A_3 \cap \cdots$, or $\cap A_n$ or $\bigcap\limits_{n \geq 1} A_n$.

Real Numbers.

Some of you may still not be feeling comfortable with our real numbers. After all, we are used to thinking real numbers $3, 355/576, \sqrt{29}$ as 'real' things (with flesh and blood, as if). But now I said: take a set R with the operations of addition $+$, multiplication $\cdot$, and comparision $<$, satisfying the rules we prescribed; and elements of the set are called real numbers. You might feel some sheen has disappeared. Before giving further explanation, let me remind you the following. We have immediately observed that whatever we have so far done (or are doing) with numbers are all correct with our present rules and so we can continue doing without any hesitation. The geometric picture that we have in mind representing numbers as points on a line is still correct and we continue to think of numbers placed on a line. There is absolutely no cause for alarm.

To re-inforce these things, we have also agreed to denote the multiplicative identity by the usual 1; $1 + 1$ by the usual 2, $1 + 1 + 1$ by the usual 3 and so on; the same things we have been using all along. The number which when multiplied by 5 gives 1, is still denoted by $1/5$. We did not bring any unknown notation or novel objects. Thus keep away any fears you may have.

Then why did we do this? The answer lies in the first lecture. We want to understand what we are talking about. It is important to have a clear (at least, as clear as possible) idea of real numbers. Let us not enter into philosophy, but just pause for a moment to think. Afterall, you see green coloured shirt, green coloured sari, green coloured leaf and from these you abstract (without your knowledge) the concept 'green colour' and use this word. Really speaking, you never see green colour in practice (closest to seeing green colour is experiments with prism). Again by seeing green colour, red colour, yellow colour etc you make a further abstraction and invent the concept of colour. You never see something and say 'yes, this is colour'. In the same way 'three' is an abstraction. You can have three apples, you can have three goats, but you never have just three. Afterall, what is 'three'; think about it; some write it as 3 and some as III; there are still other ways of writing it.

In a sense, we have laid down the ground rules, that is all, no more. We have made clear to ourselves, what can be assumed and what can not be. Did not our school teachers lay down the rules? Whatever they taught us is completely correct, there is nothing wrong. They did an excellent job. We have been given working knowledge, we have been working with numbers without making mistakes. Did we understand fully why certain things work. Let me give an example.

One of the simplest things we learnt is addition. To add 7 and 8 we did the following: kept eight fingers in mind, opened seven fingers, Thus we have

7 and 8. we need to put them together to add. So started counting after the eight fingers $9, 10, \cdots, 15$. Suppose you wanted to add 67 and 58. Then how do we go? Strictly speaking, we should bring 67 things, say beads, and 58 beads; put them together and count the total. But some one was smart and found out that there is a simpler way. You can carry out this procedure by adding *just* single digits at a time. Thus you add $7 + 8 = 15$, put down 5 and say 1 is *carried over*. Then you add digits at the next place $6 + 5 = 11$ and now add the carry over to this to get $11 + 1 = 12$. Put it down getting the answer 125. Did we ever ask: what is this carry over? why does this method give correct answer? It was not expected at such an immature level. Of course, we are now mature and should ask. Should we not? Similarly, if we want to multiply 67 by 58 we should bring 58 bundles of beads each bundle having 67 beads; put all of them together and count. But some smart fellow found a better way, is it not?

When you learn matrices, you find that sometimes $xy = yx$ is false! But this is one of our rules for numbers. As you learn more, you will find more and more funny things.

sequences.

After understanding numbers, there are two paths one can take. Some enter the path of continuous functions, differentiable functions; and integration and then turn to sequences and series of numbers. Others enter the path of sequences, series and then turn to functions. Both paths are fine, but I prefer the second path. This is because, I believe that sequences are simpler than functions. There is something in support of the first path too. In school, you probably never used the word sequence; but you definitely used the word function. You are familiar with sine, cosine, exponential functions. So perhaps taking the first path may appear as continuation of the high school course.

I consider 'sequences' as still understanding numbers. Let us consider real numbers. How many can we see? Well, all rational numbers 4/5 or 889/101 etc. Can we name others? Yes, for example you can show that exactly one positive number satisfies $x^2 - 2 = 0$ and you can say consider *that* number. Or you can show that there are exactly two positive numbers satisfy the equation $x^4 - 5x^2 + 6 = 0$ and you can say consider *larger of the two positive* numbers that satisfy this equation. In other words, you can name algebraic numbers.

If you are allowed geometric constructions, you can show that for any circle, the ratio of circumference to diameter, does not depend on the circle and you can ask me to consider that ratio. But at this moment let us not talk about geometric constructions.

Thus, there are very few numbers which we can name. How do we name, understand, and work with other numbers. How do we discover numbers? Afterall, there are uncountably many numbers which are not algebraic. We follow the usual procedure, an unknown thing is described in terms of known things. We show that every number has a sequence of rational numbers converging to it (and only to it), thus explaining that number in terms of rationals. Of course, such a description is too general to be of value. Better descriptions give better understanding. We shall also use sequences to discover new numbers.

What is a sequence.

Definition: A sequence is a function $f : N = \{1, 2, \cdots\} \to R$. We are talking only about sequences of real numbers. Later (when we are more mature) we talk about sequences of functions etc.

Since it is a function defined on a very simple set, one uses a better notation than the abstract symbol f. Let us denote the value $f(n)$ by x_n. The sequenec is denoted as (x_n) or $\{x_n\}$ or $(x_n : n \geq 1)$ etc. The meaning is that it is the function whose value at n is the number x_n. In other words, we specify the value at n, namely, x_n call it the n-th term of the sequence. Here are some examples:

$$x_1 = 1, x_2 = 1/2, x_3 = 1/3, \cdots, x_n = 1/n \quad \text{or} \quad \{\frac{1}{n}\}.$$

$$x_1 = -1, x_2 = 1/2, x_3 = -1/3, \cdots, x_n = (-1)^n 1/n \quad \text{or} \quad \{(-1)^n \frac{1}{n}\}.$$

$$x_1 = -1, x_2 = +1, x_3 = -1, \cdots x_n = (-1)^n \quad \text{or} \quad \{(-1)^n\}.$$

$$x_1 = (1/2), x_2 = (1/2)^2, x_3 = (1/2)^3, \cdots x_n = (1/2)^n \quad \text{or} \quad \{\frac{1}{2^n}\}.$$

Remember the word sequence has within it an implicit order on the terms. You have to distinguish it from the set which you may think of by putting all the values x_n together. For example, for the first sequence above, if you make a set by putting all the terms you get the set $\{1, 1/2, 1/3, \cdots\}$ but there is no implicit order on the members of this set. Just because you have written in a particular order, you should not be under the illusion that there is an order and that everyone would write in the same order. A set is simply a collection of objects and there is no a priori order for its elements. If you consider the third sequence above, the set consists of just two points $\{\pm 1\}$. The first sequence above is different from the sequence

$$x_1 = 1/2, x_2 = 1, x_n = 1/n \quad \text{for } n \geq 3.$$

There is something interesting with the first sequence. If you plot them on the number line, then the numbers of the sequence are getting closer and closer to zero. So is the second sequence. Sometimes it is to the left of zero and sometimes it is to the right of zero. But no matter which side it is, the numbers are getting again closer and closer to zero. On the other hand, the third sequence does not get closer to any number what-so-ever. We shall make this concept of 'getting closer to something' precise and give a name to it.

Definition: A sequence (x_n) is said to converge/ approach/tend to a number x if given any $\epsilon > 0$ there exists an integer n_0 such that $|x_n - x| < \epsilon$ for all $n \geq n_0$. That is, $x - \epsilon < x_n < x + \epsilon$ for all $n \geq n_0$. We write $x_n \to x$.

What this means is the following: No matter what the amount of error prescribed is, the terms of the sequence are for ever close to x after some stage, close meaning within the error prescribed. If the error prescribed is $\epsilon > 0$ the terms are within $x - \epsilon$ to $x + \epsilon$. The only thing to note is that we do not allow ϵ to be zero.

if you do not like error etc, you can think of $x - \epsilon$ and $x + \epsilon$ as walls built around x. Thus no matter what walls are erected around the point x, the sequence stays within the two walls eventually. Of course, you may say we have symmetrically placed walls, at $x - \epsilon$ and $x + \epsilon$. It does not matter, suppose that you give me any walls, say $x - \delta_1$ and $x + \delta_2$ around x, where $\delta_1, \delta_2 > 0$. We take $\epsilon = \min\{\delta_1, \delta_2\}$ and apply the definition to get an n_0 and observe that after the stage n_0 the sequence stays within the given walls.

Definition: We say that a sequence (x_n) is convergent if there is some number x such that $x_n \to x$.

Let us first show that a sequence can not converge to two different limits.

Fact: *Suppose that $x_n \to x$ and $x_n \to y$. Then $x = y$.*

Proof: if possible let $x < y$. Then take $\epsilon = (y - x)/2 > 0$. Then both the intervals $(x - \epsilon, x + \epsilon)$ and $(y - \epsilon, y + \epsilon)$ should contain the sequence eventually. But the intervals are disjoint and this can not happen.

Examples:
The sequence $(1/n)$ converges to zero.

In fact given any $\epsilon > 0$, we already knew that there is an integer n_0 such that $n_0 > 1/\epsilon$, that is, $1/n_0 < \epsilon$. But then $-\epsilon < 1/n < \epsilon$ for all $n \geq n_0$.

Fact: *Fix number $0 < a < 1$. The sequence (a^n) converges to zero.*
In particular. the sequence $(1/2^n)$ converges to zero.

Let $1/a = 1 + h$ where $h > 0$. Remember, $(1 + h)^n \geq nh$, by binomial expansion, so that

$$a^n = (\frac{1}{1+h})^n \leq \frac{1}{n}\frac{1}{h}.$$

Given $\epsilon > 0$ we can choose n_0 so that $1/n < \epsilon h$ for all $n \geq n_0$. For this n_0 we see $a^n < \epsilon$ for all $n \geq n_0$.

Sometimes careful observation of the proof gives better results.
If $0 < a < 1$, the sequence (na^n) converges to zero.
Use exactly same steps as above. But say $(1 + h)^n \geq \binom{n}{2} h^2$

$$na^n = n(\frac{1}{1+h})^n \leq \frac{n}{\binom{n}{2}h^2} = \frac{1}{n-1}\frac{2}{h^2}$$

Given $\epsilon > 0$, choose n_0 so that $1/(n_0 - 1) < \epsilon h^2/2$. this will do.

If $a > 1$, then the sequence $(\sqrt[n]{a})$ converges to 1.

Since $\sqrt[n]{a} > 1$, let $\sqrt[n]{a} = 1 + h_n$ where $h_n > 0$. Thus

$$a = (1 + h_n)^n \geq n h_n; \quad h_n \leq \frac{a}{n} \to 0.$$

Thus $\sqrt[n]{a} = 1 + h_n \to 1$. We have used the simple fact that $1 + h_n \to 1$ using that $h_n \to 0$.

When $a = 1$, clearly $\sqrt[n]{a} = 1$ and hence converges to 1. *This happens also for* $0 < a < 1$. This follows by a rseult we prove soon.

Example: $\sqrt[n]{n} \to 1$.

This follows from careful observation of the earlier proof. Let $\sqrt[n]{n} = 1 + h_n$ where $0 < h_n < 1$ for $n > 1$. To see that $h_n < 1$ just note that $2^n > n$. Thus for $n \geq 2$ we have

$$n = (1 + h_n)^n \geq \binom{n}{2} h_n^2; \quad h_n^2 \leq \frac{n}{\binom{n}{2}} = \frac{2}{n-1} \to 0.$$

Thus $h_n^2 \to 0$. From this we can deduce that $h_n \to 0$ as follows. Let $\epsilon > 0$ Choose n_0 so that $h_n^2 \leq \epsilon^2$ for $n \geq n_0$. Clearly then, $0 < h_n < \epsilon$ for $n \geq n_0$.

Definition: We say that a sequence (x_n) is increasing if $x_1 \leq x_2 \leq x_3 \leq \cdots$. Of course, if you want to avoid using dots, then you can rephrase to say: $x_n \leq x_{n+1}$ for every $n \geq 1$.

We say that a sequence (x_n) is decreasing if $x_1 \geq x_2 \geq x_3 \geq \cdots$. Of course, if you want to avoid using dots, then you can rephrase to say: $x_n \geq x_{n+1}$ for every $n \geq 1$.

A sequence is monotone if it is either increasing or decreasing sequence. Note that in the definition of increasing sequence we did not use strict inequalities. Thus a constant sequence is both increasing and decreasing. In fact, if a sequence (x_n) is both increasing and decreasing, then there is a number a such that $x_n = a$ for every $n \geq 1$.

Definition: A sequence is bounded above if the set consisting of terms of the sequence is bounded above. Similarly, we can define the concept of bounded below and bounded.

The sequence $x_n = 1/n$ is a decreasing sequence. The sequence $y_n = (-1)^n$ is not monotone. Both sequences are bounded. Note that for the sequence (y_n) the set consisting of its terms has only two members in it, namely $+1$ and -1. If $z_n = 1/n$ when n is odd, $z_n = -n$ when n is even; then the sequence (z_n) is bounded above but not bounded below. Here is an important fact.

Fact:*Every increasing sequence which is bounded above is convergent, in fact, it converges to its supremum.*

Proof: Let (x_n) be an increasing sequence and s be its supremum. Let $\epsilon > 0$. We show n_0 such that $x_n \in (s-\epsilon, s+\epsilon)$ for all $n \geq n_0$. This is easy. Since $s - \epsilon$ is not an upper bound for the sequence, there is a term of the sequence, say $x_{n_0} > s - \epsilon$. The sequence being increasing we conclude that $x_n > s - \epsilon$ for all $n \geq n_0$. Of course, $x_n \leq s < s + \epsilon$ for all n.

Fact:*Every decreasing sequence which is bounded below is convergent, in fact, it converges to its infimum.*

Example: Let

$$x_1 = 1;\ x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}, \quad n \geq 1.$$

Then the sequence (x_n) is increasing and bounded above and hence converges.

$$x_{n+1} = x_n + \frac{1}{n!}$$

shows that the sequence is increasing. Since $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \geq 2^{n-1}$, we see using sum of finite geometric series,

$$x_n = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-2}} \leq 3.$$

The limit of this sequence is denoted by e.

Example: Let

$$y_n = (1 + \frac{1}{n})^n.$$

Then the sequence (y_n) is increasing and bounded above and hence converges.

Proof: Using binomial theorem

$$y_n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + +\frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots$$

$$+\frac{1}{n!}(1 - \frac{1}{n}) \cdots (1 - \frac{n-1}{n}).$$

Observe that y_{n+1} is obtained by replacing n by $n+1$. In particular, we see that y_{n+1} has one extra positive term than y_n. Moreover each of the preceding terms are more than the corresponding terms of y_n. Thus the sequence is increasing. We also see that $y_n \leq x_{n+1}$ and since the (x_n) sequence is bounded above, we conclude that the same bound shows that (y_n) sequence is bounded above too.

Soon we shall see that the limit of this sequence is also same as the earlier number e.

First let us ask if the notion of convergence is friendly with the operations we have. For example if two sequences converge, would their 'sum sequence' converge to the expected number? Yes.

Fact: *Suppose $x_n \to x$ and $y_n \to y$.*
Then $x_n + y_n \to x + y$; $-x_n \to -x$; $x_n - y_n \to x - y$
Heere by the sequence $(x_n + y_n)$ we mean the obvious thing, namely, the sequence whose n-th term is $x_n + y_n$.

Proof: Let $\epsilon > 0$. We show n_0 such that $|(x_n + y_n) - (x + y)| < \epsilon$ for all $n \geq n_0$. To do this, get m such that $|x_n - x| < \epsilon/2$ for $n \geq m$ and get k such that $|y_n - y| < \epsilon/2$ for $n \geq k$. Take $n_0 = \max\{m, k\}$. If $n \geq n_0$, then

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $|(-x_n) - (-x)| = |x_n - x|$ second statement is clear. Either you can repeat the above proof for $x_n - y_n$ or use the first two statements toi conclude the third statement.

The concept of convergence is friendly with multiplication also. First we need to observe an auxiliary fact of independent interest.

Fact: *A convergent sequence is bounded.*

Proof: Let $x_n \to x$. Taking $\epsilon = 1$, first get n_0 so that $|x_n - x| \leq 1$ for $n \geq n_0$. Thus we have $|x_n| \leq |x_n - x| + |x| \leq |x| + 1$ Thus the number

$$M = \max\{|x|_1, |x|_2, \cdots, |x|_{n_0}, |x| + 1\}$$

will show boundedness of the sequence.

Fact: *Suppose $x_n \to x$ and $y_n \to x$. Then $x_n y_n \to xy$.*
If each $x_n \neq 0$ and $x \neq 0$, then $1/x_n \to 1/x$.
As earlier $(x_n y_n)$ is the sequence whose n-th term is the product $x_n y_n$. Similarly $(1/x_n)$.

Proof: Let $\epsilon > 0$. We show n_0 such that for $|x_n y_n - xy|\epsilon$ for $n \geq n_0$. Furst observe that

$$|x_n y_n - xy| \leq |x_n y_n - x_n y + x_n y - xy| \leq |x_n||y_n - y| + |y||x_n - x|.$$

Since the convergent sequence (x_n) is bounded, fix M such that $|x_n| \leq M$ for all n. Choose m and k so that

$$|y_n - y| \leq \frac{\epsilon}{2(M+1)}, \quad n \geq m; \text{ and } |x_n - x| \leq \frac{\epsilon}{2(|y|+1)}, \quad n \geq k.$$

Choose $n_0 = \max\{m, k\}$ and verify this does.

To prove the second part, observe

$$|\frac{1}{x_n} - \frac{1}{x}| = \frac{|x_n - x|}{|x_n||x|}.$$

First we show that there is a number $a > 0$ such that $|x_n| \geq a$ for all n and $|x| \geq a$. If this is done we can argue required convergence as follows. Fix

$\epsilon > 0$. Since $x_n \to x$, fix n_0 such that $|x_n - x| \leq a^2\epsilon$ for $n \geq n_0$. This will do in view of the equality displayed above and the fact that $1/(|x_n x|$ is at most $1/a^2$.

But to get such an a is easy. Let us consider the case $x > 0$. get n_0 such that $x_n \in (x/2, 3x/2)$ for $n \geq n_0$ (if you take $\epsilon = x/2$ then this interval is just $(x - \epsilon, x + \epsilon)$). Thus $x_n > x/2$ for all $n \geq n_0$. Now take a as

$$a = \min\{|x_1|, |x_2|, \cdots, |x_{n_0}|, \frac{x}{2}\}.$$

This completes the proof.

Finally, does convergence respect order relation? It respects $\leq$ and $\geq$ but not $<$ and $>$.

Fact: *Suppose $x_n \to x$ and $y_n \to y$.*
If $x_n \leq y_n$ for each n then we have $x \leq y$.
If $x_n < y_n$ for each n, we can not in general conclude that $x < y$.

Proof: The second statement is easily observed by taking $x_n = 0$ for all n and $y_n = 1/n$. To prove the first statement, it is enough to show that if $z_n \to z$ and each $z_n \leq 0$, then $z \leq 0$. If $z > 0$, then you see that the interval $(z/2, 3z/2)$ does not contain any point of our sequence, because $z/2 > 0$. Now take $z_n = x_n - y_n$and conclude the result.

Fact: Limit of the sequence $y_n = (1 + \frac{1}{n})^n$ is also e.

Proof: Recall that e is the name given to the limit of the sequence (x_n) where

$$x_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}.$$

We have seen that $y_n \leq x_{n+1}$ so that $\lim y_n \leq \lim x_{n+1} = e$. If we show that $\lim y_n \geq e$ we can conclude equality. For this it is enough to show that , for each k the following holds.

$$\lim y_n \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}.$$

Because, the right side is x_{k+1} and if it is smaller than the number $\lim y_n$ for each k then so will be the sup of the sequence x_{k+1} and this is precisely e.

For every $n > k$ we have

$$\begin{aligned} y_n &= 1 + 1 + +\frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots \\ &\quad + \frac{1}{n!}(1 - \frac{1}{n}) \cdots (1 - \frac{n-1}{n}). \\ &\geq 1 + 1 + +\frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots \\ &\quad + \frac{1}{k!}(1 - \frac{1}{n}) \cdots (1 - \frac{k-1}{n}). \end{aligned}$$

Since k is fixed, let us denote the last expression by z_n. Thus we have $y_n \geq z_n$ for every $n > k$. Whatever be n, the number z_n is sum of a fixed number $k + 1$ terms and each term converges , so

$$\lim z_n = 1 + 1 + +\frac{1}{2!}) + \frac{1}{3!} + \cdots + \frac{1}{k!}.$$

Since $y_n \geq z_n$ for all $n \geq k$ we conclude (?) that $\lim y_n \geq \lim z_n$ completing the proof.

We have only proved that if $x_n \geq y_n$ for every n, and if limits exist, then $\lim x_n \geq \lim y_n$. But is this true if we only have $x_n \geq y_n$ only for $n > k$? Yes, consider the sequences $u_n = x_{k+n}$ and $v_n = y_{k+n}$ for $n \geq 1$. then we see $u_n \geq v_n$ for every n. The question is whether these new sequences have the same limits as the original ones? Yes. This is easy.

Fact: *Limits will not change if we delete or add or alter finitely many terms.* That is Let $x_n \to x$.

Put $u_n = x_{n+10000}$ for $n \geq 1$. Then $u_n \to x$.

Let $v_n = x_{n-10000}$ for $n > 10000$ and any numbers of your choice for $n \leq 10000$. then $v_n \to x$.

Let $w_n = x_n$ for $n > 10000$ and any numbers of your choice for $n \leq 10000$ Then $w_n \to x$.

Proof: Do it.

Fact: *Given any real number x, there is a sequence of rational numbers (r_n) such that $r_n \to x$.*

You can take a rational r_n in the interval $I_n = (x - \frac{1}{n}, x + \frac{1}{n})$. This will do because, given any $\epsilon > 0$, you get n_0 such that $1/n_0 < \epsilon$ then for any $n > n_0$ the interval I_n is contained in I_{n_0} which is contained in $(x - \epsilon, x + \epsilon)$.

Alternatively you can consider the decimal expansion. For example if $x > 0$ and $x = n \cdot \epsilon_1 \epsilon_2 \cdots$ and consider the sequence of rationals

$$n, \quad n + \frac{\epsilon_1}{10}, \quad n + \frac{\epsilon_1}{10} + \frac{\epsilon_2}{10^2}, \quad \cdots.$$

Actually, each of the expansions, including the continued fraction expansion, are providing a simple sequence of rationals converging to the number x.

If you wish you can get an increasing sequence of rationals as follows. Pick a rational r_1 in $I_1 = (x - 1, x)$. If $r_1 < x - 1/2$ then choose a rational r_2 in $I_2 = (x - 1/2, x)$. If $r_1 > x - 1/2$, then choose r_2 in (r_1, x) and continue. You should be able to write down and show that the resulting sequence is increasing and converges to x.

You can also do the following. Choose a rational $s_n \in (x - 1/n, x)$. Of course, the sequence (s_n) is definitely converging to x but need not be increasing. so put $r_n = \max\{s_1, s_2, \cdots, s_n\}$. This sequence would do.

Thus every real number can be explained using appropriate sequence of rational numbers. This will be a way of discovering new numbers as happened with e.

Yu must keep in mind that the theorem above is only a reassurance that every number has rational numbers as close to it as we please. It is only theoretical, in the sense, each of them assumes that you know the number before hand. for example, the first method asks you to take a number within $x - 1/n$ and $x + 1/n$. This is possible only if you already knew the number x. For example how do you pick up such a rational in $(e - 0.000001, e + 0.000001)$.

Similarly, the decimal expansion method works if you already knew the decimal expansion. Of course we do not know. so this theorem is only a reassurance that the rational numbers we all know can be used to describe other numbers.