discontinuities:

We have discussed several examples of functions with a point of discontinuity illustrating several possibilities. Essentially, two phenomena can occur: the function may be well behaved getting close to something or it may behave in a wiggly manner without getting close to any particular value. Accordingly, we classify the point of discontinuity into two types.

Of course the behaviour mentioned above can happen to the right of the point under consideration or to the left of the point or on both sides. But we shall not undertake this minute classification. Before making definition, let us make an observation.

Fact: Let $f : R \to R$ be a function and $a \in R$. Let L be a real number. The following two statements are equivalent.
(i) Whenever $x_n \uparrow a$ and each $x_n < a$ we have $f(x_n) \to L$
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < a - x < \delta$ we have $|f(x) - L| < \epsilon$.

Proof is simple and similar to the corresponding statements we made while introducing continuity. Let us go through it once again.

Suppose (ii) holds. We shall prove (i). So take a sequence (x_n) where $x_n < a$ for every n and $x_n \to a$. We show $f(x_n) \to L$. So fix $\epsilon > 0$. We show a natural number n_0 such that $|f(x_n) - L| < \epsilon$ whenever $n \geq n_0$. With the given $\epsilon > 0$ in hand, use (ii) and fix $\delta > 0$ as stated. Since $x_n \to a$, fix n_0 so that for $n \geq n_0$, we have $|x_n - a| < \delta$. Now if $n \geq n_0$, using the fact that $x_n < a$ we conclude that $0 < a - x_n < \delta$ and hence choice of δ shows $|f(x_n) - L| < \epsilon$.

Suppose (ii) fails. We show that (i) fails. Fix $\epsilon > 0$ for which we can not find $\delta > 0$ as stated. Thus taking $\delta = 1$ we get an x_1 such that $0 < a - x_1 < 1$ and $|f(x_1) - L| \geq \epsilon$. Taking $\delta = 1/2$ we get an x_2 such that $0 < a - x_2 < 1/2$ and $|f(x_2) - L| \geq \epsilon$. In general, taking $\delta = 1/n$ we get an x_n such that $0 < a - x_n < 1/n$ and $|f(x_n) - L| \geq \epsilon$. Put $y_n = \max\{x_1, x_2, \cdots, x_n\}$. Then $y_n \uparrow$; $0 < a - y_n < 1/n$ so that $y_n \uparrow a$; y_n being one of the x_i we have $|f(y_n) - L| \geq \epsilon$ for every n. This completes the proof.

Note that a number L as above may not exist at all, like in case of the examples involving $\sin(1/x)$. However if such a number exists then it is unique. There can not be two such numbers. Indeed, if L and L' are two such numbers, then $f(a - \frac{1}{n})$ converges to both L and L'. Since a sequence can not converge to two different points we conclude that $L = L'$.

If any one of the above two things happens we say that f has a left limit at the point a and the value of the left limit equals L. We express it as

$$\lim_{x<a;x\to a} f(x) = L; \quad or \quad f(a-) = L.$$

Note that we are not evaluating the function f at $a-$; there is no number called $a-$. It is only a notational convenience to express in that fashion.

Fact: Let $f : R \to R$ be a function and $a \in R$. Let L be a real number. The following two statements are equivalent.
(i) Whenever $x_n \downarrow a$ and each $x_n > a$, we have $f(x_n) \to L$
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < x - a < \delta$ we have $|f(x) - L| < \epsilon$.

As in the case of left limits, note that such an L, if exists, is unique. If any one of the above two things happens we say that f has a right limit at the point a and the value of the right limit equals L. We express it as

$$\lim_{x>a;x\to a} f(x) = L \quad or \quad f(a+) = L.$$

Note that we are not evaluating the function f at $a+$; there is no number called $a+$. It is only a notational convenience to express in that fashion.

Suppose that the point a is a discontinuity point of the function f. We say that it is a simple discontinuity if both $f(a-)$ and $f(a+)$ exist. that is, the left and right limits exist at the point a. In other words simple discontinuity is first of all a discontinuity point, but the function has right and left limits at that point. Simple discontinuity is also called discontinuity of the first kind.

If a is a point of discontinuity and if a is not a discontinuity of the first kind, we say that f has discontinuity of the second kind at the point a. Thus at a discontinuity of the second kind, either the right limit or the left limit does not exist. Of course, when one of these limits does not exist, then the function is discontinuous at that point.

Fact: f is continuous at a point a iff both $f(a-)$ and $f(a+)$ exist and equal $f(a)$.

If f is continuous at a, then whenever $x_n \to a$, we have $f(x_n) \to f(a)$. In particular this happens when $x_n \uparrow a$ or $x_n \downarrow a$. Thus $f(a-)$ and $f(a+)$ exist and equals $f(a)$.

Conversely, if both the limits exist and equal $f(a)$, then we show f is continuous at a. Let $\epsilon > 0$. We show $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x-a| < \delta$. Since both left and right limits at a equal $L = f(a)$, fix one $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta$ and also whenever $0 < x - a < \delta$. This δ will do. Note that when $x = a$ we have $f(x) - f(a) = 0$. This completes the proof.

There is one class of functions which have discontinuities of only first kind. Let us say that a function $f : R \to R$ is monotone increasing if $x < y$ implies $f(x) \leq f(y)$. Say that f is monotone decreasing if $x < y$ implies $f(x) \geq f(y)$. A function is monotone if it is either monotone increasing or monotone decreasing.

Note that the word increasing/decreasing is not used in the sense of strictly increasing or strictly decreasing. That is why sometimes a function which is monotone increasing in the above sense is also referred to as 'monotone non-decreasing'. Similarly a function which is monotone decreasing in the above sense is referred to as 'monotone non-increasing'. But we shall not do that.

Fact: Let $f : R \to R$ be monotone. Then the following are true.
(i) At every point a, both $f(a+)$ and $f(a-)$ exist.
(ii) f is continuous at a point a iff $f(a-) = f(a+)$.
(iii) f is continuous at all but countably many points.

Proof: Let us assume that f is monotone increasing.
(i) Let $a \in R$. Put

$$L = \lim f(a - \frac{1}{n}).$$

This limit exists because $f\left(a - \frac{1}{n}\right)$ is an increasing sequence, bounded above by $f(a)$. In fact $L \leq f(a)$. We now show that if we take any sequence $x_k \uparrow$, each $x_k < a$, then $f(x_k) \to L$. Since $x_k \uparrow$ we conclude that $f(x_k) \uparrow$ and hence the limit $\lim f(x_k)$ exists. Denote it by L'.

Fix any k. Since $x_k < a$ we see that for all sufficiently large n, $x_k < a - \frac{1}{n}$ so that $f(x_k) \leq f\left(a - \frac{1}{n}\right)$. remember this is true for all large n. Hence

$$f(x_k) \leq \lim f\left(a - \frac{1}{n}\right) = L.$$

Remember this is true for every k. Hence $\lim f(x_k) \leq L$. Thus $L' \leq L$.

Now fix any n. Since $x_k \uparrow a$ we have $x_k > \left(a - \frac{1}{n}\right)$ for all sufficiently large k. Hence $f(x_k) \geq f\left(a - \frac{1}{n}\right)$ for all sufficiently large k. Hence

$$L' = \lim f(x_k) \geq f\left(a - \frac{1}{n}\right).$$

Remember this is true for every n. Hence

$$L' \geq \lim f\left(a - \frac{1}{n}\right) = L.$$

Thus $L' = L$. Thus $f(a-)$ exists and $f(a-) \leq f(a)$. Similarly $f(a+)$ exists and $f(a) \leq f(a+)$. In fact $f(a+) = \lim f\left(a + \frac{1}{n}\right)$.

(ii) Since we have observed above that $f(a-) \leq f(a) \leq f(a+)$, equality of the extremes implies that they both equal $f(a)$.This, combined with earlier fact proves (ii).

(iii) Let D be the set of discontinuity points of f. Of course, if D is empty there is nothing to argue. Suppose $a \in D$. Then from (ii) $I(a) = (f(a-),\ f(a+))$ is a non-empty interval. Moreover if $a < b$ are in D, then $f(a+) \leq f(\frac{a+b}{2}) \leq f(b-)$ so that the intervals $I(a)$ and $I(b)$ are disjoint. Thus we have a family of non-empty disjoint intervals of the type (a, b) with $a < b$. But any such interval contains a rational number and if D were uncountable, we would be getting uncountably many rational numbers leading to a contradiction.

Similar proof applies when f is decreasing. The proof is complete.

To define left limit, it is not necessary to take the sequence $(a - \frac{1}{n})$. We can take any sequence $a_n \uparrow a$ with each $a_n < a$ and consider $L = \lim f(a_n)$. The argument above shows precisely this, namely, limit $\lim f(x_n)$ does not depend on the particular sequence, as long as each $x_n < a$ and $x_n \uparrow a$, you get the same answer. Of course, we could also have defined $L = \sup\{f(x) : x < a\}$, thus avoiding sequences altogether.

If you consider the function $f(x)$; one or zero according as x is rational or irrational, you see that each point $a \in R$ is a point of discontinuity. Of course, this function is not monotone.

We just now saw that the set of discontinuity points of a monotone function is a countable set. In fact given any countable subset $D \subset R$, we can define $F : R \to R$ which is monotone and the given set D is precisely its set of discontinuity points.

discontinuities continued:

If you have understood the above arguments, you see that in the definition of the left limit $f(a-)$ the value of f at a, namely $f(a)$ did not play any role. In fact no value $f(x)$ for $x > a$ played any role. Thus you can talk about left limit at the point a as long as the function is defined on $(-\infty, a)$.

Also, values of f at points far below a did not play any role. What does this mean? Suppose you have two real valued functions f and g defined on $(-\infty, a)$. Suppose $f(x) = g(x)$ for all x with $a - 0.0001 < x < a$. the functions may be different outside this smll interval. It is easy to see that $f(a-)$ exists iff $g(a-)$ exists and then they are equal. This leads to the following definition. We start with an observation first and then make the definition.

Let f be a real valued function defined on some set $D_f \subset R$ and $a \in R$, Suppose that $(a_0, a) \subset D_f$ for some $a_0 < a$. Then a number L satisfies statement (i) below iff it satisfies statement (ii).
(i) $x_n \uparrow a$, each $x_n < a$, each $x_n \in D_f \Rightarrow f(x_n) \to L$.
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < a - x < \delta; \quad x \in D_f \Rightarrow |f(x) - L| < \epsilon.$$

If any one of the above two statements holds, we say L is left limit of f at a or simply $f(a-) = L$. Similarly we define right limit $f(a+)$ whenever $(a, a_1) \subset D_f$ for some $a_1 > a$. Of course these limits may not exist. If both exist and are equal, then we can define $f(a)$ to be the common value. Then f, so defined at a would be continuous at a. Of course, if $a \in D_f$ already, then f is continuous at a iff $f(a)$ equals this common value.

We can define monotone function, not necessarily taking the domain of definition to be R as we did earlier. A real valued function f with domain D_f is monotone increasing if $x, y \in D_f$ and $x < y$ imply $f(x) \leq f(y)$. Similarly we can define monotone decreasing function. Exactly the same proof given earlier shows the following. Let f be a monotone function defined on an interval I. Then at every point $a \in I$ except at the end points (if any) the left limit $f(a-)$ and $f(a+)$ exist and the set of discontinuities of f is at most countable.

At the end points one needs to be careful. For example the left end point of I is finite, say a, we can only talk about the right limit $f(a+)$ and not left limit $f(a-)$. Further the right limit $f(a+)$ is finite iff the function is bounded in an interval (a, a_1). Even if the left end point a is ∞ one can still talk about right limit at $-\infty$. But when you need this you will realize without much problem. It is unnecessary to confuse ourselves with utmost generalities at the beginning stage.

differentiation:

How to draw tangent to a curve at a point. Suppose the curve is given by $y = f(x)$ and at the point $(a, f(a))$ on the curve we are required to draw tangent. The first question is: what do you mean by tangent. well, one intuitive idea is that it is a straight line pasing through the given point and meets the curve at that point only. This is not quite correct, you can try to draw tangent to the curve $y = \sin x$ at several points and see what happens. Of course, for a circle, tangents are precisely straight lines that meet the circle at exactly one point. (why?)

Another idea is the following. Take the point $(a, f(a))$ on the curve and a nearby point $(x, f(x))$ join these two points by a straight line. This is a 'chord'. A natural question is whether this chord has a limiting form as the point x approaches a. At first sight this explanation appears too complicated because we are talking about limiting form of straight lines. But it is not difficult to understand.

Afterall, a straight line is determined by its slope — apart from a point through which it passes. In our case the point is $(a, f(a))$. Thus chords we are talking about or the tangent we are looking for , all pass through this point. So we only need to specify the slope. Thus the question amounts to asking whether the slopes of the chords have a limiting value. of course, slope of the chord we have mentioned above is nothing but $[f(x) - f(a)]/[x - a]$.. Thus we need to see if this has a limiting value as x approaches a.

To start an entirely different second line of thought, suppose a particle is starting at the origin at time zero and travelling along a path. How do we understand its velocity? If it is travelling so that at time t the particle is at $5t$, then matters are simple. At time instants $t_0 < t_1$ it is at $5t_0$ and $5t_1$ respectively, so that the distance travelled during this time duration $t_1 - t_0$ is $5t_1 - 5t_0$ and so the velocity, distance divided by time, equals 5 and does not depend on the two time points we have taken. But in practice particles accelerate and do not travel with 'uniform speed' as above.

Suppose that the particle is travelling along the curve $y = t^2$, again starting at zero. The distance travelled during the time period 1 to 2 is $4-1 = 3$ while the distance travelled during time period 10 to 11 equals $121-100 = 21$ and you can try to find out the distance travelled during time period 1000 to 1001. You see that the particle is going faster and faster as time elapses. Thus the concept of velocity does not make sense unless you specify a time point and ask: what is the velocity at this time. In other words one needs to talk about instant velocity.

Suppose a time point $t = 5$ is given. What is th velocity t time instant 5? Naturally it should depend on what is happening around this time instant and one idea is to take the distance travelled during the time period 5 to $5+h$ and then take the ratio. In other words, if the path is given by $f(t)$, to understand the velocity at time 5 we need to calculate $[f(5+h) - f(5)]/h$ and then see if there is any particular value to which this ratio is getting close as h gets closer to zero, that is, whether the intuitive idea of distance travelled/time approaches any aprticular value as we consider durations nearer to the time point of interest. To change notation, if the time point under consideration is a, then we need to look at the ration $[f(a) - f(x)]/[a - x]$ and see if it has a limiting value as the (time) point x approaches a.

To start a yet different, third line of thinking, let us understand complexity of functions. The simplest functions are constant functions. Suppose we are given a function f on R and a point a, Which simplest function best approximates f near a. Obviously the constant function $\varphi(x) \equiv f(a)$ is the best. When x is close to a, $\varphi(x)$ is close to $f(x)$, simply because $f(x)$ is close to the number $f(a)$ when x is close to a. In symbols, $f(x) - \varphi(x) \to 0$ as $x \to a$.This means, $f(x_n) - \varphi(x_n) \to 0$ whenever $x_n \to a$.

Suppose you are allowed to use a little more complicated functions than constant functions, namely, straight line functions. Can you do better? so what is meant by better? Earlier we simply said $f(x) - \varphi(x)$ should get close to zero as x gets close to a. We did not demand any quantitative measurement on how this quantity gos to zero. Now having allowed functions general than constant functions, we demand that not only this error gets close to zero, the ratio $[f(x) - \varphi(x)]/[x - a]$ should get closer to zero as x gets closer to a.

Note that this implies, in particular, that $f(x) - \varphi(x)$ gets closer to zero as x gets closer to a. But since both f and φ are continuous, this difference is getting closer to $f(a) - \varphi(a)$. In other words $\varphi(a) = f(a)$. Thus the straight line passes through $(a, f(a))$. If the straightline we are thinking is $\varphi(x) = mx + c$, then what we concluded just now amounts to $ma + c = f(a)$ or $c = f(a) - ma$. Thus the straight line is

$$\varphi(x) = mx + f(a) - ma = f(a) + (x - a)m.$$

Thus

$$\frac{f(x) - \varphi(x)}{x - a} = \frac{f(x) - f(a)}{x - a} - m.$$

Thus the demand that this ratio gets closer to zero as x gets closer to a, simply amounts to saying that the ratio $[f(x) - f(a)]/[x - a]$ should get closer to the number m, slope of the line we are looking for.

All these thought processes lead to one common conclusion: the rate of change of the function is an important quantity. Before defining this precisely, we make an observation.

Fact: Let $f : R \to R$ and $a \in R$. For a number $m \in R$, the following two statements are equivalent.
(i) If $x_n \to a$ and $x_n \neq a$ for each n; then

$$\frac{f(x_n) - f(a)}{x_n - a} \to m.$$

(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - m \right| < \epsilon.$$

Proof is simple and we have come across such a situation several times earlier — connection between discrete and continuous formulations of an idea.

Suppose (ii) holds. To prove (i), take $x_n \to a$ and $x_n \neq a$ for every n. Shall show n_0 such that

$$n \geq n_0 \Rightarrow \left| \frac{f(x_n) - f(a)}{x_n - a} - m \right| < \epsilon.$$

Use (ii) with the given ϵ to get a $\delta > 0$ and then use $x_n \to a$ to get n_0 so that $n \geq n_0 \Rightarrow |x_n - a| < \delta$. This will do.

Conversely, if (ii) is false, we exhibit a sequence for which (i) fails. Since (ii) is false, fix $\epsilon > 0$ for which we can not find $\delta > 0$ stisfying the stated condition. With $\delta = 1/n$ get x_n so that $0 < |x_n - a| < 1/n$ and yet

$$\left| \frac{f(x) - f(a)}{x - a} - m \right| \geq \epsilon.$$

This sequence shows failure of (i).

Definition: Let $f : R \to R$ and $a \in R$. We say that f is differentiable at the point a if there is a number l satisfying the above conditions. In such a case l is called derivative of f at a. There are several notations.

$$f'(a) = l; \quad \frac{df}{dx}(a) = l; \quad D_x f(a) = l.$$

Another way of saying the same thng is the following: Define the function $\varphi(x) = [f(x) - f(a)]/[x - a]$. Of course, this is defined on R except at the point a. If the right limit and left limit of this function exist at a and equal, then we say that the function f is differentiable at the point a and this limit is called the derivative of the function f at the point a. If the function is differentiable at every point, then we say that f is differentiable. In this case, we can define f' on all of R.

Observe that the number l, when exists, is unique.

Fact: Let $f : R \to R$. if f is diferentiable at a, then f is continuous at a.

Proof: Let $x_n \to a$. We need to show $f(x_n) \to f(a)$. If all the x_n are different from a, then

$$f(x_n) - f(a) = \frac{f(x_n) - f(a)}{x_n - a} (x_n - a) \to f'(a) \cdot 0 = 0.$$

Suppose that there is an n_0 such that $x_n \neq a$ for all $n \geq n_0$. Then you can write the equation above for $n \geq n_0$ obtaining the result. if there is an n_0 such that $x_n = a$ for all $n \geq n_0$, then there is nothing to do.

Finally suppose that there are infinitely many n such that $x_n \neq a$ and infinitely many n such that $x_n = a$. Then you will get two subsequences corresponding to each and the above argument applies for the two corresponding subsequences of $f(x_n)$. Using an earlier observation, we conclude that $f(x_n) - f(a) \to 0$.

Here is another way to argue. If only finitely many x_n are different from a, then after some stage $f(x_n) = f(a)$ and so we are done. If infinitely many $x_n \neq a$; let $n_1 < n_2 < \cdots$ are precisely those integers. Then $[f(x_{n_i}) - f(a)]/[x_{n_i} - a]$ converges to $f'(a)$ and hence is a bounded sequence, say bounded by C. Since $x_{n_i} - a \to 0$, there is n_k such that for all $i \geq k$ $|x_{n_i} - a| < \epsilon/C$. But then $|f(x_{n_i}) - f(a)| < \epsilon$. And of course, if $n > n_k$ and not in the subsequence then $f(x_n) = f(a)$ and hence for all $n > n_k$, we see $|f(x_n) - f(a)| < \epsilon$.

It is instructive to argue using $\epsilon - \delta$. That is, fix $\epsilon > 0$ and show $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Fact: if $f : R \to R$ and $g : R \to R$ are differentiable at a then so are $f+g$ and $39f$ and fg. In fact

$$(f+g)'(a) = f'(a) + g'(a); \qquad (39f)'(a) = 39f'(a);$$

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

If $g(x) \neq 0$ for all x, then f/g is also differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Proof: if you take any sequence $x_n \to a$, $x_n \neq a$ for all n, then

$$\frac{(f+g)(x_n) - (f+g)(a)}{x_n - a} = \frac{f(x_n) - f(a)}{x_n - a} + \frac{g(x_n) - g(a)}{x_n - a}$$

$$\to f'(a) + g'(a).$$

$$\frac{(39f)(x_n) - (39f)(a)}{x_n - a} = 39\frac{f(x_n) - f(a)}{x_n - a} \to 39f'(a).$$

$$\frac{(fg)(x_n) - (fg)(a)}{x_n - a} = \frac{f(x_n)g(x_n) - f(x_n)g(x) + f(x_n)g(x) - f(x)g(x)}{x_n - a}$$

$$\to f(a)g'(a) - f'(a)g(a).$$

Here we used continuity of f at a.
We show $1/g$ is differentiable and calculate its derivative, then the above

multiplication rule can be used for the product $f \cdot (1/g)$. Take a sequence $x_n \to a$, $x_n \neq a$ for all n.Then

$$\frac{(1/g)(x_n) - (1/g)(a)}{x_n - a} = \frac{g(a) - g(x_n)}{x_n - a} \frac{1}{g(x_n)g(a)} \to \frac{g'(a)}{g^2(a)}.$$

here we have used continuity of g at a.

Fact: Let $f : R \to R$ and $g : R \to R$ and $a \in R$. Assume that f is differentiable at a and g is differentiable at the point $f(a)$. Let $h(x) = g(f(x))$, the composition. then h is differentiable at a and

$$h'(a) = g'(f(a))f'(a).$$

Proof: Take $x_n \to a$, $x_n \neq a$ for all n.

Case 1: Assume that there is an n_0 such that for all $n \geq n_0$ we have $f(x_n) = f(a)$. Then note that

$$f'(a) = \lim \frac{f(x_n) - f(a)}{x_n - a} = 0$$

so that $g'(f(a))f'(a) = 0$. Of course, $h(x_n) = h(a)$ for all $n \geq n_0$ so that $h'(a) = 0$ proving existence of limit (of $[h(x_n) - h(a)/[x_n - a])$ as well as the stated equality in the case under consideration.

Case 2: Assume that there is an n_0 such that $f(x_n) \neq f(a)$ for all $n \geq n_0$. Then for $n \geq n_0$ we have

$$\frac{h(x_n) - h(a)}{x_n - a} = \frac{g(f(x_n)) - g(f(a))}{f(x_n) - f(a)} \frac{f(x_n) - f(a)}{x_n - a}$$

$$\to g'(f(a))f'(a).$$

Here we have used that f is continuous at a, thus $f(x_n) \to f(a)$. This proves existence of limit (of $[h(x_n) - h(a)/[x_n - a])$ and also the stated formula.

Case 3. There are infinitely many n such that $f(x_n) = f(a)$ and infinitely many n such that $f(x_n) \neq f(a)$. Enumerate the integers satisfying the first condition as $n_1 < n_2 < n_3 < \cdots$ and the second kind as $l_1 < l_2 < l_3 < \cdots$ and apply the above cases for the subsequences

$$\frac{h(x_{n_i}) - h(a)}{x_{n_i} - a}; \qquad \frac{h(x_{l_i}) - h(a)}{x_{l_i} - a}$$

and conclude the result by employing earlier fact about subsequences.

So far we have not calculated derivative of any specific function.

Fact: if $f(x) = 55$ for all x then $f'(a) = 0$ for all a.
if $f(x) = x$ for all x, then $f'(a) = 1$ for every a.
if $f(x) = x^{39}$ for all x, then $f'(a) = 39a^{38}$ for every a.
Proof: First two statements are easy (should be done to say this).

Last statement follows from

$$\frac{x^{39} - a^{39}}{x - a} = x^{38} + x^{37}a + x^{36}a62 + \cdots + a^{38}$$

$$\to 39a^{38}.$$

Using the rules above, you can now show that polynomials are differentiable and be able to calculate their derivatives. Also for rational functions you should be able to do. What is a rational function? a function of the form $P(x)/Q(x)$ where P and Q are polynomials and Q is not the zero polynomial. Where is this function defined? On R minus finitely many points; which is a finite unon of intervals.

if you have understood the earlier calculations, then you can do some improvements.

Why should we have functions defined on all of the real line? suppose that f is a function defined on an interval $I = (\alpha, \beta)$ where $-\infty \leq \alpha < \beta \leq +\infty$. Let $a \in I$. We say that f is differentiable at a iand $f'(a) = l$ if any one of the following two equivalent conditions hold;
(i) $x_n \in I$ for all n, $x_n \neq a$ for all $n \Rightarrow \frac{f(x_n)-f(a)}{x_n-a} \to l$.
(ii) Given $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - a| < \delta; x \in I \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - l \right| < \epsilon.$$

Of course the case $\alpha = -\infty$ and $\beta = +\infty$ corresponds to $I = R$, discussed above.

All the facts remain true. For sum, product and ratio of functions (defined on the interval I now) with exactly the same proofs. the composition rule also holds with exactly the same proof as follows: let f be real valued function defined on an interval $I = (\alpha, \beta)$. let g be a rael valued function defined on another interval $J = (\gamma, \delta)$. let us assume that range of the function f is contained in J. that is, $f(x) \in J$ for every $x \in I$. Then the composition makes sense: $h(x) = g(f(x))$ defined on I. let now $a \in I$. Assume that f is differentiable at a and g is differentiable at $f(a)$. The h is differentiable at a and $h'(a) = g'(f(a))f'(a)$.

This is not a novel. You are expected to pause and convince yourselves that what was said in the para above is true. if you have trouble, that means you have not understood the earlier calculations for functions defined on all of R. You are advised to go back and work it out taking pen and paper. then you should return to the para above and not take it for granted.

Some of you were asking about functions defined on a closed interval. Suppose that f is defined on the interval $[23, 33]$. You can define the concept of differentiability at all points in the interval $(23, 33)$. if you take a point a in this interval, there is a small $\delta > 0$ so that $(a - \delta, a + \delta) \subset (23, 33)$ — you can take $\delta = \min\{a - 23, 33 - a\}$.

Can you define derivative at the points 23 and 33? Yes, at the point 23, you can define righthand derivative, namely,limit of $[f(x) - f(23)]/[x - 23]$ as $x \downarrow 23$ as was done in discussing left and right limits of functions. of course, this limit may not exist. similarly, you can define left derivative at the point 33. some of you who are curious can think about this. However, most of you should concentrate on understanding the above discussion thoroughly and clearly.