While "pleasure" and "enjoyment" are often used to characterize one's efforts in science, failures, frustrations, and disappointments are equally, if not the more. common ingradeints of scientific experience. Overcoming difficulties, undoubtedly, contributes to one's final enjoyment of success.

S. Chandrasekhar.

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

John von Neumann

Before beginning:

We are going to learn about real numbers and functions. After all, now a days you learn some calculus in school. Let us start at the beginning.

What are real numbers?

??

Am I a real number? Is this table a real number?

??

We appear unsure, Ok, let us see. What is the colour of your shirt?

Green.

Good. What is 'colour"? What is 'green colour'?

??

Again we appear unsure. Let us be clear, we know very few things. Indeed, colour is a difficult concept. We have learnt to identify green colour without really knowing what exactly it is. There is nothing wrong with it, except that sometimes we might go wrong.

If some one tried to explain what green colour is, he would probably start saying: you are able to see the shirt because light reflected from the shirt is reaching your eyes; you know light comes in several sizes, call it wave lengths or frequencies; the shirt material you have is absorbing all except a particular size. So just that size light is reflected reaching your eye, and this corresponds to green colour.

We would probably say to him: forget it, I know better, I do not need your explanation. I do not understand sizes of light and reflection and all that nonsense.

The same thing happens with real numbers too. We are familiar with them, we know how to use them and so on. Let us ask ourselves some simple questions.

Is there a number whose square is 3?

Yes. $\sqrt{3}$.

You are absolutely right, there is such a number, but I do not understand what you said.

It is $\sqrt{3}$.

Yes, I heard, but what is that number?

It is a number whose square is 3.

But this is precisely our question, whether there is such a number at all. If you first show me that there is such a number, then we all can name it $\sqrt{3}$.

Actually, we can write it: $1 \cdot 732 \cdots$.

If you multiply with itself, will you get 3?

Yes.

Can you write that number on the board and multiply with itself and show me you will get 3.

We can not write.

But do you know the number?

Yes.

Write it on the board.

It has infinitely many decimal places.

But do you know all the decimal places?

No.

You mean to say you do not know the number fully.

It is there, but we can not write it down.

How can I believe that it is there. You did not show that there is such a number: you can do this by writing the number OR by convincing me, by an argument, that there is such a number. You did neither.

??

OK, let us leave this discussion, accept that there is a number whose square is 3.

Is there a number whose square is -3?

No.

How come both questions appear similar, but for the first the answer is 'yes', while for the the second question it is 'no'.

Because square of any number is non-negative.

Good, but why is it so.

Because product of two positive numbers is positive and product of two negative numbers is also positive.

Very Good, we are heading some where. But why is this so?

We can actually multiply and see.

How do you multiply? We do not have any specific numbers before us.

We can take some numbers, multiply and see.

If you take some numbers and see the truth, why should others believe that it is *always* true?

??

Obviously we use certain properties of the collection of real numbers to arrive at these answers. What are those properties? Do we keep on bringing in new properties every time you want to answer a question or are there some properties listed once and for all (not depending on the question asked) which are used forever in the analysis? If so what are those rules? Does everything you know follow from those rules?

You have learnt in school, the meanings of 2^4, multiply 2 with itself four times (equals 16); or $(\sqrt{3})^5$, multiply $\sqrt{3}$ with itself five times (equals $9\sqrt{3}$ But what is the meaning of $(\sqrt{3})^{(\sqrt{5})}$? You can not say multiply $\sqrt{3}$ with itself $\sqrt{5}$ times!

We define it through limits.

Yes, So some of you know clearly, but many seem to be unsure.

Let us ask ourselves another question. you seem to know square roots exist for non-negative numbers. But do fifth roots exist? Is there a number x such that $x^{55} = 4$?

Yes, we use least upper bound to define.

Good, again some of you know exactly how we get those numbers we are looking for, but many are not sure.

You know how to add two numbers or a billion numbers. You also know

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = 2.$$

What is the meaning of adding infinitely many numbers?

We can keep adding, we get closer to 2, so the sum is nearly 2.

Wait a minute, is the sum exactly 2 or nearly 2?

Very close to 2.

How close?

Very close.

I do not understand what is meant by very close.

We can define using limits.

Yes, you are right. Our basic rules only tell us how to add finitely many numbers, adding infinitely many numbers is *not* part of basic rules. We have developed. the above series is simple. You can keep on adding and get

$$1,\ 3/2,\ 7/4,\ 15/16,\ 31/32\cdots$$

or equivalently

$$2-1,\ 2-\frac{1}{2},\ 2-\frac{1}{4},\ 2-\frac{1}{8},\ 2-\frac{1}{16},\ 2-\frac{1}{32}\cdots$$

the terms differ less and less from 2 as you keep adding. Here we are lucky, we could add numbers. Some times we can not add like this. Do you know if the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\cdots=?$$

converges?

Yes, it converges.

Now it is diffficult to keep on adding terms and see what is happening. Do you know the value of the sum.

Yes, $\pi^2/6$.

Very Good, you are right. Some of you have done extra reading in analysis, very nice.

You read about sine and cosine functions. Why do we need functions, why are they useful?

If you have a rod and heat it, you would like to know its temperature at time t, as a function of t.

Yes, good. You can think of many other things. Simplest is the simple pendulum. a bob hanging (nearly frictionless manner). You displace from vertical a little bit and see what happens. Its displacement at time t is an interesting quantity. Or when you discuss planetary motion, you describe using functions.

One of the first calculations made by people, was to find areas of plane figures. Archimedes (287 BC - 212 BC) — Greek mathematician, physicist, inventor — discussed the following problem, even before the concepts of real number, function, limits were invented.

Consider the plane, with x-axis and y-axis drawn as you did in high school. Now consider the region bounded by the positive x-axis, the line $x = 1$ and the parabola $y = x^2$ (between $x = 0$ and $x = 1$). What is its area?

This is how he argued.

[You should draw the x-axis, y-axis and the curve $y = x^2$ between $x = 0$ and $x = 1$.] We need the area, A, below this curve. Since we know areas of rectangles, let us see if we can get rectangles to approximate the area. Draw rectangles with bases

$$[0, 1/10],\ [1/10, 2/10],\ [2/10, 3/10], \cdots\cdots, [9/10, 10/10]$$

and heights

$$(1/10)^2,\ (2/10)^2,\ (3/10)^2,\ \cdots\cdots, (10/10)^2$$

respectively. You see that the area under the curve is covered by these rectangles. So

$$A \le \frac{1}{10} \times (\frac{1}{10})^2 + \frac{1}{10} \times (\frac{2}{10})^2 + \frac{1}{10} \times (\frac{3}{10})^2 + \cdots + \frac{1}{10} \times (\frac{10}{10})^2.$$

Instead of ten rectangles, if you selected any integer n and covered by n rectangles with bases each of length $1/n$, we would obtain

$$A \le \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}.$$

Archimedes knew

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Thus

$$A \le \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

Archimedes knew that, whatever non-zero number is given, $1/n$ is smaller than that number for all large values of n. Using this he deduced that the right side above can be made arbitrarily close to 1/3. Thus $A \leq 1/3$. With a similar and smart argument, using the rectangles with the same bases but heights

$$(0/10)^2,\ (1/10)^2,\ (2/10)^2, \cdots\cdots, (9/10)^2.$$

he could deduce $A \geq 1/3$. He concluded that the area must equal 1/3.

Beginning:

Let us now start with the properties of real numbers that we shall use. Anything that we say must follow from these basic rules we agree now.

We have a set R with two operations + (addition: associates with every pair of elements of R an element of R) and . (multiplication; associates with every pair of elements of R an element of R) and a comparision relation $<$ (less than: this is not an operation as the other two, this only compares two elemets of R). The system is denoted by $(R, +, \cdot, <)$.

Axiom set I: Four rules for for addition (+).
(i) $x + y = y + x$;
(ii) $(x + y) + z = x + (y + z)$;
(iii) there is an element $a \in R$ such that $x + a = a + x = x$;
(iv) for each $x \in R$ there is an element y, depending on x, such that $x + y = y + x = a$.

The conditions above hold for all x, y, z in R. The a asserted in (iii) is unique. Indeed if there are two such, say a and b, then $a = a + b = b$. Here the first equality is by property of b and the second is by property of a. Since there is only one such, we denote it by 0. Also, for every x there is a unique y such that $x + y = 0$. For a given x, this y is denoted by $-x$.

Axiom set II: Four rules for multiplication $(\cdot)$.
(i) $x \cdot y = y \cdot x$;

(ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(iii) there is an element $c \in R, c \neq 0$ such that $x \cdot c = c \cdot x = x$;
(iv) for each $x \in R, x \neq 0$ there is an element y, depending on x, such that $x \cdot y = y \cdot x = c$.

As above we can show that the c asserted above is unique. This will be denoted by 1. For a given $x \neq 0$, the y asserted above is unique and is denoted by $1/x$ or $\frac{1}{x}$.

Axiom set III: one rule, says $(+, \cdot)$ are friendly.
$x \cdot (y + z) = x \cdot y + x \cdot z$.

Axiom set IV: two rules for $(<)$.
(i) For any x, y exactly one of $x < y$, $x = y$, $y < x$ holds.
(ii) If $x < y$ and $y < z$ then $x < z$.

Second rule says that the comparision is consistent, in the sense, if x is smaller than y and y smaller than z, then x is smaller than z. In practice, if three persons A, B, C are playing matches, it is quite possible that A wins over B and B wins over C, but this does not automatically imply that A wins over C. First rule says that any two elements can be compared. Taking y same as x, we see that we never have $x < x$. We use $x \leq y$ as abbreviation for '$x < y$ or $x = y$'. We also sometimes write $x > y$ to mean $y < x$. Similarly $x \geq y$ means $y \leq x$.

Axiom set V: Two rules, say $(<)$ is friendly with $(+)$ and $(\cdot)$.
(i) $y < z$ implies $x + y < x + z$;
(ii) $0 < x,\ 0 < y \Rightarrow 0 < x \cdot y$.

Finally Axiom set VI: one rule, says least upper bounds exist.
Let $S \subset R$ be non-empty.
Suppose S has an upper bound — which means $(\exists y)(\forall x \in S)(x \leq y)$.
Then S has a least upper bound — which means
$(\exists z)\ \{[\forall x \in S,\ x \leq z]\ \&\ [(\forall x \in S,\ x \leq y) \Rightarrow z \leq y]\ \}$.

This axiom is called least upper bound axiom or completeness axiom or continuity axiom. This axiom tells us that our geometric picture of real numbers as a line without breaks/gaps is justified.

Fix such a system $(R, +, \cdot, <)$ once and for all. Elements of R are called real numbers. sometimes we just xy for $x \cdot y$.

There is only one such system, in the sense, if there are two such systems you can establish a bijection between the elements of the two systems so that the operations as well as the order are preserved. Will such a system allow us to do everything we are used to do? Yes, instead of trying to convince you of *everything*, we shall do a few as a sample. There are two aspects you are used to. Algebraic (or arithmetic) manipulation of numbers and geometric visualization of numbers as a line. First we see some algebraic manipulations.

Arithmetic of numbers.

Fact 1. (for plus): (i) $x + y = x + z$ implies $y = z$. (ii) $x + y = x$ implies $y = 0$. (iii) If $x + y = 0$ then $y = (-x)$. (iv) $(-(-x)) = x$.

Here are the proofs: (i)

$$y = y + 0 = y + (x + (-x)) = (y + x) + (-x) = (x + y) + (-x)$$

$$= (x + z) + (-x) = (z + x) + (-x) = z + (x + (-x)) = z + 0 = z.$$

Decipher the rules used at each equality.

To see (ii) use (i) with $z = 0$ and to see (iii) use (i) with $z = -x$. To show (iv) observe $(-x) + x = 0$ and use (iii).

Fact 2. (for multiplication): (i) $x \cdot y = x \cdot z$ and $x \neq 0$ imply $y = z$. (ii) $x \cdot y = x$ and $x \neq 0$ imply $y = 1$. (iii) If $x \cdot y = 1$ and $x \neq 0$ then $y = (1/x)$. (iv) If $x \neq 0$, $(1/(1/x)) = x$.

Proof of this is similar to the above, carefully replace $(+)$ by $(\cdot)$. *Please* remember that just because I said that the proof is similar, you can not say *unless* you have gone through the proof and convinced yourself that it is indeed similar.

Fact 3. (1) $0 \cdot x = 0$. (ii) If $x \neq 0$ and $y \neq 0$ then $x \cdot y \neq 0$. (iii) $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$ (iv) $(-x) \cdot (-y) = xy$.

Proof: (i). We shall omit putting the symbol $\cdot$. Observe $0x = (0+0)x = 0x + 0x$ and use appropriate part of fact 1. To prove (ii), suppose $xy = 0$. Using $x \neq 0$

$$y = y1 = y(x\frac{1}{x}) = (yx)\frac{1}{x} = 0\frac{1}{x} = 0.$$

But $y \neq 0$. Thus $xy \neq 0$. To prove (iii), we need to show that $(-x)y$ is the additive inverse of xy. This follows from

$$xy + (-x)y = (x + (-x))y = 0y = 0$$

and appropriate part of fact 1. Other part of (iii) is similar. To prove (iv) use (iii) repeatedly and finally Fact 1.

$$(-x)(-y) = -(x(-y)) = -(-(xy)) = xy.$$

Fact 4. (for order): (i) $x > 0$ if and only if $-x < 0$. (ii) If $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$. If $x < 0$ and $y < z$ then $x \cdot y > x \cdot z$. (iv) $x \neq 0$ implies $x^2 > 0$. In particular $1 > 0$. (v) if $0 < x < y$ then $0 < (1/y) < (1/x)$.

Proof: Using an axiom, add (-x) to both sides of $x > 0$ to see $0 > -x$, to see part of (i). Other part is similar. Add $-y$ to both sides of $y < z$ to see $0 < (z - y)$. Now use $0 < x$ and axiom to see $0 < x(z - y) = xz - xy$ (why). Now add xy to both sides to get (ii). The other part of (ii) is similar. To see (iii), note that if $0 < x$ then axiom does it. If $x < 0$, then (i) tells us $(-x) > 0$. Hence $(-x)(-x) > 0$. But this is $x \cdot x$ by the last part of Fact 3. Since $0 < x$, $0 = x$ and $x < 0$ are the only possibilities this shows $x^2 > 0$. Since $1 = 1 \cdot 1$, and $1 \neq 0$, we get $1 > 0$.

Finally let $0 < x < y$. If $1/x < 0$, then $(-1/x) > 0$ so that $x(-1/x) = -(x \cdot 1/x) = -1 > 0$ which implies $1 < 0$, but both $1 > 0$ and $1 < 0$ can not simultaneously hold. Thus both $1/x > 0$ and $1/y > 0$ hold. Multiply both sides of $x < y$ with $1/x \cdot 1/y$ to see $1/y < 1/x$.

You should by now be convinced that much of the algebra you are using is actually a consequence of the few rules we have listed about the set R of real numbers.

Geometry of the number line.

Now let us see the geometric picture. We usually draw the number line as below.

We plot the number zero arbitrarily.

0

If $0 < x$ we say x is positive, If $x < 0$ we say x is negative. Fact 4 tells us that x is positive if and only if $-x$ is negative. We make up our mind that numbers to the right of zero are positive and increase as you go away from zero. If you do not like, you can plot positive numbers to the left of zero.

So we plot 1 to the right of zero. Since $1 > 0$ we see $1+1 > 1+0 = 1$. We denote $1+1$ by 2, plot it to the right of 1. Since $2-1 = 1-0$, we plot 2 at the same distance from 1 as 1 is from zero. Now argue in the same way that $1+1+1 > 1+1$. Denoting $1+1+1$ by 3, wehave $3 > 2$. I leave it to your imagination now for negative numbers. There is one main problem: are all positive numbers somewhere on the right side of zero between the $1, 2, 3 \cdots$ we have plotted or are there numbers beyond all things we plotted? Luckily there is nothing beyond all these. To formulate this precisely, let us give the name

$$N = \{1, 2, 3, \cdots\cdots\}.$$

We denote elements of N by n, m etc for now. These are called natural numbers.

There is only one problem with this definition of the set N. I have used $\cdots\cdots$. But what do these dots denote, do we know what exactly is our set N? We should define N as the smallest subset of R which has the following two properties: (i) $1 \in N$ and (ii) $m \in N$ implies $m+1 \in N$. I think it is better to leave it at this stage. You can keep your mental picture that N consists of 1, $1+1$, $1+1+1$ etc etc (without understanding what is the meaning of etc). We shall return to this point at some stage later.

Fact 5. (Archimedean property of R) Let $x > 0$. Then there is an $n \in N$ such that $x < n$.

Suppose this is false. Then for every $n \in N$ we have $n \leq x$. Thus the set N is bounded above. Let s be its lub. But then $s - 1$ can not be upper bound of the set N. So $s - 1 < m$ for some $m \in N$ which means $s < m + 1$. But $m \in N$ tells us that $m + 1 \in N$. But then s can not be upper bound of N. This contradiction proves the result.

So there is no number beyond all the elements of N. We have the following picture.

_________-2_________-1_________0_________1_________2_________

We denote

$$Z = \{n : n = 0 \quad \text{or} \quad m \in N \quad \text{or} \quad - m \in N\}.$$

Elements of Z are called integers. We define

$$Q = \{\frac{m}{n} : m \in Z; n \in N\} = \{\frac{m}{n} : m, n \in Z, n \neq 0\}.$$

Elements of Q are called rational numbers. Just to remind you, m/n means $m \cdot (1/n)$ and of course $1/n$ is the multiplicative inverse of n. Now we start plotting these numbers on the line. Since $(1/2) + (1/2) = 1$ we plot $1/2$ midway between zero and 1. Now I leave the rest to your imagination.

_____-$\frac{3}{2}$_____-1_____-$\frac{1}{2}$_____0 _____$\frac{1}{2}$_____1_____$\frac{3}{2}$_____

The main point is that the mental picture about the arithmetic and geometry (whatever it may mean) of the number line we have, agrees with the picture given by the few rules we made above.

There are two very important questions now.
Does every real number correspond to a point on the line? Yes.
Does every point of the line correspond to a number? Yes.

The first answer comes out of the fact that every real number is lub of a set of rational numbers and the line is continuous, it has no gaps. The second answer comes out of the lub axiom. We shall not spend any more time on these matters.

From now on, we shall sometimes be brief with proofs. You should not just copy, you should understand the truth of each of the steps of the argument.

Fact 6: Given any two numbers $a < b$, there is a rational number x such that $a < x < b$.

Proof: Since $b - a > 0$ get, by archemidean property (apply with $x = 1/(b-a)$), an integer $n \in N$ such that $(b - a) > 1/n$. There is at least one integer k such that $k/n > a$ (apply archimedean peoperty with $x = na$, if $a < 0$, then $k = 0$ would do). Let us take least such integer m. Thus $m/n > a$ but $(m-1)/n \leq a$ Can $m/n > b$? If it were so then,

$$b - a \leq \frac{m}{n} - \frac{m-1}{n} = \frac{1}{n}.$$

But $(b - a) > 1/n$ leading to a contradiction. Thus we have $a < (m/n) < b$.

Are there numbers which are not rational? Let us assume for the time being, there is a positive number x such that $x^2 = 2$. Denote such a number by $\sqrt{2}$.

Fact 7. $\sqrt{2}$ is not a rational number.

You know this, please try to write a proof of this fact.

Fact 8. Given any two numbers $a < b$, there is an irrational number x such that $a < x < b$.

Apply fact 6 to the numbers $a/\sqrt{2}$ and $b/\sqrt{2}$ and argue.

So between any two numbers $a < b$, we can find both rational numbers as well as irrational numbers. Can we find out how many numbers of each kind are there? Both are infinite. What does this mean?

Let us say that emptyset has zero number of elements, $|\emptyset| = 0$. If A is a non-empty set, we say that A has n elements, in symbols $|A| = n$ if we can establish a map $f : A \to \{1, 2, \cdots, n\}$ which is one-to-one and onto. Here $n \in N$. We say that a set A is finite if $|A| = n$ for some $n = 0, 1, 2 \cdots$. Otherwise, we say that the set A is infinite.

We say that a set A is countably infinite if we can establish a map $f : A \to N$ which is one-to-one and onto. We say A is countable if either it is finite or countably infinite. Otherwise, we say that A is uncountable.

Here is a fact which you know and we shall prove soon:

Fact 9. The set of rational numbers is countably infinite, the set of irrational numbers is uncountable.

Given any number x, we define x^n for every $n \in N$ by induction as follows: $x^1 = x$. If we have defined x^n, we define $x^{n+1} = x^n \cdot x = x \cdot x^n$.

Fact 10. If $0 < x < y$ then $x^n < y^n$.

Prove it by induction. As suggested by one of you, you can also use the fact that product/sum of positive numbers is positive along with the formula

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}).$$

But since we need induction later on, it is better to know. Here it is:

Fact 11. (Mathematical induction)
Suppose that for every $n \in N$ we have a mathematical statement, say, P_n.
Suppose that P_1 is true.
Suppose that for every integer k, truth of $P_1, P_2, \cdots, P_k$ implies truth of P_{k+1}.
THEN: For every $n \in N$, the statement P_n is true.

There are two issues. Why should we believe this? How does it help us? Suppose that P_n is the statement $x^n < y^n$. Here x and y are fixed numbers

$0 < x < y$. This is a statement depending on n. Clearly P_1 is true. If we assume P_k is true, then

$$x^{k+1} = x^k x < y^k x < y^k y = y^{k+1}.$$

Mathematical induction is true for the following reason. Let

$$A = \{n : P_n \text{ is not true.}\}$$

If $A = \emptyset$, then P_n is true for every n. If $A \neq \emptyset$, then it has a first element (why?), say m. Can $m = 1$? No, because P_1 is true. Thus $m > 1$. Also m being the first element of A, we see $P_1, P_2, \cdots, P_{m-1}$ are true. But then P_m must also be true. Thus $m \notin A$ leading to a contradiction.

Fact 12. Fix $n \in N$. Let $x > 0$. Then there is a unique $y > 0$ such that $y^n = x$.

Proof: We show two things
(i) If $z > 0$ and $z^n < x$, then we can increase z a little bit so that its n-th power is still smaller than x. That is, there is $h > 0$ so that $(z+h)^n < x$.
(ii) If $z > 0$ and $z^n > x$, then we can decrease z a little bit so that its n-th power is still larger than x. That is, there is $h > 0$ so that $0 < z - h < z$ and $(z-h)^n > x$.

Let us see what happens if we have done this. Let

$$A = \{z > 0 : z^n < x\}$$

Is A non-empty? Yes, because if we take $z = x/(1+x)$, then $0 < z < 1$ so that $z^n < z < x$ and hence $z \in A$.

Is this set bounded above? Yes, because if $u = 1 + x$, then $u > 1$ so that $x < u < u^n$ and hence $u \notin A$. But then of course $v \notin A$ for every $v > u$; by fact 10. In other words u is an upper bound of the set A.

Thus A has lub, name it s.

Can $s^n < x$? No, because then by (i) above we can get $(s+h)^n < x$, which means $s + h \in A$. But then $s < s + h \in A$, contradicting that s is an upper bound of A.

Can $s^n > x$? No, because then by (ii), there is $h > 0$ with $0 < s - h < s$ so that $(s-h)^n > a$. This implies that $z < s - h$ for every $z \in A$. Thus $s-h$ is an upper bound of A contradicting that s is least upper bound.

There are only three possibilities: $s^n < x$, $s^n = x$, $s^n > x$. Since two of them are ruled out above, we must have $s^n = x$. completing the proof of Fact 12. Of course, the proof is complete provided we show the truth of the two statemens we made at the beginning of the proof.

Let us prove (i). Let $z > 0$ and $z^n < x$. We need $h > 0$ so that $(z+h)^n < x$. Let us make up our mind that we shall choose $0 < h < 1$.
Let us see what we want: $(z+h)^n - z^n < x - z^n$
That is $h[(z+h)^{n-1} + (z+h)^{n-2}z + \cdots + z^{n-1}] < x - z^n$.
This will be true if $hn(z+h)^{n-1} < x - z^n$.
This in turn will be true if we can choose $0 < h < 1$ so that $hn(z+1)^{n-1} < x - z^n$.
We are done with rough calculations. We say now, choose

$$0 < h < 1; \quad \text{so that} \quad h < \frac{x - z^n}{n(z+1)^{n-1}}.$$

Such a choice is possible, for example

$$h = \frac{1}{2} \min \left\{ 1, \frac{(x - z^n)}{n(z+1)^{n-1}} \right\}.$$

You can convert the above rough calculation into a proof of the fact that, with this choice of h, we do have $(z+h)^n < x$.

Let us finally prove (ii). Let $z > 0$ and $z^n > x$. We need h, so that $0 < z - h < z$ and $(z-h)^n > x$. Again let us do rough calculation.
We need $(z-h)^n - z^n > x - z^n$.
That is, $z^n - (z-h)^n < z^n - x$.
That is, $h[z^{n-1} + z^{n-2}(z-h) + \cdots + (z-h)^{n-1}] < z^n - x$.
This will be true if $0 < z - h < z$ and $hnz^{n-1} < z^n - x$.

Now rough calculation is over, we say choose

$$0 < h < z; \qquad \text{and also} \quad h < \frac{z^n - x}{nz^{n-1}}.$$

As earlier, such a h can be chosen, for example, half of minimum of the two bounds we have above. The rough calculations above can be converted to showing that with this choice of h, (ii) is verified.