improper integrals:

We shall continue our discussion of improper integrals. Let us again recall that there is nothing improper about them. It so happens that the recipe of making Riemann sums and taking limits is not done.

if the function is unbounded, we take integral over a smaller interval where the function is bounded and let the smaller interval grow to the large interval of interest. Similarly, if the function is bounded but we are integrating over an infinite interval, we calculate integrals over a smaller finite interval and take limit as the smaller interval grows to the large interval of interest.

Of course if the function is both unbounded and interval of integration is infinite, we take smaller bounded intervals where the function is bounded and let the interval grow to the large interval of integration. Of course, there are several other possibilities, we may not always integrate over intervals, we may have to integrate over union of intervals. For example, this happens for functions like

$$f(x) = \frac{1}{\sqrt{x(1-x)}} \quad 0 < x < 1; \qquad f(1) = 0;$$

$$f(x) = \frac{1}{\sqrt{(x-1)(2-x)}} \quad 1 < x < 2$$

defined on the interval $(0, 2)$. Or

$$f(x) = \frac{1}{\sqrt{1-x}} \quad 0 < x < 1; \qquad f(1) = 0;$$

$$f(x) = \frac{1}{\sqrt{(x-1)}} \quad 1 < x < 2$$

We shall not discuss all these possibilities partly because we do not need them. And also because, if you understood these simple cases you know how to define and calculate these integrals too.

(i) *Consistency:*

The first questions that arises is whether for bounded functions on bounded intervals this recipe agrees with the earlier definition. This we have already seen earlier. If you have a bounded function on a bounded interval (a, b), you can follow two procedures and they both lead to the same answer.

The first method is to take partitions of the open interval (a, b), calculate the lower and upper Riemann sums; say that the function is integrable if sup of lower sums agrees with inf of upper sums and declare the common value as integral.

Second method is to take integral over $[a + \epsilon, b - \epsilon]$ and take the limit as $\epsilon \downarrow 0$. Both lead to the same answer.

We can also take $a < \alpha < \beta < b$, calculate integral over $[\alpha, \beta]$ and then take limit as $\alpha \downarrow a$ and $\beta \uparrow b$. These are called double limits and we have not discussed. so you should carefully understand what such things mean. This means, there is a number c such that the following happens: Whenever $\epsilon > 0$ is given, there is a $\delta > 0$ so that

$$a < \alpha < a + \delta; b - \delta < \beta < b \Rightarrow |\int_\alpha^\beta f - c| < \epsilon.$$

This also leads to the same answer as above. This is how we defined, in case f were unbounded at both end points.

(ii) *usual rules:*

The second question that needs to be looked into is whether the usual simple rules —for sum, constant multiple, etc — apply. Yes, it is just a matter of using them for the case we know and applying limits. We discuss some examples, rather than stating general theorems.

(a) If f and g are unbounded but the improper integrals $\int\limits_a^b f$ and $\int\limits_a^b g$ exist then so does $\int\limits_a^b (f + g)$ and

$$\int\limits_a^b (f + g) = \int_a^b f + \int_a^b g.$$

This is easy.

$$a < \alpha < \beta < b \Rightarrow \int_\alpha^\beta (f + g) = \int_\alpha^\beta f + \int_\alpha^\beta g.$$

Now use the fact that limit of sum is sum of limits.

You must carefully notice that in the above statement, it is quite possible that f is unbounded and hence $\int f$ is an improper integral, but g is a bounded integrable function.

(b) similarly

$$\int_a^b (10f) = 10 \int_a^b f.$$

This means that if the integral on one side of the equation above exists, then so does the integral on the other side and the equality holds.

Of course, similar argument applies for integrals over infinite intervals.

(c) If $a < c < b$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

This means if both integrals on the right side exist then so does the integral on the left side; if the integral on the left side exists then so do both integrals on the right side; and then equality holds. Remember that we are assuming f is unbounded only near the end points, either both end points or only one. In other words, if we take any $a < \alpha < \beta < b$, then f is bounded on $[\alpha, \beta]$.

you can prove again by taking $a < \alpha < c < \beta < b$ and using known result over the interval $[\alpha, \beta]$ and then taking limits.

Same argument applies over infinite intervals too.

(3) *integration by parts:*

The integration by parts formula is valid even for improper integrals. Suppose that (a, b) is a bounded interval. Let F and G be two C^1 functions on this intervals. Assume that F and G have limits at a and b. That is,

$$\lim_{x \downarrow a} F(x); \qquad \lim_{x \uparrow b} F(x); \qquad \lim_{x \downarrow a} G(x); \qquad \lim_{x \uparrow b} G(x)$$

all exist. Denote these lints by $F(a)$, $F(b)$, $G(a)$, $G(b)$ respectively. Then

$$\int_a^b fG = [F(b)G(b) - F(a)G(a)] - \int_a^b Fg.$$

again this means: if the integral on one side exists, then so does the other and equality holds.

Again, if left side exists, write the equation with a, b replaced by α, β and then take limits. Similar argument applies if the right side exists.

The same argument applies even if both a , b or one of them is infinite. Of course you need to assume as above that the limits at the end points exist for the functions.

(4) *substitution:*

Recall that the method of substitution says the following.

$$\varphi : (a, b) \to (c, d)$$

strictly increasing C^1 function with bounded derivative and f is a continuous function on (c, d). Then

$$\int_a^b f(\varphi(x))\varphi'(x) = \int_c^d f(y).$$

The same result holds for improper integrals too.

There are several possibilities. For example the intervals may be bounded and f may also be bounded but φ' may be unbounded. For instance, both the intervals are $(0, 1)$; $\varphi(x) = \sqrt{x}$, $f(x) = 1$. Then of course, the right side is usual integral and the left side is improper integral.

The function f may be unbounded and both sides could be improper integrals.

If you have understood the spirit of earlier arguments, there is nothing new in proving this.

example:

$$\int_0^\infty \frac{1}{1+x^2} = \frac{\pi}{2}$$

Consider the functions

$$\varphi(x) = \tan x : [0, \pi/2) \to [0, \infty); \quad f(x) = \frac{1}{1+x^2} : [0, \infty) \to R$$

Then $\varphi'(x) = 1 + \tan^2 x$ and $f(\varphi(x))\varphi'(x) = 1$.

To calculate $\int\limits_{-\infty}^{\infty}$, either you can use the same argument. Or, you can also write this as sum of two integrals, over $(-\infty, 0)$ and $(0, \infty)$ and add. To calculate the integral over $(-\infty, 0)$ substitute $y = -x$ to see this integral is same s the integral over $(0, \infty)$.

gamma integral:

The integral

$$\int_0^\infty e^{-x} x^{a-1} dx$$

is called Gamma integral. It converges for $a > 0$ and does not converge for $a \leq 0$. For $a > 0$, the value of the integral is denoted by $\Gamma(a)$. We ahve already seen that for $a = 1$, the integrand is just e^{-x} and the integral converges and has value one.

First note that the integrand is positive. If $a \geq 1$ the integrand is bounded at zero. It is improper only because range of integration is infinite interval. We show that

$$\lim_{A \to \infty} \int_0^A e^{-x} x^{a-1}$$

exists. Denote this integral by I_A. Since integrand is positive, we conclude that I_A increases with A. It suffices to show that $\{I_A : A > 0\}$ is bounded.

If this is done, then I_A converges to the supremum. Indeed let

$$c = \sup\{I_n : n = 1, 2, 3, \cdots\}.$$

Then $I_n \uparrow c$. We argue that $I_A \uparrow c$ as follows. Let $\epsilon > 0$ be given. Choose m so that $I_m > c - \epsilon$. Then for any $A > m$, we have $I_A \geq I_m > c - \epsilon$, more precisely, $c - \epsilon \leq I_A \leq c$ for $A > m$.

Let $k > a - 1$ be any integer. We know that $e^{-x/2} x^k \to 0$ as $x \to \infty$ and hence

$$e^{-x/2} x^{a-1} \to 0; \qquad \text{as} \quad x \to \infty.$$

Say, it is smaller than one for $x \geq \alpha$. On the interval $[0, \alpha]$ the function $e^{-x/2} x^{a-1}$ is continuous and hence bounded; remember that $a \geq 1$. Thus there is a number M so that

$$e^{-x/2} x^{a-1} \leq M; \qquad \forall x \geq 0.$$

Thus

$$e^{-x}x^{a-1} \leq e^{-x/2}M; \quad \forall x \geq 0.$$

So for any $A > 0$

$$\int_0^A e^{-x}x^{a-1} \leq \int_0^A Me^{-x/2} = M2[1 - e^{-A/2}] \leq 2M.$$

showing that the set $\{I_A : A > 0\}$ is bounded.

Let $0 < a < 1$. In this case the integrand is unbounded at zero and also the range of integration is unbounded. For any $0 < \alpha < 1$, we have

$$\int_\alpha^1 e^{-x}x^{a-1} \leq \int_\alpha^1 x^{a-1} = \frac{1}{a} - \frac{\alpha^a}{a} \leq \frac{1}{a}.$$

Thus the integral converges over the interval $(0, 1)$. More precsiely, there is a number c_1 such that

$$\int_\alpha^1 e^{-x}x^{a-1} \to c_1; \quad \text{as} \quad \alpha \to 0.$$

Also for any $\alpha > 1$, noting that $a - 1 < 0$ and hence $x^{a-1} < 1$ for $x > 1$, we have

$$\int_1^\alpha e^{-x}x^{a-1} \leq \int_1^\alpha e^{-x} = e^{-1} - e^{-\alpha} \leq 1.$$

Thus the integral converges over the interval $(1, \infty)$. More precsiely, there is a number c_2 such that

$$\int_1^\alpha e^{-x}x^{a-1} \to c_2; \quad \text{as} \quad \alpha \to \infty.$$

Thus the integral converges over $(0, \infty)$. In fact, if we take $\alpha_n \to 0$ and $\beta_n \to \infty$, then after some stage $\alpha_n < 1 < \beta_n$. Thus after this stage,

$$\int_{\alpha_n}^{\beta_n} e^{-x}x^{a-1} = \int_{\alpha_n}^1 e^{-x}x^{a-1} + \int_1^{\beta_n} e^{-x}x^{a-1} \to c_1 + c_2.$$

An integration by parts (and induction) shows that for integers $n = 1, 2, \cdots$

$$\Gamma(n) = (n - 1)!.$$

More generally, integartion by parts shows that for $a > 0$,

$$\Gamma(a + 1) = a\Gamma(a).$$

This simple equation has interesting consequences. For example, when $a = -1/2$, which we are not supposed to take because the equation above holds for $a > 0$, since $\Gamma(-\frac{1}{2}+1) = \Gamma(1/2)$ is defiined we can take the above equation to define

$$\Gamma(-1/2) = -2\Gamma(1/2).$$

Now we can define $\Gamma(-3/2)$ etc. In other words the equation above which is a theorem for $a > 0$ can be taken used for defining $\Gamma(a)$ for negative values of a. Thus one can define the Gamma function for all real values *except* for $a = 0, -1, -2, \cdots$., that is, except for non-positive integers. In fact it can be extended for complex numbers a also. Such an extension is not only fun, but also has interesting consequences.

beta integral:

The integral

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

is cvalled beta integral. The integral converges for $a > 0$ and $b > 0$. It does not converge fi either $a \leq 0$ or $b \leq 0$. The value of the integral is denoted by $\beta(a, b)$. Thus this is defined only when both a and b are strictly positive.

When $a = 1, b = 1$, then the integrand is one and hence so is the value of the integral. For $a \geq 1, b \geq 1$ the integrand is a nice continuous function on the closed interval $[0, 1]$ and the integral is the usual one and is not improper. You may say it is a proper integral.

If $b \geq 1$ but $a < 1$ then the integrand is unbounded at zero, and the integral is improper. Let now $0 < a < 1$. Then for any $0 < \alpha < a$ we have

$$\int_\alpha^1 x^{a-1}(1-x)^{b-1} \leq \int_\alpha^1 x^{a-1} \leq \frac{1}{a} - \frac{\alpha^a}{a} \leq \frac{1}{a}.$$

and hence the integral converges. the same arguments as in the case of gamma integral would do. In fact the set

$$\{\int_\alpha^1 x^{a-1}(1-x)^{b-1} : 0 < \alpha < 1$$

is a bounded set and its supremum is the value of the integral from zero to one.

Let us continue assuming that $b \geq 1$ but now $a \leq 0$. Thsn

$$\int_\alpha^1 x^{a-1}(1-x)^{b-1} \geq \int_\alpha^{1/2} x^{a-1}(1-x)^{b-1} \geq (1/2)^{b-1} \int_\alpha^1 x^{a-1}$$

$$= (1/2)^{b-1}\frac{1}{a} - \frac{\alpha^a}{a} \le \frac{1}{a}.$$

Since $a < 0$ we see that this last quantity increases to infinity as $\alpha \to 0$. Thus the integral does not converge.

Exactly similar arguments apply when $a \ge 1$ and $0 < b < 1$.

When $0 < a < 1$ and $0 < b < 1$, the integrand is unbounded at both end points of the interval $(0, 1)$. We argue that integral over $(0, 1/2)$ converges and integral over $(1/2, 1)$ converges and argue as earlier (see gamma function discussion) that the integral over $(0, 1)$ convergs and actually it converges to the sum of the above two, namely, integral over $(0, 1/2)$ and integral over $(1/2, 1)$.

Finally when both $a < 0$ and $b < 0$, the integral does not converge. Ideas needed are already present in the above argument.

There is a close relation between the beta and gamma integrals. These are two important integrals that arise in practice. As you see in both cases we have positive integrands. here is one interesting specific improper integral where the integrand takes negative and positive values. This integral also appears in several discussions, especially Fourier series and integrals.

$\int \frac{\sin \mathbf{x}}{\mathbf{x}} = \frac{\pi}{\mathbf{2}}$:

1. We first show that the integral is convergent. That is,

$$I_A = \int_0^A \frac{\sin x}{x}$$

has a finite limit as $A \to \infty$. Note that the integral is improper only at infinity, th integrand is bounded near 0.

We first show that given $\epsilon > 0$, there is a number A_0 such that $\int\limits_A^B \frac{\sin x}{x} < \epsilon$ for $B > A > A_0$. Indeed take $A_0 > 4/\epsilon$. Let $B > A > A_0$. Then

$$\int_A^B \frac{\sin x}{x} = -\frac{\cos B}{B} + \frac{\cos A}{A} + \int_A^B \frac{\cos x}{x^2}$$

so that

$$\left| \int_A^B \frac{\sin x}{x} \right| \le \frac{2}{A} + \int_A^B \frac{1}{x^2} \le \frac{4}{A} < \epsilon.$$

Just as Cauchy sequences have limits, such functions have limits and is argued as follows. Take any sequence increasing to ∞, say $\{n\}$. Clearly, the above inequality shows that $\{I_n\}$ is a Cauchy sequence and hence ha s a finite limit, say c. We show that $I_A \to c$. For this, fix $\epsilon > 0$. First choose A_0 so that

$$B > A > A_0 \Rightarrow \left| \int_A^B \frac{\sin x}{x} \right| < \frac{\epsilon}{2}.$$

Choose $k > A_0$ so that

$$n \geq k \Rightarrow |I_n - c| < \frac{\epsilon}{2}.$$

Now

$$A > n_0 \Rightarrow |I_A - c| \leq |I_A - I_k| + |I_k - c| < \epsilon.$$

We now need to show that this limit equals $\pi/2$.

2. Did we evaluate any integral involving sine functions? Yes $\sin nx$ etc. But is there anything with denominator? Yes

$$\int_0^\pi \frac{\sin(n + \frac{1}{2})x}{2\sin\frac{x}{2}} = \frac{\pi}{2}.$$

This is very simple. The sine and cosine formulae give

$$2\sin\frac{x}{2}\cos kx = \sin(k + \frac{1}{2})x - \sin(k - \frac{1}{2})x.$$

Adding from $k = 1$ to $k = n$, we see

$$2\sin\frac{x}{2}\sum_1^n \cos kx = \sin(n + \frac{1}{2})x - \frac{\sin}{x}2.$$

or

$$\frac{1}{2} + \sum_1^n \cos kx = \frac{\sin(n + \frac{1}{2})x}{2\sin\frac{x}{2}}$$

integrating and noting that cosine integrals vanish, we get

$$\int_0^\pi \frac{\sin(n + \frac{1}{2})x}{2\sin\frac{x}{2}} = \frac{\pi}{2}.$$

3. It is sufficient, in view of (1), to show that

$$\int_0^{(n+\frac{1}{2})\pi} \frac{\sin x}{x} \to \frac{\pi}{2}.$$

that is, should show

$$\int_0^\pi \frac{\sin(n+\frac{1}{2})x}{x} \to \frac{\pi}{2}.$$

4. It is sufficient to show, in view of (2) and (3), that

$$\int_0^\pi \left(\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right)\sin(n+\frac{1}{2})x \to 0$$

This follows from the following two claims:

$$\int_a^b \varphi(x)\sin\lambda x \to 0 \qquad \text{for any } C^1 \text{ function } \varphi\ . \quad (*)$$

$$\left(\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right) \quad \text{a } C^1 \text{ function on } [0,\pi]. \quad (**)$$

Proof of (*) is again integration by parts as in (1).

$$\int_a^b \varphi(x)\sin\lambda x = \frac{\varphi(a)\cos\lambda a - \varphi(b)\cos\lambda b}{\lambda} + \int_a^b \varphi' \frac{\cos\lambda x}{\lambda}$$

So

$$|\int_a^b \varphi(x)\sin\lambda x| \le \frac{2M}{\lambda} + \frac{M(b-a)}{\lambda} \to 0.$$

as $\lambda \to 0$. Here M is a bound for φ and φ' on the closed bounded interval $[a,b]$.

Proof of (**) is L'Hospital's rule. Of course as it stands value of the function at zero is not defined but one shows that the function converges to zero as $x \to 0$ and hence by defining the value at zero to be ero we see it is a continuous function on $[0,\pi]$. To see that the limit at zero is zero, differentiate numerator and denominator of

$$\frac{2\sin\frac{x}{2} - x}{2x\sin\frac{x}{2}}$$

twice and see.

Similarly, one shows that the function has derivative at every point of $[0,\pi]$ and it is continuous. This is again by L'Hospital's rule.

This completes the proof.