Continuous functions:

We proved that a continuous function f defined on a closed bounded set S has a maximum and minimum. That is, first of all the values of the function form a bounded set. If M denotes the supremum of all the values of f and m denotes the infimum of all values of f then there are points x_0 and x_1 in S such that $f(x_0) = m$ and $f(x_1) = M$. Thus the infimum and supremum are actually attained. All other values $f(x)$ of the function are between $f(x_0)$ and $f(x_1)$.

Here is another useful property of continuous functions. Instead of closed bounded set, it should now be defined on an interval. If α and β are in the range of the function, then so is any value in between. In other words the function can not skip values. This is same as saying that the range of the function is an interval. This is known as the intermediate value property.

This is a very useful result as we see later. But for now, you can imagine the following. consider the function $f(x) = x$ for $-1 \leq x \leq 0$ and $f(x) = 1 + x$ for $0 < x \leq 1$. Imagine drawing the graph of this function. you can draw the curve from $x = -1$ to $x = 0$ without lifting your pen, however to proceed further you have to lift your pen at $x = 0$ and then continue. That is because the function after reaching vallue zero at $x = 0$ skips numbers a little above zero and starts assuming values beyond one. On the other hand imagine drawing the curve $f(x) = x$ for $-1 \leq x \leq 1$. You can do so without lifting your pen.

Of course, we have already seen that we can not draw graphs of many functions. But imagine, you are still not having the modern definition of function. You still have geometry to guide you and think of function as the graph. Then continuous function should mean a function whose graph you can draw continuously. But what is meant by being able to draw continuously? One way of interpreting is that, we should be able to draw the curve smoothly without breaks, or without lifting our pen.

Suppose that a function assumes a certain value α at a point and then immediately afterwards it starts assuming values larger than $\beta > \alpha$, missing all numbers in between. Then, you can feel that while drawing its graph, you need to necessarily lift your pen at that point. In other words, if you can draw a graph without lifting your pen, then the following happens: whenever the pen reaches a height (from x-axis) of α at some stage and later reaches a height β, the pen must have passed through all heights in between. This is, of course, an intuitive feeling. The intermediate value theorem make this precise and assures us that for a continuous function this holds good, whether you can draw the graph or not.

Fact: Let f be a continuous function defined on an interval. Suppose $f(a) = \alpha$ and $f(b) = \beta > \alpha$. Let γ be a number $\alpha < \gamma < \beta$ then there is a number c such that $a < c < b$ and $f(c) = \gamma$..

Proof is simple, but first note the following. Since we assumed that f is defined on an interval, every number between a and b is in the domain of f and hence it makes sense to talk of value of f at such points. Consider

$$c = \sup\{x : a \leq x \leq b; f(x) < \gamma\}.$$

Denote the set on right by S. Then $S \neq \emptyset$ because, $f(a) = \alpha < \gamma$; S is bounded above by b. Thus supremum makes sense.

$\underline{a < c}$. Since $f(a) < \gamma$, there is $\delta > 0$ so that $f(x) < \gamma$ for x in domain of f with $a - \delta < x < a + \delta$. To see this, just take $\epsilon = (\gamma - \alpha)/2$ in the definition of continuity. In any case for all points a little above a we have $f(x) < \gamma$. More precisely, if $\delta' = \min\{\delta, b - a\}$, then $\delta' > 0$ and we have $f(x) < \gamma$ for $a < x < a + \delta'$ showing that c must at least be $a + \delta'$.

$\underline{c < b}$. Since $f(b) > \gamma$ we argue as above to get $\delta' > 0$ so that $f(x) > \gamma$ for $b - \delta' < x \leq b$. Thus c is at most $b - \delta'$.

$\underline{\neg(f(c) < \gamma)}$. If $f(c) < \gamma$ then by continuity, we get $\delta' > 0$ so that $f(x) < \gamma$ for $c \leq x \leq \delta'$; showing that c can not be upper bound of S.

$\underline{\neg(f(c) > \gamma)}$. If $f(c) > \gamma$ then by continuity, we get $\delta' > 0$ so that $f(x) > \gamma$ for $c - \delta' \leq x \leq c$. Since c is upper bound of S, points above c are not in S and the present inequality shows $c - \delta'$ is also an upper bound of S contradicting that c is least upper bound of S.
Thus $f(c) = \gamma$ and the proof is complete.

You should be careful. We only said that a continuous function satisfies above property. We did not say that a function which satisfies above property is continuous.

Consider the function $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Then this function has the intermediate value property, that is, the statemnent of the theorem above holds. However the function is not continuous;

$$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi} \cdots \to 0$$

but the value of f at all these points equals 1. In fact, you can take $f(0)$ to be any number in $[-1, 1]$. Then f is not continuous but has intermediate value property.

We saw that polynomials are continuous functions. Actually, polynomials of infinite degree are also continuous. We only do a special case now. Before that, we make an observation about convergent series.

Fact: Let $\sum a_n$ be a convergent series. Given $\epsilon > 0$ there is n_0 such that $|\sum\limits_{i \geq n_0} a_i| \leq \epsilon$ for $n \geq n_0$.

Just as sums of the form $\sum\limits_{i \leq n} a_i$ are called partial sums, sums of the form $\sum\limits_{i \geq n} a_i$ are called tail sums, simply because, this is sum of a 'tail' of the series. Partial sums always make sense because they are finite sums. However tail sums make sense only when the original series converges. Because then by our observation about deletion/insertion of terms, this sum also converges.

To prove the fact stated above, fix $\epsilon > 0$, get n_0 so that $|s_n - s_m| < \epsilon$ for $n, m \geq n_0$. This will do.
Now take any $n \geq n_0$. To show that $|\sum\limits_{i \geq n} a_i| \leq \epsilon$, we only need to show each of its partial sums obey this inequality. But k-th partial sum of this series is $a_n + a_{n+1} + \cdots + a_{n+k-1}$ which is nothing but $s_{n+k-1} - s_n$.

Fact: The function $e(x)$ is a continuous function on R.

Proof: Let $x_k \to x$. Need to show $e(x_k) \to e(x)$. Let $\epsilon > 0$. We exhibit k_0 so that $|e(x_k) - e(x)| < \epsilon$ for $k \geq k_0$.

The idea is the following. The infinite sum $e(x_k)$ is close to finite sum. Since polynomials are continuous, such a finite sum is close to the corresponding finite sum of $e(x)$, which in turn is close to the infinite sum. This works out fine if we can choose 'one tail' so that the tail sum is small for all $e(x_k)$ as well as for $e(x)$. Here are the details.

Since a convergent sequence is bounded, fix C so that $|x| \leq C$ and also $|x_k| \leq C$ for all k. Since the series $\sum\limits_{i\geq 0} \frac{C^i}{i!}$ converges, choose N so that

$$\sum_{i\geq N} \frac{C^i}{i!} \leq \epsilon/4.$$

This is made possible by the previous observation. In particular we have

$$\left|\sum_{i\geq N} \frac{x^i}{i!}\right| \;\leq\; \sum_{i\geq N} \left|\frac{x^i}{i!}\right| \;\leq\; \sum_{i\geq N} \frac{C^i}{i!} \;\leq\; \epsilon/4$$

Same inequality holds for each of the x_k too. Incidentally, we used that $|\sum \alpha_i| \leq \sum |\alpha_i|$. However we have proved this only for finite sums, can we use for infinite sums? Yes, use for each partial sum and then properties of limits of sequences. Of course, you need to assume convergence of the series $\sum \alpha_i$.

We also have,

$$\sum_{i<N} \frac{x_k^i}{i!} \to \sum_{i<N} \frac{x^i}{i!}.$$

Note these are finite sums and the hypothesis $x_k \to x$ makes this true. So fix k_0 so that

$$\left|\sum_{i<N} \frac{x_k^i}{i!} - \sum_{i<N} \frac{x^i}{i!}\right| < \epsilon/4, \qquad k \geq k_0.$$

Let now $k \geq k_0$. Then

$$|e(x_k) - e(x)| \;=\; \left|\sum_{i<N} \frac{x_k^i}{i!} + \sum_{i\geq N} \frac{x_k^i}{i!} - \sum_{i<N} \frac{x^i}{i!} - \sum_{i\geq N} \frac{x^i}{i!}\right|$$

$$\leq\; \left|\sum_{i<N} \frac{x_k^i}{i!} - \sum_{i<N} \frac{x^i}{i!}\right| + \left|\sum_{i\geq N} \frac{x_k^i}{i!}\right| + \left|\sum_{i\geq N} \frac{x^i}{i!}\right| \;<\; 3\epsilon/4.$$

We used the earlier inequalities in the last step. This completes the proof.

This is one of the nice techniques. The same type of argument shows that $\sin x$ and $\cos x$, are continuous functions too. Note that they are also defined as sum of infinite series. This technique achieves much more than what we said just now. But this we see later.

Fact: If f and g are two continuous functions on R and if $f(x) = g(x)$ for every rational number x then $f(x) = g(x)$ for every real number x.

This is easy, because if we take a real number x we can get rational numbers $r_n \to x$ and we know $f(r_n) = g(r_n)$ for every n so that continuity tells us

$$f(x) = \lim f(r_n) = \lim g(r_n) = g(x).$$

The interesting point is the following. You only need to show equality for countably many rationals. Then the equality holds for all the uncountably many irrationals too. Of course, you could have taken, instead of rationals, any other countable set D satisfying the condition: given a real number x there is a sequence d_n in the set D such that $d_n \to x$.

Discontinuity:

Let us consider $f : R \to R$. If f is not continuous at a point a, then we say that it is discontinuous at a or say that a is a point of discontinuity of f. Is there anything interesting worth studying about such points. Firstly, what exactly can happen or not happen if the function has a discontinuity at a point a. As always, it is best to look at some examples and try to get a feel.

We shall describe several functions. They are simple and you should draw graphs of all these functions.
Let $f(x) = x$ for $x \leq 0$ and $f(x) = +1$ for $x > 0$.
$g(x) = -1$ for $x < 0$ and $g(x) = x$ for $x \geq 0$.
$h(x) = x - 1$ for $x < 0$, $h(x) = x + 1$ for $x > 0$ and $h(0) = 0$.

All these functions are discontinuous at the point $a = 0$. The function f fails because to the right of zero, no matter how close you look, the values of f are not close to $f(0)$. Interestingly, to the left of zero they are close. In other words given $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for $x \in (-\delta, 0)$.

The function g fails because now values $f(x)$ to the left of zero are not close to $f(0)$. However, given $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for $x \in (0, \delta)$.

For the function h we can do neither. Values to the right or to the left, no matter how close, are not close to $f(0)$.

Let us now coonsider the following functions.
$f_1(x) = x$ for $x \leq 0$ and $f_1(x) = \sin(1/x)$ for $x > 0$.
$g_1(x) = \sin(1/x)$ for $x < 0$ and $g_1(x) = x$ for $x \geq 0$.
$h_1(x) = \sin(1/x)$ for $x \neq 0$ and $h_1(0) = 0$.

The function f_1 has exactly the same property as f. But there is a difference. Eventhough f to the right of zero takes values far from $f(0)$, those values are close to 1. In other words, given $\epsilon > 0$ we can indeed find $\delta > 0$ so that $|f(x) - 1| < \epsilon$ for $0 < x < \delta$. It just so happens that the number 1 is not $f(0)$. In other words the values to the right of zero *are* close to some thing but not $f(0)$. To put it differently, as x approaches zero, staying above zero the values approach 1, it behaves smoothly. Consider the function f_1. To the right of zero it is wiggly. As x stays above zero but goes closer and closer to zero the values of the function are not approaching any particular number.

Exactly the same kind of difference is seen between g and g_1. The function h_1 is wiggly both to the right as well as to the left of zero.

Thus when a function is not continuous at a point there are several possibilities. The values may be approaching something or wiggly when you look at the right; the values may be approaching some number or wiggly to the left. Finally even if the values on a side approach a number, it may be different from the value of the function at a.

We shall take up these issues later. Of course, we do not spend too much time on discontinuities. We spend enough time to convince that some interesting things can be said even about discontinuities.

Home Assignment:

Q: How do you show that there is no bijection between N and its power set $P(N)$. Here

$$N = \{1, 2, 3, \vdots \cdots\}, \qquad P(N) = \{A : A \subset N\}.$$

We can identify $P(N)$ with the set S of infinite sequences consisting zeros and ones. If $A \subset N$, you can identify with the sequence (x_n) where $x_n = 1$ if $n \in A$ and $x_n = 0$ if $n \notin A$. Note that given any such sequence (x_n) it corresponds to the set $\{n : x_n = 1\}$. This is a bijection between $P(N)$ and S.

Thus we need ot show that there is no bijection between N and S. But this is just Cantor diagonal argument. Suppose there is a function $f : N \to S$. Let us make a sequence (z_n) as follows. We first look at the sequence $f(n)$, then look at its n-th term. If this is 1 we take $z_n = 0$; if this is zero we take $z_n = 1$. Thus z_n is different from the n-th term of the sequence $f(n)$. In particular the sequence (z_n) can not be any of the sequences $f(1), f(2), f(3), \cdots\cdots$. Thus whatever f you take, it can not be onto S. In particular, there is no bijection.

Here is another argument without passing through sequences of zeros and ones. Suppose there is a function $f : N \to P(N)$. We make a subset of N as follows. Take an n. Thus $f(n)$ is a subset of N. There are exactly two possibilities: either $n \in f(n)$ or $n \notin f(n)$. We make a set consisting of the second kind of integers. More precisely, we define $A = \{n : n \notin f(n)\}$. Thus $A \subset N$. You may get unnecessary (irrelevant) doubts like A may be empty, it may be all of N. Do not get distracted. Consider the set A. No matter what it consists of, it is a subset of N. We say that there is no k such that $f(k) = A$.

Suppose you say that there is an $k \in N$ such that $f(k) = A$. Where is this integer k. Surely, $k \in A$ or $k \notin A$.

If you say $k \in A$. We look at the criterion for an integer to be in A. Remember $n \notin f(n)$. So if you say $k \in A$, then remembering that $A = f(k)$, you conclude that $k \notin A$.
If you say $k \notin A$, then remembering again that $A = f(k)$, you are saying $k \notin f(k)$. But then by our criterion, $k \in A$, that is, $k \in f(k)$.
In either case there is a contradiction and one of them must occur. Thus our assumptiuon that there is an k such that $f(k) = A$ is false.

Thus there is no function on N onto $P(N)$. In particular, there is no bijection.

The second proof has an advantage. it works for any set!. Let S be any set. There is no function on S onto $P(S)$ and in particular there is no bijection between S and $P(S)$. By the way, $P(S)$ is the collection of all subsets of S, that is $\{A : A \subset S\}$. the proof above applies verbatim.

Closer look tells you that, in case of N, both the above proofs are exactly (yes, exactly) the same! Decipher.

Q: Some of you have confusion regarding limit point of a set and limit point of a sequence.

For example, take the sequence: $1, 2, 1, 2, 1, 2, 1, 2, \cdots$ and the set $A = \{1, 2\}$ which consists points of the sequence. Clearly both the numbers 1 and 2 are limit points of the sequence, simply because if you take any interval around 1 or 2, there are infinitely many n such that x_n is in that interval. However the set A has no limit points, because no interval around any point has infinitely many points of the set A; afterall A is a finite set. So A has no limit point.

There are two ways you could have invited this confusion. Firstly, a is limit pont of a sequence (x_n), if for any $\epsilon > 0$, the interval $(a - \epsilon, a + \epsilon)$ contains x_n for infinitely many n. You were careless and shortened this to say $(a - \epsilon, a + \epsilon)$ contains infinitely many numbers of the sequence; naturally the sequence has only finitely many numbers (namely 1 and 2); you concluded that the sequence has no limit point. You should read the two sentences carefully, they do not convey the same meaning. You have no business to replace a definition with something which is not equivalent.

Second way you invited confusion is by thinking of the sequence as the set A and since A has no limit point, you concluded that the sequence has no limit point. This is again wrong. 'sequence' and 'set' are as different as chair and table. As I mentioned, a sequence has an order: first term of the sequence, second term of the sequence etc. On the other hand when you say set, there is no order on the elements of the set. You can say that a point is in the set and another point is not in the set. But it makes no sense to say that a point is the first point of the set! So you must not identify sequence with the set of points that the sequence consists of.

Q30: If $a_{n+1}/a_n \to L > 0$, then $\sqrt[n]{a_n} \to L$. All a_n are strictly positive.

Quick way of seeing this is to say $\log a_{n+1} - \log a_n \to \log L$ and hence their averages also converge to L.

$$\frac{1}{n+1} \log a_{n+1} = \frac{n}{n+1} \frac{\log a_{n+1} - \log a_1}{n} + \frac{\log a_1}{n+1} \to \log L.$$

Of course, you need not use logarithm etc. This is 'geometric' analogue of the Cesaro limit we considered. Obviously, one is tempted to use that idea. Hre it is.

Let $\epsilon > 0$. We show after some stage $\sqrt[n]{a_n} < L + \epsilon$. Choose k so that

$$n \geq k \Rightarrow \frac{a_{n+1}}{a_n} < L + \frac{\epsilon}{2}.$$

Now for any $n > k$

$$\sqrt[n]{a_n} = \sqrt[n]{a_k \, \frac{a_{k+1}}{a_k} \, \frac{a_{k+2}}{a_{k+1}} \, \cdots \, \frac{a_n}{a_{n-1}}} \leq \sqrt[n]{a_k} \;\; \sqrt[n]{\left(L + \frac{\epsilon}{2}\right)^{-k}} \;\; \left(L + \frac{\epsilon}{2}\right).$$

Using that $\sqrt[n]{\alpha} \to 1$ choose $n_0 > k$ so that for $n \geq n_0$

$$\sqrt[n]{a_k} \;\; \sqrt[n]{\left(L + \frac{\epsilon}{2}\right)^{-k}} \;\; \leq \;\; \frac{L + \epsilon}{L + \frac{\epsilon}{2}}.$$

This will do. Similarly, you can choose n_0 so that $L - \epsilon < \sqrt[n]{a_n}$ for $n \geq n_0$.

exponentiation. (continued).

At the expense of repetition, we shall recall exponentiation and complete that discussion. The reason I defined x^a earlier already is that it is simple and should not wait till we do sequences and continuous functions. One smart and very useful way is to say

$$x^a = e^{a \log x}.$$

This appears still worse to me because you need to wait till you learn e^x and natural logarithm etc. Generally one does not pay attention to this; worse than that, one assumes he/she knows everything — the definition and all properties. For example, did you ever understand the meaning and prove the equation,

$$(\sqrt{7})^{\sqrt{2}+\sqrt{35}} = (\sqrt{7})^{\sqrt{2}} \times (\sqrt{7})^{\sqrt{35}}$$

In the earlier discussion some details were left out because we anyway need to return for a comprehensive discussion.

<u>Step 1:</u> $x \neq 0$. To define x^n for $n = 1, 2, 3, \cdots$.

This is defined by induction: $x^1 = x$ and if we have defined x^n for $n = 1, 2, \cdots k$ then we put $x^{k+1} = x^k \cdot x$. Here are two facts.

$$x^{n+m} = x^n \cdot x^m; \qquad (xy)^n = x^n y^n \qquad m, n \in N$$

Usually this is mentioned but never proved in high school. It is understandable because at the high school level, concept of 'proof' is difficult. It may even be uninteresting and counter productive. Having heard it several times from your teacher, you take it as a fact that needs no proof! The old adage — familiarity breeds contempt — fits here well.

If you never saw a proof, now is the time to write a proof of this fact. Some of you felt that to prove $x^{20+30} = x^{50}$ is simple because left side is $x \times x \times x \cdots$ 50 times and the first 20 make x^{20} and the remaining make x^{30}. Do you see why this is not acceptable? Firstly, you have only restated what is to be proved, but did not prove anything. Secondly, this 'dot dot dot' is perfect in thinking but it is not the definition we adapted. Thirdly, even if someone accepts your dot dot dot, are you going to write one sentence for each pair $(20, 30)$, $(21, 33)$, $(44, 89)$, etc. Then your proof will never end.

Step 2: $x \neq 0$. To define x^n for $n \in Z$.

For $n \in N$ it is defined above. For $n = 0$, we put $x^0 = 1$. For $n < 0$ it is defined as $x^n = (1/x)^{-n}$. Prove the law of indices.

$$x^{n+m} = x^n \cdot x^m; \qquad (xy)^n = x^n y^n \qquad m, n \in Z$$

Step 3: $x > 0$. To define $x^{1/n}$ for $n = 1, 2, 3, \cdots$.

We proved in class the existence of exactly one number $y > 0$ such that $y^n = x$. We define this y as $x^{1/n}$, also denoted as $\sqrt[n]{x}$. We proved,

$$0 < x < y \Rightarrow x^{1/n} < y^{1/n}; \qquad (xy)^{1/n} = x^{1/n} y^{1/n}.$$
$$x > 1 \Rightarrow x^{1/n} > 1; \quad x < 1 \Rightarrow x^{1/n} < 1; \quad x = 1 \Rightarrow x^{1/n} = 1.$$

More precisely,

$$x > y > 1 \Rightarrow x^{1/n} > y^{1/n} > 1; \quad x < y < 1 \Rightarrow x^{1/n} < y^{1/n} < 1.$$

This last statement is expressed by saying that n-th root is monotone increasing on the set $(1, \infty)$ and monotone decreasing on the set $(0, 1)$.

Now that we know a little more about numbers and functions, we can give a smart argument. Consider the function $f(y) = y^n$ defined on $[0, \infty)$. It is a continuous function, strictly increasing, $f(0) = 0$ and the sequence $f(1), f(2), f(3), \cdots \to \infty$. You can now use the intermediate value property of continuous functions to see that range of f is indeed all of $[0, \infty)$. In other words, given $x > 0$, there is an y so that $f(y) = x$. That such a y is unique follows from the fact that the function f is strictly increasing.

Step 4: $x > 0$. To define x^r for $r \in Q$.

Let $r = m/n$ where m, n are integers and $n \geq 1$. we put $x^r = (x^m)^{1/n}$. This makes sense because $x^m > 0$ whatever be $m \in Z$. We have proved earlier that this is a good definition, in the sense, it does not depend on how you write the rational number — 2/3 or 4/6 or 6/9 etc. Prove

$$x^{r+s} = x^r x^s; \qquad (xy)^r = x^r y^r; \qquad x^r = (1/x)^{-r}.$$

For example, if r and s are two given rationals, you can write them as fractions with commmon denominators. Since the definition does not depend on how you write the rational as a fraction, let $r = m/n$ and $s = k/n$ where $n \geq 1$. Then $r + s = (m + k)/n$.

$$x^{(r+s)} = [x^{m+k}]^{1/n} = [x^m x^k]^{1/n} = [x^m]^{1/n}[x^k]^{1/n} = x^r x^s.$$

Here the first and last equalities use the definition; second equality uses law of indices proved for integers (step 2); third equality uses what was proved above (step 3).

Similarly

$$(xy)^{m/n} = [(xy)^m]^{1/n} = [x^m y^m]^{1/n} = [x^m]^{1/n}[y^m]^{1/n} = x^r y^s.$$

You should justify each of these equalities.

$$x^{m/n} = [x^m]^{1/n} = [(1/x)^{-m}]^{1/n} = (1/x)^{-m/n}.$$

$$x > 1, r < s \Rightarrow x^r < x^s; \qquad x < 1, r < s \Rightarrow x^r > x^s.$$

This follows from the fact that if $x > 1$, then $x^m > x^k$ whenever $m > k$ and property of $(1/n)$-th power (step 3) now shows $x^{m/n} > x^{k/n}$. See how we started expressing r and s with the same denominator. Similarly, we can argue for $x < 1$.

This last statement is expressed by saying that for $x > 1$, x^r is increases with r whereas for $x < 1$ it decreases as r increases.

$$r_n \to r \quad \Rightarrow \quad x^{r_n} \to x^r.$$

More precisely, if $x > 1$, the following holds. If $r_n \uparrow r$, then $x^{r_n} \uparrow x^r$, while $r_n \downarrow r$ implies that $x^{r_n} \downarrow x^r$.

If $x < 1$, the following holds. When $r_n \uparrow r$, then $x^{r_n} \downarrow x^r$, while $r_n \downarrow r$ implies that $x^{r_n} \uparrow x^r$.

The more precise statement about $\uparrow$ and $\downarrow$ follow from monotonicity observed just now. So we only need to prove convergence. Again we only need to consider the case $r = 0$. This is because $r_n - r \to 0$ and so the specail case, if proved, tells

$$x^{r_n - r} \to 1$$

so that by properties of convergence of sequences

$$x^{r_n - r} x^r \to 1 \cdot x^r$$

and the law of indices completes proof.

To prove the special case, let $r_n \to 0$. Let $\epsilon > 0$. We show n_0 so that $1 - \epsilon < x^{r_n} < 1 + \epsilon$ for $n \geq n_0$. Since $\sqrt[n]{x} \to 1$, get k_0 so that

$$1 - \epsilon < \sqrt[k]{x} < 1 + \epsilon; \quad 1 - \epsilon < \sqrt[k]{1/x} < 1 + \epsilon; \quad k \geq k_0$$

Since $r_n \to 0$, get n_0 so that

$$-\frac{1}{k_0} < r_n < \frac{1}{k_0}; \qquad n \geq n_0.$$

This n_0 will do. Check. Remember r_n may be negative or positive.

$$x > 1 \Rightarrow x^r = \sup\{x^s : s \in Q; s \leq r\} = \sup\{x^s : s \in Q; s < r\}.$$

First equality is obvious by monotonocity. For the second equality, observe that

$$x^{r-(1/n)} \uparrow x^r$$

and each of the numbers $x^{r-(1/n)}$ is in the last set.

Step 5: $x > 1$. To define x^a for $a \in R$.

For $x > 1$, taking a clue from the last observation of the previous step, we define $x^a = \sup\{x^r : r \in Q, r \leq a\}$.

If we take any rational t, with $a - 1 < t < a$, then x^t is in the above set; if we take any rational s with $a < s < a + 1$ then x^s is an upper bound for that set. Thus supremum is sensible. Also if a happens to be rational then this definition gives the answer: x^a as defined in step 4, by monotonicity (or the last observation of step 4.

The definition is also equivalent to $x^a = \sup\{x^r : r \in Q, r < a\}$. The difference is strict inequality. This is easy. If a is irrational, there is no difference between the sets. If a is rational, this is precisely the statement proved in step 4.

$$r_n \uparrow a \quad \Rightarrow \quad x^{r_n} \uparrow x^a.$$

If a is rational, this is already done in step 4. Enough to consider a irrational. That x^{r_n} increases is by monotonicity. Let the limit be α. Let $A = \{x^r : r \text{ rational}; r < a\}$. Thus we need to show $\sup A = \alpha$. Each x^{r_n} is in A and so $x^{r_n} \leq \sup A$ for each n. Hence so is their limit giving $\alpha \leq \sup A$. By monotonicity, each x^r is smaller than some x^{r_n} and hence smaller than α. Thus α is an upper bound for A showing $\sup A \leq \alpha$. This shows $\sup A = \alpha$ as required.

$$x^{a+b} = x^a x^b; \qquad (xy)^a = x^a y^a.$$

Note that at this moment we have defined x^a only for $x > 1$. Thus in the second equality above, it is assumed that both x and y are larger than one. Then of course $xy > 1$ too. To prove the first equality, take rationals $r_n \to a$ and $s_n \to b$. From the fact proved just now and step 4, we get

$$x^{a+b} = \lim x^{r_n + s_n} = \lim x^{r_n} x^{s_n} = x^a x^b.$$

The second equality is similar.

$$a < b \quad \Rightarrow \quad x^a < x^b.$$

Since the set whose sup defines x^a increases with a the inequality $\leq$ is clear. To show strict inequality, fix rationals r, s so that $a < r < s < b$ and $x^a \leq x^r < x^s \leq x^b$.

$$a_n \uparrow a \quad \Rightarrow \quad x^{a_n} \uparrow x^a.$$

That x^{a_n} increases is clear. Proof that it increases to x^a is exactly as in the corresponding statement in step 4; need to prove special case $a_n \to 0$ etc. Similarly,

$$a_n \downarrow a \quad \Rightarrow \quad x^{a_n} \downarrow x^a.$$

x^a for $x > 0$ and $a \in R$

If $x > 1$ the above clause defines x^a. If $x = 1$, we put $x^a = 1$ whatever be a. If $0 < x < 1$, we put $x^a = (1/x)^{-a}$. This makes sense because, $1/x > 1$

and above clause applies.

If you have survived so far, you can prove the following properties. if you can not prove, return and start from step 1 and understand.

$$a_n \to a \quad \Rightarrow \quad x^{a_n} \to x^a.$$
$$x^{a+b} = x^a x^b; \qquad (xy)^a = x^a y^a.$$
$$x^a \uparrow \;\text{ as }\; a \uparrow \;\text{ for }\; x > 1; \quad x^a \downarrow \;\text{ as }\; a \uparrow \text{for }\; 0 < x < 1.$$

Theorem: Fix $x > 0$. Consider the function $f(a) = x^a$. Then $f : R \to (0, \infty)$.
(i) f is a continuous function satisfying two conditions: $f(a+b) = f(a)+f(b)$ and $f(1) = x$.
(ii) f is the only continuous function on R to $(0, \infty)$ satisfying the two conditions above.
(iii) If $x > 1$, then

$$f(a) \to \infty \quad \text{as} \quad a \to \infty; \qquad f(a) \to 0; \quad \text{as} \quad a \to -\infty.$$

This means the following. Given any number c, there is A so that $f(a) > c$ for all $a \geq A$. Similarly, givcen any number c there is an A so that $f(a) < c$ for all $a \leq A$.
If $x < 1$, then

$$f(a) \to 0; \quad \text{as} \quad a \to \infty; \qquad f(a) \to \infty; \quad \text{as} \quad a \to -\infty.$$

(iv) Suppose that f is any continuous function on R to R such that $f(a+b) = f(a) + f(b)$ holds for all $a, b \in R$. Then f necessarily takes values in $[0, \infty)$. Either it is zero for all a or it is never zero. In the second case, it must be one of the functions $f(a)$ listed above, namely, $f(a) \equiv x^a$ for some $x > 0$.

Proof: (i) Continuity was already shown above. The equation is just law of indices.
(ii) Since $f(1) = x$, the conditions imply, by induction, that $f(n) = x^n$ for $n \in N$ and then for $n \in Z$. Since $(1/2) + (1/2) = 1$ we see $[f(1/2)]^2 = x$ and since $f(1/2) > 0$ we conclude that $f(1/2)$ must be $\sqrt{x}$. You can now show by induction that $[f(1/n)]^n = f(1)$ and since $f(1/n) > 0$ conclude that $f(1/n) = x^{1/n}$. Now it follows that $f(r) = x^r$ for every $r \in Q$. Since f is given to be continuous and $a \mapsto x^a$ is shown to be continuous function and

since they agree at every $r \in Q$, we conclude that they agree at every $a \in R$.
(iii) If $x > 1$, then we knew, $x^n \to \infty$ and hence by (monotonocity in a) we conclude $x^a \to \infty$ in the sense described above. The part $a \to -\infty$ follows from noting $x^a = (1/x)^{-a}$ and again monotonocity. The case $0 < x < 1$ is similar.
(iv) Since for any a, $f(a) = [f(a/2)]^2$ we see that $f(a) \geq 0$ for every $a \in R$. Suppose that $f(1) = 0$. then $f(a) = f(1)f(a-1)$ shows that $f \equiv 0$. On the other hand when $f(1) > 0$ we have already discussed in part (ii) above.

We have regarded x^a as a function of a for every fixed number $x > 0$. We can also regard it as a function of x on $(0, \infty)$ for every fixed $a \in R$.

Theorem: Let $a \in R$. Define $g(x) = x^a$. Then $g : (0, \infty) \to (0, \infty)$. Then g is continuous and satisfies $g(xy) = g(x)g(y)$.

In defining x^a we have fixed x and defined for every a. So it is not clear as to what happens if you were to change x. We could not have fixed a and defined x^a for every x. So to understand properties when x is changed, we have to retrace our definition and make observations step by step.

step 1: Fix $n \in N$. Let $g(x) = x^n$ defined on $(0, \infty)$. Then g is continuous. We knew this already, in fact we knew polynomials are continuous, in fact, on all of R.

step2: Fix $n \in Z$. Let $g(x) = x^n$ defined on $(0, \infty)$. Then g is continuous. Follows from properties of sequences.

step 3: Fix $n \in N$. Let $g(x) = x^{1/n}$ defined on $(0, \infty)$. Then g is continuous. This can be seen in two ways, one method is what Uma suggested. Take $x_k \to x$. The sequence $(y_k) = (\sqrt[n]{x_k})$ is bounded, it has at least one limit point. Let z be a limit point. Take a subsequence, say $z_{k_1}, z_{k_2}, \cdots$ converging to z. their n-th powers must converge to z^n. But their n-th powers form a subsequence of (x_k) and hence must converge to x. Thus $z^n = x$. Of course $z \geq 0$, since all our numbers are non-negative, to start with. By definition of n-th root, we see that $z = x^{1/n}$. Thus the bounded sequence $(x^{1/n})$ has only one limit point and hence must converge to it, completing the proof.

Here is another way of proving the above result. It will be useful in other situations. Let us prove a theorem first and apply to the current problem. Later we see other uses too.

Theorem: Let $f : (0, \infty) \to (0, \infty)$ be a continuous strictly increasing function with $f(1/n) \to 0$ and $f(n) \to \infty$. Then for every $y \in (0, \infty)$ there is a unique $x \in (0, \infty)$, to be denoted $g(y)$ such that $f(x) = y$. In other words g is inverse of f. Moreover g is a strictly increasing continuous function on $(0, \infty)$.

Proof: The intermediate value theorem and hypothesis tell us that range of f is all of $(0, \infty)$. Since f is strictly increasing, it is one-to-one map. Thus there is an inverse map g. If $g(y_1) < g(y_2)$ then $y_1 = f(g(y_1)) < f(g(y_2)) = y_2$. So g is strictly increasing. Pause and think, we used proof by contradiction.

To see that g is continuous, let b and $\epsilon > 0$ be given. Let $g(b) = a$. Need to show $\delta > 0$ so that $|g(y) - g(b)| < \epsilon$ whenever $|y - b| < \delta$. There is no loss to assume that $0 < \epsilon < b$. Since $f(a) = b$, using f is strictly increasing, get $\delta_1, \delta_2 > 0$ so that $f(a - \delta_1) = b - \epsilon$ and $f(a + \delta_2) = b + \epsilon$. Take $\delta = \min\{\delta_1, \delta_2\}/2$. This will do and is easy from monotonicity.

If you apply the theorem above to the function $f(x) = x^n$, you see that $g(y) = y^{1/n}$ is a continuous function.

step 4: Fix rational r. Then the function $g(x) = x^r$ is continuous on $(0, \infty)$. Indeed, if $r = m/n$ then g is composition of two continuous functions, namely, the maps $x \mapsto x^m$ and $u \mapsto u^{1/n}$.

step 5: Let $a \in R$. Then the map $g(x) = x^a$ is continuous on $(0, \infty)$. Enough to consider the case $a > 0$. In fact if $a = 0$ this is the constant function 1 and there is nothing to do. If $a < 0$, then the function $x \mapsto x^a$ is composition of two functions, namely, $x \mapsto 1/x$ and $u \mapsto u^{-a}$. Note that $-a > 0$.

So let $a > 0$. We start with an observation. Let $\beta > \alpha > 0$ and $\epsilon > 0$. Then there is an rational r so that $|x^r - x^a| < \epsilon$ for all $x \in [\alpha, \beta]$. Let us see what happens if this is done.

Let $x_n \to x$, all of them in $(0, \infty)$. Shall show that $x_n^a \to x^a$. Let $\epsilon > 0$. Propose to exhibit n_0 so that $|x_n^a - x^a| < \epsilon$ for $n \geq n_0$. Firstly, since all x_n and x are strictly positive, you can fix $0 < \alpha < \beta$ so that all of these points are in the interval $[\alpha, \beta]$. Fix rational r so that for all points x in this interval $|x^a - x^r| < \epsilon/4$. Since we know that $x \mapsto x^r$ is continuous, fix n_0 so that

$|x_n^r - x^r| < \epsilon/4$ for all $n \geq n_0$. Clearly, for $n \geq n_0$,

$$|x_n^a - x^a| \leq |x_n^a - x_n^r| + |x_n^r - x^r| + |x^r - x^a| < 3\epsilon/4.$$

Returning to the proposal made at the beginning, we are given the following: $a > 0$; $0 < \alpha < \beta$ and $\epsilon > 0$. Need to locate rational r so that $|x^r - x^a| < \epsilon$ for all $x \in [\alpha, \beta]$. Note that $x^a \leq \alpha^a + \beta^a = M$ (say) for all $x \in [\alpha, \beta]$. In fact by monotonicity, if $x > 1$ then $x^a < \beta^a$ while if $x < 1$, then $x^a < \alpha^a$. Thus if we can get a rational r so that

$$|x^{r-a} - 1| < \epsilon/M; \quad \text{for all} \quad x \in [\alpha, \beta] \qquad (\spadesuit)$$

we are done, because then

$$|x^r - x^a| \leq x^a |x^{r-a} - 1| \leq M \frac{\epsilon}{M} = \epsilon.$$

Finally, to choose rational r satisfying ($\spadesuit$) we only need to make sure

$$|\alpha^{r-a} - 1| < \epsilon/M; \quad |\beta^{r-a} - 1| < \epsilon/M.$$

But this is easy. you only need to choose r close to a. Check.