Cantor intersection Theorem:

One way to get numbers is to produce convergent sequences. Thus we can regard a convergent sequence as defining that point to which it converges. Here is another way of producing numbers. Any decreasing sequence of closed bounded intervals with length decreasing to zero would all have exactly one common point. Thus such a sequence of intervals can be regarded as defining that unique number which is common to all those intervals.

Fact: Let $[a_n, b_n]$ for $n \geq 1$ be a sequence of intervals, where a_n, b_n are all real numbers. Suppose that the sequence of intervals is decreasing, that is, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for $n \geq 1$. If $b_n - a_n \to 0$, then there is exactly one point which is common to all these intervals.

Proof: First note the following. If $[a, b] \supset [c, d]$, then both c and d are in the interval $[a, b]$ so that $a \leq c \leq d \leq b$. Thus we have

$$a_1 \leq a_2 \leq a_3 \cdots\cdots \leq b_3 \leq b_2 \leq b_1.$$

Thus any b_i is an upper bound for the increasing sequence (a_n). So $a_n \to s$ where s is supremum of the sequence (a_n). In particular $a_n \leq s$ for each n. Moreover, each b_i being an upper bound for the sequence (a_n) we see that $s \leq b_n$ for each n. In other words $a_n \leq s \leq b_n$ for each n, showing that the point s is in all the intervals. Let x be any other point, say $s < x$. since $b_n - a_n \to 0$, fix k large so that $b_k - a_k < (x - s)/4$. Since $s \in [a_k, b_k]$ if $x \in [a_k, b_k]$ then we would have $x - s \leq b_k - a_k$, contradiction.

Just like Cantor diagonal argument which we used to show uncountability of real number system, the above argument is also powerful and appears in several contexts.

Cauchy sequences:

How do we know if a given sequence (x_n) converges? One way is to take each real number x and see if $x_n \to x$. If the answer is yes for some number x, then stop and say that the sequence converges. If the answer is no for every x, say that the sequence does not converge. But it is difficult to test every x. Moreover, in this procedure we are looking outside the sequnece.

Is there a way to tell whether the sequence converges, by just looking at the terms of the sequence. Yes. Here it is. If a sequence converges, then the terms are getting close to something. And hence, they must be getting close among themselves. the interesting point is that if the terms of a sequence are getting closer among thmeselves, then the terms are actually getting close to some thing.

Fact: A sequence (x_n) converges iff the following holds. Given $\epsilon > 0$, there is an integer $n_0 \geq 1$ such that $|x_m - x_n| < \epsilon$ for every $n, m \geq n_0$.

Proof: Let $x_n \to x$. Let $\epsilon > 0$. Get n_0 so that $|x_n - x| < \epsilon/2$ for $n \geq n_0$. Clearly, if $m, n \geq n_0$ we have

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, suppose that the sequence satisfies the given condition. We exhibit a number x and show that $x_n \to x$.

Step 1: Take $\epsilon = 1$ and get n_1 so that $|x_m - x_n| < 1$ for $m, n \geq n_1$, in particular, we have $|x_{n_1} - x_n| < 1$ for every $n \geq n_1$. Thus if $c_1 = x_{n_1} - 1$ and $d_1 = x_{n_1} + 1$, then the interval $[c_1, d_1]$ includes all points x_n of the sequence with $n \geq n_1$.

In general, take $\epsilon = 1/k$ and get n_k so that $|x_m - x_n| < 1/k$ for $m, n \geq n_k$. Thus if $c_k = x_{n_k} - 1/k$ and $d_k = x_{n_k} + 1/k$, then the interval $[c_k, d_k]$ includes all points x_n of the sequence with $n \geq n_k$.

Observe that the interval $[c_k, d_k]$ has length $2/k$ so that lengths of thses intervals are converging to zero. However,these intervals may not be decreasing. So let us put

$$a_k = \max\{c_1, c_2, \cdots, c_k\} \quad b_k = \min\{d_1, d_2, \cdots, d_k\},$$

so that

$$[a_k, b_k] = [c_1, d_1] \cap [c_2, d_2] \cap \cdots \cap [c_k, d_k].$$

Of course, it is not clear if this is meaningful, for example, is $a_k \leq b_k$? Yes, in fact if n is larger than all of $n_1, n_2, \cdots, n_k$, then x_n is in all the intervals $[c_1, d_1], \cdots, [c_k, d_k]$ showing that $a_k \leq b_k$. Not only that, the interval $[a_k, b_k]$ contains all x_n with n larger than $p_k = \max\{n_1, n_2, \cdots, n_k\}$. By construction, these intervals are decreasing and lengths converge to zero. Thus Cantor theorem applies to provide us with a unique point x.

Step 2. The sequence (x_n) converges to x. To see this, let $\epsilon > 0$. Choose k so that $[a_k, b_k] \subset (x - \epsilon, x + \epsilon)$. For example, as soon as $b_k - a_k < \epsilon/4$ this will be true because $x \in [a_k, b_k]$. But then for all $n > p_k$ we have $x_n \in [a_k, b_k]$

and hence $x_n \in (x - \epsilon, x + \epsilon)$. This completes the proof.

The property of sequences that appeared in the above result is important enough to warrant a name.

Definition: A sequence (x_n) is Cauchy sequence if the following holds. Given $\epsilon > 0$, there is n_0 so that $|x_m - x_n| < \epsilon$ for all $m, n \geq n_0$.

This means that terms of the sequence are getting closer among themselves. Given any closeness (that is, $\epsilon > 0$) there is a stage after which any two terms of the sequence are close up to the given ϵ. With this notation, the fact observed above can be restated as follows:

Fact: *A sequence converges if and only if it is a Cauchy sequence.*

This is one way to verify the convergence of a sequence. Just show it is Cauchy sequence. Sometimes, this is much simpler than searching for a number and showing that the sequence converges to that number. This is precsiely the point. Without knowing where it converges, we will be able to conclude that it converges. sometimes the limit may be a number that we have not seen before, that is, a new number is discovered.

more on lub axiom:

We used the lub axiom to show that every bounded monotone sequence converges, which helped in the Cantor intersection theorem, which, in turn, lead to the conclusion that Cauchy sequences converge. Actually all these statements are equivalent. You may feel that the lub axiom is unnatural, or you do not like it. But you may like one of the consequences that we just mentioned or you may feel it is more natural. Then you can take that as an axiom in place of the lub axiom. We make it precsie now. You can ignore this section, its intent is exactly what I said just now, nothing more.

Fact: Let us delete the lub axiom from the set of axioms for real number system. Instead, assume the following.

If $[a_n, b_n]$ for $n \geq 1$ is a decreasing sequence of intervals
and if $b_n - a_n \leq 1/n$, then
there is exactly one point common to all these intervals

Conclusion: The lub axiom holds.

Proof:

First observe that the hypothesis implies the Archimedean property. This is because we know that the intervals $[-1/(2n), +1/(2n)]$ all contain the point zero and hence can not contain any other point. Thus, given $x > 0$ we see that the number $1/x$ is outside one of these intervals which provides you an integer $2n$ larger than x.

Let S be a non-empty set which is bounded above. Need to exhibit a number s which is an upper bound of S and any other upper bound of S is larger than s.

We take an upper bound b of S and a point $a \in S$. Thus the interval $[a, b]$ has points of S. We denote this interval as $[a_1, b_1]$. That is, set $a_1 = a$ and $b_1 = b$. Note that

$[a_1, b_1]$ contains points of S and no point of S is larger than b_1.

Consider the two intervals (left half) $[a_1, (a_1 + b_1)/2]$ and (right half) $[(a_1 + b_1)/2, b_1]$. If right half has points of S take it as $[a_2, b_2]$. Otherwise take the left half as $[a_2, b_2]$. Since $[a_1, b_1]$ has points of S, we conclude that if the right half does not contain points of S, then left half must necessarily contain points of S. Thus we have

$[a_2, b_2]$ contains points of S and no point of S is larger than b_2.

If we have obtained $[a_i, b_i]$ for $1 \leq i \leq k-1$ such that for each $i \leq k-1$

(i) The interval $[a_i, b_i]$ is either the left half or right half of $[a_{i-1}, b_{i-1}]$.
(ii) $[a_i, b_i]$ contains points of S and no point of S is larger than b_i.

Then, we consider the right half of $[a_{k-1}, b_{k-1}]$ if it contains points of S, otherwise consider the left half and designate it as $[a_k, b_k]$. Then the condition above holds for $i = k$ also showing that we can continue constructing a sequence of intervals satisfying the above two conditions for every i. Eventhough their lengths may not satisfy the required hypothesis, we can get a point common to all these as follows. The n-th interval has length $(b - a)/2^{n-1}$. Using $2^n > n$ and Archimedean property get k so that k-th interval has length smaller than 1 and consider only intervals after this stage. You see the hypothesis on lengths is satisfied.

so that we have exactly one point s common to all therse in tervals.

We shall now show that s is an upper bound of S. Let $x > s$. So fix i

such that $x \notin [a_i, b_i]$. Of course, s is in this interval, x is not, $x > s$ will force $x > b_i$. But no point of S is larger than b_i. Thus $x \notin S$. Similarly, if $x < s$, then you get an i such that, $x < a_i$. Since there are points of S in this interval, x can not be an upper bound of S. Thus s is lub.

Fact: Let us delete the lub axiom from the axioms of real number system. Instead, assume the following:

Every increasing sequence bounded above converges.

Conclusion: The lub axiom holds.

Proof: Again, we start showing that Archimedean property holds. Since the sequence $(x_n = n;\ n \geq 1)$ is increasing. It can not converge because for any x, the interval $(x - 1/4, x + 1/4)$ can contain at most one integer. Thus, (x_n) is not bounded above. In other words given any $x > 0$, there is n such that $n > x$.

Now we take a non-empty set S which is bunded above. Proceed as in the earlier proof, get $[a_i, b_i]$. Observe that the sequence (a_i) is increasing and bounded above; each b_i is an upper bound. Use hypothesis to get its limit s. You only need to note that $a_i \leq s \leq b_i$ for each i to conclude that s is lub of S.

Fact: Let us delete the lub axiom from the axioms of real number system. Instead, assume the following:

(i) Archimedean property holds and (ii) every Cauchy sequence converges.

Conclusion: The lub axiom holds.

Proof: Again start with a non-empty set S bounded above and do the construction to get intervals $[a_i b_i]$ for $i \geq 1$. We only need to show that (a_i) is a Cauchy sequence. If this is done it converges to a pont s and one can show that $a_i \leq s \leq b_i$ for all i and then conclude that s is lub of S. But length of the k-th interval is $(b - a)/2^{n-1}$ and by Archimedean property this sequence of lengths converges to zero. Note that $\{a_i : i \geq n\} \subset [a_n, b_n]$. These two comments can be used to show that (a_i) is a Cauchy sequence.

discussion of HA:

Q1: $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$. Need to show $n \sum x_i y_i \geq (\sum x_i)(\sum y_i)$.

Since $(x_i - x_j)$ and $(y_i - y_j)$ have the same sign we have

$$\sum_{i,j}(x_i - x_j)(y_i - y_j) \geq 0, \text{that is,}$$

$$\sum_{i,j} x_i y_i + \sum_{i,j} x_j y_j \geq \sum_{i,j} x_i y_j + \sum_{i,j} x_j y_i.$$

The two terms on left side are same, each equals $n \sum x_i y_i$. The two terms on right side are also equal, each equals $(\sum x_i)(\sum y_i)$.

You can also prove by induction. For $n = 1$ it is trivial. You can try $n = 2$ also just to understand. Assume for n.

$$(\sum_1^{n+1} x_i)(\sum_1^{n+1} y_i) = (\sum_1^n x_i)(\sum_1^n y_i) + x_{n+1}(\sum_1^n y_i) + y_{n+1}(\sum_1^n x_i) + x_{n+1}y_{n+1}.$$

$$\leq n \sum_1^n x_i y_i + \sum_1^n (x_i y_{n+1} + y_i x_{n+1}) + x_{n+1} y_{n+1}$$

$$\leq n \sum_1^n x_i y_i + \sum_1^n (x_i y_i + x_{n+1} y_{n+1}) + x_{n+1} y_{n+1}$$

Q 18. any two non-empty open intervals have the same number of elements.

Given (a, b) with $(a < b)$ establish bijection with $(0, 1)$ by the map $F(x) = (x - a)/(b - a)$.

Q 23. To solve $x^2 - x - 1 = 0$.

We setup, $x_0 = 1$ and for $n \geq 1, x_n = 1 + (x_{n-1})^{-1}$ If $3/2 \leq x_{n-1} \leq 2$, then $1 + 1/2 \leq 1 + (x_{n-1})^{-1} \leq 1 + 2/3$.

$$|x_{n+1} - x_n| = \left| \frac{1}{x_n} - \frac{1}{x_{n-1}} \right| = \frac{|x_n - x_{n-1}|}{x_n x_{n-1}} \leq \left(\frac{2}{3}\right)^2 |x_n - x_{n-1}|.$$

Thus, letting $c = 4/9$, we have $|x_3 - x_2| \leq c|x_2 - x_1|$; $|x_4 - x_3| \leq c^2|x_2 - x_1|$ and $|x_5 - x_4| \leq c^3|x_2 - x_1|$. More generally, letting $|x_2 - x_1| = A$

$$|x_n - x_{n-1}| \leq c^{n-2} A.$$

This will help showing that (x_n) is Cauchy sequence. Indeed, take any N, then for $m > N$

$$|x_n - x_N| \leq |x_{N+1} - x_N| + |x_{N+2} - x_{N+1}| + \cdots + |x_{n-2} - x_{n-1}||x_{n-1} - x_n|$$

$$\leq \sum_{N-1}^{?} c^i A \quad \leq \sum_{N-1}^{\infty} c^i A \quad \leq \left(\frac{A}{1-c}\right) c^{N-1}.$$

If you make this last quantity small, by good choice of N, then any two terms after the N-th stage are small too.

Q 24. If you have a rational m/n, as you implement the algorithm for the expansion, each time you have smaller remainder than the previous one and sonn you will hit one and stop the process. Conversely, if the expansion is terminating, then inducting on its length, you can show it is rational.

Q 28. You already know $\sum a_i \sqrt{n} = 0$, whatever be n. Thus it suffices to show that $\sum a_i [\sqrt{n+i} - \sqrt{n}] \to 0$. Each term converges to zero and we have a fixed number of terms. You can use sum of sequences.

limit points:

it is, in general, difficult to decide whether a sequence converges or not. Even if we argue that it converge, it is difficult, in general, to find out the number to which it converges. If the sequence does not converge what can we do. Well, if it is not converging, then it is not staying close to any number; it may probably be going close to several numbers. For example consider the sequence: $+1, -1, +1, -1, \cdots$. It does not stay near any number, but it goes to $+1$ infinitely often and also goes to -1 infinitely often,. Here is another example.

$$\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{4}, \frac{1}{8}, \frac{9}{8}, \frac{1}{16}, \frac{17}{16}, \frac{1}{32}, \frac{33}{32}, \frac{1}{64}, \frac{65}{64}, \cdots\cdots$$

This sequence visits as close to zero as you please infinitely often, but does not stay there; it visits as close to one as you please infinitely often. Such numbers where the sequence goes close infinitely often are called limit points of the sequence. The interesting point is that for bounded sequences we may not be able to talk about limit unless the sequence converges; but we can always talk about limit points. The useful aspect is that we can hang on to the largest and smallest limit points to get a feel of how the sequence behaves. If both are same, that is, if there is only one limit point for a bounded sequence, then the sequence necessarily converges to that point.

Definition: Let (x_n) be a bounded sequence. A number a is a limit point of the sequence if the following holds. Given any $\epsilon > 0$, for infinitely many values of n we have $x_n \in (a-\epsilon, a+\epsilon)$. If L denotes the set of all limit points of the bounded sequence, then the following fact shows that L is a non-empty bounded set. Its supremum is called the limit superior or limsup of the sequence. The infimum of L is called the limit inferior or liminf of the sequence.

Fact: Let (x_n) be a bounded sequence. Then L, the set of its limit points is non-empty and bounded.

Proof: Since the sequence is bounded, suppose that $c \leq x_n \leq d$ for all n. If you take $a < c$, then for $\epsilon = (c-a)/4$ we see that $(a-\epsilon, a+\epsilon)$ does not contain any point of the sequence. That no number smaller than c can be a limit point. Similarly, no number larger than d can be a limit point. Thus L is bounded. We now show that $L \neq \emptyset$. This is a standard argument, imitating Cantor intersection theorem.

Denote $a_1 = c, b_1 = d$. Let $[a_2, b_2]$ be the left half or right half whichever contains x_n for infinitely many n. Since the entire sequence lives in $[a_1, b_1]$, one of these two halfs must contain infinitely many terms of the sequence. Of course, both may contain infinitely many terms of the sequence, in which case, you take the left half. If we have obtained intervals $[a_i, b_i]$ for $1 \leq i \leq k$ such that each interval is either left half or right half of the preceeding interval and each interval contains x_n for infinitely many n, then we define $[a_{k+1}, b_{k+1}]$ as the left half of $[a_k, b_k]$ if it contains infinitely many terms of the sequence; otherwise we degine it to be right half. These intervals have lengths decreasing to zero and so have exactly one number a in common.

We show that a is a limit point of the sequence. Take $\epsilon > 0$. Take k so that $[a_k, b_k] \subset (a-\epsilon, a+\epsilon)$ (this happpens if, for example, $b_k - a_k < \epsilon/4$). By construction of the intervals we see that $(a-\epsilon, a+\epsilon)$ contains infinitely many terms of the sequence. This completes the proof.

Fact: $\liminf x_n \leq \limsup x_n$.
This is easy.

Fact: If a is a limit point of (x_n), then there are integers

$$1 \leq n_1 < n_2 < n_3 < n_4 < \cdots$$

such that if we define

$$y_1 = x_{n_1}, y_2 = x_{n_2}, y_3 = x_{n_3}, \cdots$$

then $y_i \to a$.

Proof: Since infinitely many terms of the sequence are in the interval $(a-1, a+1)$, take n_1 so that $x_{n_1} \in (a-1, a+1)$. Using the same argument, pick $n_2 > n_1$ so that $x_{n_2} \in (a-1/2, a+1/2)$. Having got $n_1 < n_2 < \cdots < n_{k-1}$ so that $x_{n_i} \in (a-1/i, a+1/i)$ for $1 \leq i \leq k-1$; using the fact that infinitely many terms of the sequence are in the interval $(a-1/k, a+1/k)$ pick $n_k > n_{k-1}$ so that x_{n_k} is in this interval. This will do. Observe that all the (y_i) are in $(a-1, a+1)$; all but the first are in $(a-1/2, a+1/2)$; all but the first two terms are in $(a-1/3, a+1/3)$ etc. This shows $y_i \to a$.

on limits and convergence:

I have presented the concept of convergence as a way to discover new numbers, like e. Actually, this concept is much more basic, much more fundamental than I made it out. *Everything* that we do (well, almost everything) depends on this limit concept — continuous functions, derivative, integral etc. Even to define functions rigorously we resort to infinite sums, essentially we consider polynomials of infinite degree. Limits arise in all disciplines where maths makes its appearance.

Of course, you are having a course in physics and know how it appears. Right now, you are probably using derivatives and integrals which are defined through limits. Even if you want to discuss about the air in this lecture hall, you will not write down equations of motion for each of the paticles. Instead you will try to understand the question when there are only n particles and then take limits.

If you are doing computations and providing an algorithm for finding root of an equation or for finding minimum of a cost function over a certain region, typically your algorithm would give an iterative procedure and runs like this: start with x_1, do some thing to end up with x_2, then keep on repeating the instructions to get $x_2, x_3, \cdots$. Hope is that you will be heading to the quantity you are looking for.

Even in biological sciences it makes its appearance in an overwhelming manner. For example, you might be interested in: what happens to the species in the long run? (do they survive or become extinct) what happens to the mutant gene, in the long run, over generations (will it spread to the whole population?). Of course, most pertinent question is: what happens in the long run if we keep on using available natural resources at the current rate!

The rigorous development of Calculus was initiated by Newton and Leibnitz. But they were using infinitesimals, very small quantities etc. It was Cauchy and Bolzano who provided clear meaning of limits, though, of course, Euler, Fourier, Gauss, Poisson and several others before them already used infinite series and products. Did we not see that already Archimedes used the concept of limit, without saying so.

You should pay attention to this concept and think about it till you feel comfortable and till the concept appears *natural* to *you*.

The main difference between maths and reality is the following. In maths we can think of the following sequence: First trillion terms are zero and all later terms are equal to one. Obviously, this sequence is converging to one, but looking at the first trillion terms gives no indication of this fact. However, such a thing does not occur in nature. That is, sequences that arise in most of the practical problems do give an indication of what is happening — with its first ten thousand or so terms. This can be considered as the gift of nature to us.

series:

By virtue of the rules regarding real numbers, we can add any finitely many real numbers. But many times, we do see that infinitely many real numbers can be 'theoretically' added, that is, we feel that the sum must be a particular number.

For example, let us consider adding

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots.$$

Of course, we do not have time to add all these infinitely many numbers one by one. But, if we do start adding, we get

$$1, 2 - \frac{1}{2}, 2 - \frac{1}{2^2}, 2 - \frac{1}{2^3}, 2 - \frac{1}{2^4}, 2 - \frac{1}{2^5}, 2 - \frac{1}{2^6}, \cdots.$$

Eventhough we are unable to add all the infinitely many numbers above, we see that the successive sums are nearing 2. To put it differently, it seems reasonable to *define* the infinite sum to be 2. This is what we are going to do now.

Definition: A series is simply an expression $a_1 + a_2 + a_3 + \cdots$ or $\sum a_n$ or $\sum_{n \geq 1} a_n$.

Here the a_i are numbers. Actually, as with sequences, we are only talking about series of real numbers. The number a_n is called the n-th term of the series.

There is no meaning yet for a series. Here are some examples of series:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots.$$

$$1 + 1 + 1 + 1 + \cdots.$$

$$1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots.$$

This series is usually written as

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 \cdots.$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots.$$

The first series appears meaningful and we should get the number 2, if we add *all* those terms. It is not obvious, but true and will be proved, that the last series also has a meaning.

However, the second series has the following sums if we keep on adding the terms: $1, 2, 3, 4, 5, 6, 7, \cdots$. The value of the sum, as we keep on adding, exceeds anything that you can think of — remember the Archimedean principle, namely, given any number x, after some stage all the integers are larger than x. Since there is a particular direction in which we are going — on the number line — if we keep on and on adding, it is tempting to say that the sum is infinity. Yes, it is convenient and we shall do that later. This needs introduction of funny animal $+\infty$, and for symmetry another animal $-\infty$. But remember, it is not just a matter of introducing these characters into the drama, we should also agree upon the rules as to what we can and cannot do with these new things. Otherwise, there may come a stage when we do not understand each other.

The third series is interesting. If we keep on adding we get $1, 0, 1, 0, 1, 0, \cdots$. The addition of successive numbers is not leading us away, in a particular direction or to a particular number. The sums oscillate between zero and one. In any case, if we keep on adding the sums are not approaching any number.

Thus a series is simply a sugggestion to add certain numbers. Whether we can really add or not is a different matter. Sometimes we have a feeling as to what we should get if we really add. We shall make this feeling precise (Most mathematical definitions are just precise way of expressing some feeling we have).

Definition: Let $\sum a_n$ be a series. We define its partial sums to be the sequence of numbers,

$$s_1 = a_1; \quad s_n = a_1 + a_2 + \cdots + a_n \quad n \geq 1.$$

We say that the series $\sum a_n$ converges if the sequence (s_n) converges. If $s_n \to a$, we say that the values of the series is a. We express this by writing in any of the following ways.

$$\sum a_n = a; \quad \sum_{n \geq 1} a_n = a; \quad a_1 + a_2 + a_3 + \cdots = a.$$

You should keep in mind that we said the value of the series *is* a. We did not say the series approaches a or nearly a or close to a etc. It *is* a.

You should note the distinction between series and sequence. Both have numbers in a particular order. The order is important in both of them. If you change the order, it becomes a different thing. In a sequence, we simply have numbers standing in a line or in a row. In a series, we have numbers again in a row with a suggestion to add them. This suggestion is shown by putting the plus sign between the terms. Of course, you should know, by now, that adding includes subtraction too, adding (-5) is same as subtracting 5. Whether we can execute the suggestion is a different matter and that is why we made the next definition regarding convergence or value of a series.

So, a sequence is not a series and a series is not a sequence. Of course, you may think that a series is just a sequence with plus signs between terms, instead of the commas. This is only a typographical or superficial recognition — though correct and helps you in remembering, it is not the spirit in which these concepts are to be understood.

We shall now discuss convergence of some specific series. Of course, there is no criterion to tell exactly which series converge and which do not. Instead, we have several rules that help recognize whether a particular series converges. These do not cover all possible series you cabn think of, but will include some important series we come across in practice.

Fact:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots = 2.$$

We knew this already. We are lucky, because we could actually calculate all the partial sums and see what exactly is happening.

Fact: Let $\sum a_n$ be a series where each $a_n \geq 0$. If the partial sums are bounded, then the series converges.

Since the numbers a_n are positive (non-negative), the partial sums are increasing and by hypothesis, they are bounded above. So they converge from a theorem proved earlier.

Actually, for series of positive terms, convergence holds iff the partial sums are bounded.

Fact: The series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \cdots.$$

converges. We have, of course, seen this already earlier.

Fact: Suppose that the series $\sum |a_n|$ converges. Then the series $\sum a_n$ converges.

Let s_n be the partial sums of the series $\sum a_n$. We show that the sequence (s_n) is a Cauchy sequence. Let $\epsilon > 0$. Let t_n be the partial sums of the series $\sum |a_n|$. We know that this sequence (t_n) is a Cauchy sequence, because it converges by hypothesis. So fix n_0 so that $|t_n - t_m| < \epsilon$ for $n, m \geq n_0$. This n_0 does for the sequence (s_n) as well because of the following. Let $m > n$, then

$$|s_m - s_n| = |\sum_{n+1}^{m} a_i| \leq \sum_{n+1}^{m} |a_i| = t_m - t_n.$$

Fact: The series

$$1 \pm \frac{1}{2} \pm \frac{1}{2^2} \pm \frac{1}{2^3} \pm \frac{1}{2^4} \pm \frac{1}{2^5} \pm \frac{1}{2^6} \pm \cdots.$$

converges. Here the signs are left to your choice. You need not follow any rule, make up your mind at each term whether to put plus sign or minus sign. The resulting series converges.

This follows from the previous fact. Now you see, it is not possible to produce the number to which it converges, even if there were a rule in your putting signs. For example, if you put minus sign for all prime powers and plus for others, probably I do not know the value.

Fact: Let $0 < r < 1$. Take any *bounded* sequence of numbers (α_n). Then the series

$$\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \alpha_4 r^4 + \alpha_5 r^5 + \alpha_6 r^6 + \cdots.$$

converges.

If $|\alpha_n| \leq c$, then the series $\sum |\alpha_n| r^n$ has all partial sums bounded by $M/(1-r)$ and hence converges. Now apply an earlier fact to see that the orginal series, which has no modulus signs, also converges.

Fact: Let $(\alpha_n; n \geq 0)$ be a bounded sequence as above. Then the following series converge.

$$\alpha_0 + \alpha_1 \frac{1}{2} + \alpha_2 \frac{1}{2^2} + \alpha_3 \frac{1}{2^3} + \alpha_4 \frac{1}{2^4} + \cdots.$$

$$\alpha_0 + \alpha_1 \frac{1}{10} + \alpha_2 \frac{1}{10^2} + \alpha_3 \frac{1}{10^3} + \alpha_4 \frac{1}{10^4} + \cdots.$$

In particular, if you restrict the numbers to $\alpha_0 = 0$ and $\alpha_n \in \{0, 1, 2, 3, \cdots, 9\}$ for $n \geq 1$, in the second series, then all the resulting series converge. By taking particular choices, you can get all the numbers in the interval $[0, 1]$ and no more.

Actually when we did decimal expansion, I have already used the notion of convergence. With this discussion, it is now precise. Whatever we have done there is just showing the convergence of the expansion that we suggested.

Sometimes by observing the proof, we can get a better result.

Fact: Suppose $|a_n| \leq c_n$ for each $n \geq 1$ and the series $\sum c_n$ converges. Then, the series $\sum |a_n|$ converges and hence the series $\sum a_n$ also converges.

In fact the partial sums of the series $\sum c_n$ is bounded and hence so are the partial sums of the series $\sum |a_n|$. This can be used to complete the proof.

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