We shall now discuss two generalizations of the mean value theorem and uses of these gneralizations. First let us recall the mean value theorem.

If f is a continuous function which is differentiable at every point of (a, b), then there is a point $\theta \in (a, b)$ so that

$$[f(b) - fa)] = (b - a)f'(\theta).$$

We shall complicate it now. Let g be the function $g(x) = x$. Then note that $g' \equiv 1$. Thus the above equation takes the form

$$[f(b) - f(a)]g'(\theta) = [g(b) - g(a)]f'(\theta).$$

This statement is true in general and is called the generalized mean value theorem. Here it is

gneralized MVT:

Fact: Let f and g be two continuous functions on the interval $[a, b]$, both differentiable at every point of (a, b). Then there is a number $\theta \in (a, b)$ such that

$$[f(b) - f(a)]g'(\theta) = [g(b) - g(a)]f'(\theta).$$

Proof is simple. As earlier define

$$\varphi(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Note φ is continuous on $[a, b]$ and differentiable at every point of (a, b) with

$$\varphi'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x).$$

Also $\varphi(b) = f(b)g(a) - f(a)g(b) = \varphi(a)$. Mean value theorem applies to give a point θ where $\varphi'(\theta) = 0$. This is the required conclusion.

We use this to derive a nice theorem that goes by the name of L'Hopital's rule. This theorem helps to evaluate limits of ratios of functions. Suppose f and g are two functions and we want to evaluate $\lim\limits_{x \to a}[f(x)/g(x)]$. If f and

g converge to some finite nonzeo numbers then this is easy, the ratio of the functions converges to the ratio of the two limits. However, both f and g converge to zero we would not be able to blindly apply the theorem on limits of ratios.

L'Hopital:

Fact: Let (a, b) be a bounded interval; f and g be two differentiable functions on this interval. Assume the following.

$$\lim_{x \downarrow a} f(x) = 0; \qquad \lim_{x \downarrow a} g(x) = 0$$

$$g'(x) \neq 0 \text{ for all } x \in (a, b); \qquad \lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = \alpha \in R.$$

Then $\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \alpha$.

Proof is simple. Let $\epsilon > 0$ be given. we need to show a $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \left| \frac{f(x)}{g(x)} - \alpha \right| < \epsilon.$$

first note that the ratio makes sense for all numbers close to a. Indeed, there can be at most one point x with $g(x) = 0$. This is because, if there are two such points, x_1 and x_2 where g vanishes the usual mean value theorem says that in between at some point g' vanishes which is not possible in view of the hypothesis. since there is at most one number where g vanishes, we shall from now on consider all our points smaller than that number.

Let us choose, using hypothesis, a number $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - \alpha \right| < \epsilon/2.$$

To show that the same δ serves our purpose, let us take $a < x < a + \delta$. Take also some y with $a < y < x$. Then by generalized MVT, the ratio $[f(x) - f(y)]/[g(x) - g(y)]$ equals $f'(\theta)/g'(\theta)$ where θ is in between x and y; in particular $a < \theta < a + \delta$ so that we have

$$\left| \frac{f(x) - f(y)}{g(x) - g(y)} - \alpha \right| < \epsilon/2.$$

But this is true for every $a < y < x$. If we let $y \downarrow a$ we see $f(y) \to 0$ and $g(y) \to 0$. Thus taking limits (as $y \downarrow a$) in the above inequality we get

$$a < x < a + \delta \Rightarrow \left| \frac{f(x)}{g(x)} - \alpha \right| \le \epsilon/2.$$

This completes the proof. If you are not comfortable with the phrase $y \downarrow a$, take a sequence of points $a < y_n < x$ such that $y_n \to a$. Write the inequalities only for these y_n and take limit as $n \to \infty$.

Why did we take α to be finite? It is not necessary, we can allow $\alpha = \infty$ Thus, suppose

$$\lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = \infty$$

We show that

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \infty.$$

Let A be any given number. We shall show $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \frac{f(x)}{g(x)} > A.$$

Use hypothesis to get $\delta > 0$ so that

$$a < x < a + \delta \Rightarrow \frac{f'(x)}{g'(x)} > A.$$

As ealier this δ would do with exactly the same proof — take $a < x < a + \delta$, take any $a < y < x$, argue about the ratio $[f(x) - f(y)]/[g(x) - g(y)]$ and take $y \downarrow a$.

Can we take $\alpha = -\infty$. Yes, exactly the same argument as in the case of ∞ above would do.

Why did we take a finite? Can it be $-\infty$. Yes. But then we will be talking about lim as $x \downarrow -\infty$. So we should reformulate the argument and not use $-\infty + \delta$. For example let α be finite. Given $\epsilon > 0$ we need to show a number c so that $x < c$ implies the ratio $f(x)/g(x)$ is close to α upto ϵ. Proof is exactly the same. Same holds in case α is not finite.

Why did we take limit at a? Yes we can take limit as $x \uparrow b$ and we can allow b to be ∞ too. How about taking a point $a < c < b$; assume that f and g converge to zero as $x \to c$ but the ratio $f'(x)/g'(x) \to \alpha$ as $x \to c$, Can we still say that $f(x)/g(x)$ converges to α as $x \to c$? Yes, argue in the intervals (a, c) and (c, b).

I have stated simplest case, but all these embellishments are possible and useful too. There is one question, namely, why did you assume f and g are converging to zero? what if they converged to ∞ Instead of $0/0$ form we have ∞/∞ form. is the result true? Yes, it is true, needs different proof, shall do later.

Taylor:

We shall now generalize MVT in a different direction. But this needs the concept of higher derivatves, that is, the process of repeating differentiation. Suppose f is a function on (a, b) differentiable at each point of this interval. Thus we have a new function $f'(x)$ which associates with every point x the value of the derivative at that point. if this new function is diferentiable at a point x, we denote it by $f''(x)$, called second derivative of f at x. if the function f' is differentiable at every point x of (a, b) we can define the function f'' on (a, b) and try differentiating as long as the derivatives exist.

For example if f is a polynomial then you can talk about derivatives, they are polynomials, after some stage we get the zero function.

if $f(x) = \sin x$ and $g(x) = \cos x$, we again see that we can talk about any number of derivatives. $f' = g$; $f'' = -f$ etc.

if $f(x) = e^x$, then $f' = e^x$ again and hence differentiable any number of times.

I have introduced higher derivatives, without much ado. it is possible to motivate, at least the second derivative. Geometric picture is this. If the funcion is not a straight line, then it must be curving. How is it curving? How do you define and measure it?

Particle picture is this. If the velocity is not constant, then it is changing.

What is its rate of change? Just as first derivative (velocity) is rate of change of distance travelled, second derivative is the rate of change of velocity. This is called acceleration.

Analytical picture is this. Given a function f and a point a we wanted first a constant function which approximates f near a. More precisely, wanted constant function $\varphi(x) \equiv c$ so that $f(x) - \varphi(x) \to 0$ as $x \to a$. The function is the constant function equal to the number $f(a)$. Then we allowed a little more general functions, namely straight line function φ but wanted better approximation. More precisely, wanted $\varphi(x) = \alpha + \beta x$ so that $[f(x) - \varphi(x)]/[x - a] \to 0$ as $x \to a$. of course this, in particular implies $f(x) - \varphi(x) \to 0$ as $x \to a$. We saw $\varphi(x) = f(a) + f'(a)(x - a)$.

We can allow quadratic functions but demand more better aproximation, $[f(x) - \varphi(x)]/(x - a)^2 \to 0$. Remember, this implies, in particular, $f(x) - \varphi(x) \to 0$ and also $[f(x) - \varphi(x)]/(x - a) \to 0$. All these limits are as $x \to a$. We shall not pursue the details.

Returning to our discussion on higher derivatives, you may be able to talk of only third derivative but not fourth etc. it all depends on the function. Look at the example involving $x^8 \sin(1/x)$.

If our function f is given by a power series, then the fundamental theorem on power series tells us that f' is again given by a power series and hence f'' is again given by a power series. this goes on forever; repeated application of the fundamental theorem on power series.

To denote higher derivatives f', f'', f''' is not convenient. One uses $f^{(k)}$ to denote the k-th derivative of f. Do not confuse it with k-th power of f, pay attention to the brackets. Thus for the first derivative we use $f^{(1)}$ or f'. Sometimes we use, for notational convenience, $f^{(0)}$ for f itself, zeroth derivative.

Let us now restate MVT as follows. If f is a continuous function on $[a, b]$ which is differentiable at every point of (a, b), then there is a number $\theta \in (a, b)$ such that

$$f(b) = f(a) + (b - a)f'(\theta).$$

The value of the function at b is explained in terms of $f(a)$ and derivative at some point. suppose the function has second derivative. Can we explain $f(b)$ in terms of $f(a)$, $f'(a)$ and second derivative at some point, — probably a better explanation of $f(b)$? Yes. But before doing this, let us look at proof of the MVT.

Let

$$\varphi(x) = f(x) - f(a) - C(x - a).$$

We see $\varphi(a) = 0$. the number C is so chosen that $\varphi(b) = 0$. In other words, $C(b - a) = f(b) - f(a)$. There is a number $\theta \in (a, b)$ such that $\varphi'(\theta) = 0$. If φ is constant, then any point would do, otherwise consider a point where φ has max or min. But $\varphi'(\theta) = f'(\theta) - C$. Thus there is a number θ such that

$$(b - a)f'(\theta) = (b - a)C = f(b) - f(a)$$

In other words

$$f(b) = f(a) + (b - a)f'(\theta).$$

Here is now the extension. let f be two times differentiable function on an open interval which includes two points a and b, say, $a < b$. Then there is a number $\theta \in (a, b)$ such that

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2!}(b - a)^2 f^{(2)}(\theta).$$

Proof is exactly as above. Put

$$\varphi(x) = f(x) - f(a) - (x - a)f'(a) - C\frac{1}{2!}(x - a)^2.$$

Note that $\varphi(a) = 0$. Choose C so that $\varphi(b) = 0$. that is

$$C\frac{1}{2!}(b - a)^2 = f(b) - f(a) - (b - a)f'(a).$$

We get $\xi \in (a, b)$ so that $\varphi'(\xi) = 0$. Note that

$$\varphi'(x) = f'(x) - f'(a) - C(x - a)$$

so that $\varphi'(a) = 0$ and MVT applied to φ' gives $\theta \in (a, \xi)$ such that $\varphi^{(2)}(\theta) = 0$. But $\varphi^{(2)} = f^{(2)} - C$. Thus $f^{(2)}(\theta) = C$. Hence,

$$\frac{1}{2!}(b - a)^2 f^{(2)}(\theta) = C\frac{1}{2!}(b - a)^2 = f(b) - f(a) - (b - a)f'(a).$$

That is

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f^{(2)}(\theta).$$

Suppose that f is three times differentiable on an interval which includes $a < b$. then there is a number $\theta \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f^{(2)}(a) + \frac{1}{3!}(b-a)^3 f^{(3)}(\theta).$$

Proof is exactly as above. Define

$$\varphi(x) = f(x) - f(a) - (x-a)f'(a) - \frac{1}{2!}(x-a)^2 f^{(2)}(a) - C\frac{1}{3!}(x-a)^3.$$

$$\varphi'(x) = f'(x) - f'(a) - (x-a)f^{(2)}(a) - C\frac{1}{2!}(x-a)^2.$$

$$\varphi''(x) = f''(x) - f''(a) - C(x-a).$$

$$\varphi^{(3)}(x) = f^{(3)}(x) - C.$$

Thus $\varphi(a) = \varphi'(a) = \varphi''(a) = 0$. Choose C so that $\varphi(b) = 0$. Applying MVT successively for φ to get $\xi \in (a,b)$, then for φ' to get $\eta \in (a,\xi)$, then for φ'' to get $\theta \in (a,\eta)$. This will give the desired result.

If you have understood this, there is no difficulty in proving the following.

If f is n times differentiable in an interval which includes two points $a < b$, then there is a number $\theta \in (a,b)$ so that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f^{(2)}(a) + \frac{1}{3!}(b-a)^3 f^{(3)}(a) + \cdots$$

$$+\frac{1}{(n-1)!}(b-a)^{(n-1)} f^{(n-1)}(a) + \frac{1}{n!}(b-a)^n f^{(n)}(\theta).$$

In compact notation

$$f(b) = \sum_{0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{1}{n!}(b-a)^n f^{(n)}(\theta).$$

Recall that here we used $f^{(0)}$ for f. This is convenient notation to push it under the summation sign.

For the proof, you define

$$\varphi(x) = f(x) - \sum_{0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) - \frac{1}{n!}(x-a)^n C.$$

Note again that $f^{(0)}(a) = f(a)$. Prove by induction that for $k \le n-1$,

$$\varphi^{(k)}(x) = f^{(k)}(x) - \sum_{i=k}^{n-1} \frac{(x-a)^{i-k}}{(i-k)!} f^{(i)}(a) + \frac{1}{(n-k)!}(x-a)^{n-k} C.$$

$$\varphi^{(n-1)}(x) = f^{(n-1)}(x) - (x-a)C; \qquad \varphi^{(n)}(x) = f^{(n)}(x) - C.$$

Observe $\varphi^{(k)}(a) = 0$ for $k = 0, 1, 2 \cdots, n-1$. Now let the constant C be so chosen that $\varphi(b) = 0$. That is

$$\frac{1}{n!}(b-a)^n C = f(b) - \sum_{0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a).$$

As earlier, applying MVT for φ, φ', φ'' etc you get $\theta \in (a, b)$ so that$\varphi^{(n)}(\theta) = 0$. In other words $f^{(n)}(\theta) = C$. Thus

$$\frac{1}{n!}(b-a)^n f^{(n)}(\theta) = f(b) - \sum_{0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a).$$

This completes proof of the theorem.

Of course, there is nothing special in taking $a < b$, we have done it to fix ideas. We could have taken $a > b$. Exactly, the same formula is valid. Instead of saying $\theta \in (a, b)$ we say, θ is between a and b.

Let us look at a special case. suppose that f is a function defined o $(-r, r)$ where $r > 0$, differentiable n times. Let us take $a = 0$ and b any point of this interval. we get the following. There is a number θ between zero and b so that

$$f(b) = \sum_{0}^{n-1} \frac{b^k}{k!} f^{(k)}(0) + \frac{1}{n!} b^n f^{(n)}(\theta).$$

In other words, for every $x \in (-r, r)$, there is a number θ between zero and x; (this θ depends on the point x) so that the following holds.

$$f(x) = \sum_{0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n)}(\theta)}{n!} x^n.$$

In long hand,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n)}(\theta)}{n!}x^n,$$

for some θ between zeo and x. Of course, the number θ depends on the point x.

The above formula is called Taylor expansion of f around zero, with remainder R_n; or simply, Taylor formula with remainder.

Assume, for a moment that our function f has derivatives of all orders, and they are bounded by a number M. That is, $f^{(n)}$ exists for every n and $|f^{(n)}(x)| \leq M$ for all x in the interval $(-r, r)$ and also for all integers $n \geq 1$. For example if we take the function e^x or $\sin x$ or $\cos x$ and $r = 1000$, this holds. These are not the only functions.

Then we can keep on writing the Taylor formula for every n. Interestingly,

$$|R_n(x)| \leq M\frac{x^n}{n!} \to 0,$$

because the series $\sum(x^n/n!)$ converges. Where does this lead us? For every $x \in (-r, r)$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$

simply because if you consider the n-th partial sum on the right side, it differs from left side by an amount that converges to zero.

What does all this mean? Under the conditions we assumed, the unknown function is actually a power series, no more complicated! Once you understand what is going on, you can do better. For example, you can assume that there is a number M such that the n-th derivative on the interval $(-r, r)$ is bounded by M^n instead of M. Exactly the same proof works.

you can even do better. Assume that for any given c, with $0 < c < r$; there is a number M (depending on c) so that $|f^{(n)}(x)| \leq M^n$ for every $x \in [-c, c]$ and every n. Of course theses trivial statements look complicated at this stage and you need not bother. Just ignore this para, if you feel so.

Firstly, understand that there are functions which are differentiable exactly 30 times and no more. For such functions, you can not talk of Taylor series for $n > 30$. So let us consider only functions which are differentiable n times for every n. then the above thought process leads us to guess that probably, every such function is a power series.

There are two questions now. firstly, is the above guess true? Secondly, if the function is actually a power series, does Taylor series give us the same or some different series. Answer to the first question is in the negative. There are functions which are ininitely differentiable, but do not come from a power series. We shall see.

Answer to the second question is: yes. This is simple to see. suppose that we do have a function f given by a power series.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots; \qquad -r, x < r.$$

Clearly, $f(0) = a_0$. Using the fundamental theorem on power series, we see

$$f'(0) = a_1; \quad f''(0) = 2!a_2; \cdots, \quad f^{(k)}(0) = k!a_k.$$

Thus $f^{(k)}(0)/k! = a_k$ and Taylor series agrees with the given power series.

Let us proceed to understand the first question. We give an example of a function which has all derivatives but yet the error term in Taylor expansion does not become smaller, in other words, the infinite series formula is not valid. Let us define

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

We show that f is infinitely differentiable. As you see from the formula for f, the troublesome point is zero. if $x < 0$, then the ratio $[f(x) - f(0)]/x$ is zero. if we can show that the ratio converges to zero as $x \downarrow 0$, we can conclude $f'(0) = 0$. But when $x > 0$,

$$\frac{f(x) - f(0)}{x} = \frac{1}{x} e^{-1/x^2} \to 0,$$

which is seen by recalling that $ye^{-y^2} \to 0$ as $y \to \infty$.

Unfortunately, showing that it is 30 time differentiable at zero alone, does not lead us to show that it is 31 times differentiable. we need a formula for the 30-th derivative for points near zero to calculate the 31-st derivative at zero. This is how we do.

We claim that for each $n \geq 1$, there is a polynomial $P_n(u)$ such that

$$f^{(n)}(x) = \begin{cases} P_n(\frac{1}{x})e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

For $n = 1$, $P(u) = 2u^3$. Verify directly by calculating derivative.

Suppose it is true for n. We show for $n + 1$. For $x < 0$, since $f^{(n)} \equiv 0$ implies $f^{(n+1)} \equiv 0$. To calculate derivative at zero,

$$\lim_{x \downarrow 0} \frac{1}{x} P_n(\frac{1}{x})e^{-1/x^2} = \lim_{y \to \infty} yP_n(y)e^{-y^2} = 0.$$

For $x > 0$, by product rule,

$$\frac{d}{dx} P_n(\frac{1}{x})e^{-1/x^2} = P_n'(\frac{1}{x})(-\frac{1}{x^2})e^{-1/x^2} + P_n(\frac{1}{x})e^{-1/x^2}(\frac{2}{x^3}).$$

Thus $P_{n+1}(u) = -u^2 P_n'(u) + 2u^3 P_n(u)$. will satisfy our requirements.

This shows that the above function f is infinitely differentiable. However, since the value of the function as well as all derivatives at zero equal zero, Taylor expansion of any order will return only $R_n(x)$ which must equal $f(x)$.

Exponentiation again:

We have shown that $f(x) = x^n$ is a differentiable, but not yet shown that $f(x) = x^{\sqrt{2}}$ is differentiable. We shall now discuss the differentiability of the functions x^a.

Note that we have defined x^a for each $x > 0$ and each $a \in R$. Thus we have here many functions. For each fixed $a \in R$ we have the function $f(x) = x^a$ defined for $x \in (0, \infty)$. Also for each $x > 0$ we have the function

$g(a) = x^a$ defined for every $a \in R$. All these functions are shown to be continuous. We shall now show that these are differentiable. First we relate these functions to the exponential function.

We know that $e(x) : (-\infty, \infty) \to (0, \infty)$ is a strictly increasing function. Hence has an inverse $L(x) : (0, \infty) \to R$. We also know from the fundamental theorem on power series, that the function e is a contiuous function. Hence L is also a continuous function. We argue that L is differentiable. Let $u > 0$. Take any sequence $v_n \to u$, each $v_n > 0$. We need to calculate limit of the ratios $[L(v_n) - L(u)]/[v_n - u]$. Let $L(u) = a$ so that $e(a) = u$. Similarly, let $L(v_n) = x_n$ so that $e(x_n) = v_n$.

$$\frac{L(v_n) - L(u)}{v_n - u} = \frac{x_n - a}{e(x_n) - e(a)} \to \frac{1}{e'(a)} = \frac{1}{e(a)} = \frac{1}{u}.$$

Thus L is differentiable at the point u and $L'(u) = 1/u$.

We claim $L(uv) = L(u) + L(v)$ for $u, v > 0$. Indeed if $e(x) = u$ and $e(y) = v$, then we know that $e(x + y) = e(x)e(y) = uv$ so that $L(uv) = x + y = L(u) + L(v)$. In particular, $L(u^2) = 2L(u)$ and by induction,we have $L(u^m) = mL(u)$ for each integer $m \geq 1$. This holds for $m = 0$ also because, $e(0) = 1$ giving us $L(1) = 0$. For any integer $n \geq 1$, we have $L(u^{1/n}) = L(u)/n$ because

$$L(u) = L(u^{1/n} u^{1/n} \cdots n \text{ times}) = nL(u^{1/n}).$$

Thus we deduce that for any rational number $r = m/n$ with $m \geq 0, n \geq 1$, we have $L(u^r) = rL(u)$.

We claim that $L(1/u) = -L(u)$. Indeed, if $L(u) = x$, then $e(-x) = 1/e(x) = 1/u$ so that $L(1/u) = -x$. Thus if $r > 0$ is a positive rational number,

$$L(u^{-r}) = L(1/u^r) = -L(u^r) = -rL(u).$$

Tus $L(u^r) = rL(u)$ holds for every rational, positive or negative. If we fix a number $u > 0$, then $L(u^a)$ is a continuus function of a because it is composition of the two continuous functions $a \mapsto u^a$ and $u \mapsto L(u)$. Clearly, $a \mapsto aL(u)$ is a continuous function of a. Since these two functions agree at every rational, they agree at every real number. Thus $L(u^a) = aL(u)$ for every $a \in R$. This is true for every $u > 0$. Thus

$$L(u^a) = aL(u); \qquad u > 0, \;\; a \in R.$$

The function L is given the name log or logarithm or logarithm to the base e or natural logarithm. remember, this is defined only for positive numbers.

The equation deduced above can be restated as follows.

$$x^a = e(L(x^a)) = e(aL(x)); \quad i.e. \quad x^a = e^{a \log x}.$$

Now let us fix a number $a \in R$. consider the function $f(x) = x^a$. The equation above shows that this is composition of two functions, so that by chain rule

$$f'(x) = e'(aL(x))aL'(x) = x^a \; a \; \frac{1}{x} = ax^{a-1}.$$

Thus the formula: derivative of the function x^n is nx^{n-1} holds good for all numbers, not necessarily integers. But of course, the function x^n is defined on all of R as opposed to the function x^a which is defined only for $x > 0$.

to conclude this circle of ideas, let us consider the function $g(a) = x^a$ defined on R, where $x > 0$ is fixed. Again, this is composition of functions $a \mapsto aL(x) \mapsto e(aL(x))$ and so by chain rule

$$g'(a) = e'(aL(x))L(x) = x^a \log x.$$

(I am not including our discussion of Home assignment)