series:

If you delete, or add or alter finitely many terms in a series, then convergence is unaffected.

Fact: Let $\sum a_n$ be a convergent series.
Let $b_n = a_{1000+n}$ for $n \geq 1$. Then $\sum b_n$ converges. Here we deleted the first few terms.

Let $c_n = a_{n-1000}$ for $n > 1000$., For $n \leq 1000$ let a_n be your choice. Then $\sum c_n$ converges. Here we added a few terms at the beginning of the existing series.

Let $d_n = a_n$ for $n > 1000$ and d_n for $n \leq 1000$ be your choice. Then $\sum d_n$ converges. Here we changed the first feew terms of the series, keeping the remaining as they are.

All these three statements are proved by showing that the partial sums are Cauchy. Let (s_n) be the partial sums of the series $\sum a_n$. If (t_n) are partial sums of $\sum b_n$, then $t_n = s_{n+1000} - s_{1000}$. Others are proved in the same manner.

The last statement has the following special case. Consider the first 1000 terms of the original sequence, permute them and take as d_i. In other words, if you take a convergent series and permute the first finitely many terms, the resulting series converges; in fact, it converges to the same number as the original series.

This can be made precise as follows. Let f be a bijection of $\{1, 2, \cdots, 1000\}$ to itself. Define $d_n = a_{f(n)}$ for $n \leq 1000$ and $d_n = a_n$ for $n > 1000$. Then the series $\sum d_n$ converges. In fact if $\sum a_n = a$ then $\sum d_n = a$ as well. This is clear by noting that the partial sums for both sequences coincide as soon as $n > 1000$.

Recall that $\sum a_n$ converges if the sequence of partial sums (s_n) converges, which, in turn, is same as saying (s_n) is a Cauchy sequence. This is restated as follows.

Fact: The series $\sum a_n$ converges iff the following holds. Given $\epsilon > 0$, there

is n_0 such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$ for $m > n \geq n_0$.
In particular, if $\sum a_n$ converges then $a_n \to 0$.

The first sentence follows by observing $s_m - s_n = a_{n+1} + a_{n+2} + \cdots + a_m$. The second sentence follows by taking $m = n + 1$ in the first sentence.

Fact: Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum a_n$ converges iff the series $a_1 + 2a_2 + 2^2 a_{2^2} + 2^3 a_{2^3} + \cdots$ converges.

Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_k = a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^k a_{2^k}$. Since both the series consist of non-negative terms, their convergence is equivalent to boundedness of the partial sums. Thus the stated result follows from the following two claims.
(i) For every n, there is a k such that $s_n \leq t_k$.
(ii) For every k, there is an n such that $t_k \leq 2s_n$.

We prove (i) as follows. If $2^{k-1} \leq n < 2^k$, then

$$s_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n \leq$$

$$a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k - 1})$$
$$\leq a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^{k-1} a_{2^{k-1}} = t_{k-1}.$$

where, for the inequality we used that a_n are decreasing.

We prove (ii) as follows.

$$t_k = a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^k a_{2^k} \leq$$

$$2\{a_1 + a_2 + 2a_4 + 2^2 a_8 + \cdots + 2^{k-1} a_{2^k}\} \leq$$

$$2\{a_1 + a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})\}$$
$$= 2s_{2^k}.$$

Fact: The following series converge iff $p > 1$.

$$\sum \frac{1}{n^p}; \qquad \sum_{n \geq 2} \frac{1}{n(\log n)^p}$$

For the first series, $a_n = n^{-p}$ and the terms are decreasing.

$$2^n a_{2^n} = 2^n 2^{-np} = 2^{n(1-p)}$$

Thus $\sum 2^n a_{2^n}$ is a geometric series $\sum r^n$ where $r = 2^{1-p}$.

For the second series $a_n = n^{-1}(\log n)^{-p}$. Actually, since $log 1 = 0$, it started with $n = 2$. Instead of counting that a_2 is the first term etc, to match the notation being used, take $a_1 = 0$ and a_2 is the second term etc. What we write below will be true for $n \geq 2$.

$$2^n a_{2^n} = 2^n \frac{1}{2^n (n \log 2)^p} = \frac{1}{(\log 2)^p n^p}$$

and so this series converges iff $\sum n^{-p}$ converges.

Fact: The series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

converges to e.

This follows from our discussion of the partial sum sequence earlier. We shall see it again by another method which applies not only to this series but also for several other series.

In what follows we consider series of non-zero terms, that is $a_n \neq 0$ for each n. (Is it necessary to consider series where some a_n are zero?).

Fact: Suppose that there is an α, $0 < \alpha < 1$, such that $|a_{n+1}/a_n| < \alpha$ for all large n. Then the series $\sum |a_n|$ converges and hence $\sum a_n$ converges.

Let us say that for $n \geq k$ the stated inequality holds. Then $|a_{k+1}| \leq \alpha |a_k|$, $|a_{k+2}| \leq \alpha |a_{k+1}| \leq \alpha^2 |a_k|$. In general

$$|a_{k+n}| \leq \alpha^n |a_k|, \qquad n \geq 0.$$

Since geometric series is convergent ($|\alpha| < 1$) we see, by comparison test, the series $|a_k| + |a_{k+1}| + \cdots$converges. Adding finitely many terms does not destroy convergence. Hence $\sum |a_n|$ ocnverges.

Fact: Suppose that $\lim |a_{n+1}/a_n| < 1$. Then the series $\sum |a_n|$ is convergent and hence $\sum a_n$ converges.

If this limit is denoted by l, then hypothesis says that $0 \leq l < 1$. We can fix a number α so that $l < \alpha < 1$. Then by definition of limit, the ratios are smaller than α after some stage and the previous result applies.

Fact: If $\limsup |a_{n+1}/a_n| < 1$, then the series $\sum |a_n|$ converges.

This is simply observed by looking at the earlier argument. If this limsup is denoted by s, then $0 \leq s < 1$ and you can pick $s < \alpha < 1$. Use definition (or characterization) of limsup to conclude that after some stage the ratios are smaller than α.

As an application of this we have the following.

Fact: The following series converge for every real number x.

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots.$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots.$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\cdots.$$

The sum of the first series is denoted by $e(x)$. The sum of the second series is denoted by $\sin x$. The sum of the third series is denoted by $\cos x$. Yes, these will be identified with the functions you know.

Fact: If there is a number $\alpha > 1$ such that $|a_{n+1}/a_n| > \alpha$ after some stage, then the series $\sum a_n$ does not converge.

If this happens for $n \geq k$, then by repeated application of the hypothesis we see that $|a_{n+k}| \geq \alpha^n |a_k|$ which does not converge to zero. Remember $a_k \neq 0$.

The above facts go by the name of ratio test.

Fact: If $\limsup \sqrt[n]{|a_n|} < 1$ then the series $\sum |a_n|$ converges.

Proof: If this limsup is s, then $0 \leq s < 1$ and hence we can fix $s < \alpha < 1$. Definition of limsup now says after some stage $\sqrt[n]{|a_n|} < \alpha$. That is, $|a_n| < \alpha^n$ after some stage. Now compare with geometric series $\sum \alpha^n$ as earlier.

This result has a converse too — not exactly, but nearly.

Fact: If $\limsup \sqrt[n]{|a_n|} > 1$ then the series $\sum a_n$ does not converge.

if this limsup is s then fix $1 < \alpha < s$. Again by definition of limsup, we have, after some stage $|a_n| > \alpha^n$. Since $\alpha > 1$, we conclude that $a_n \not\to 0$.

We see that $\sum 1/n$ does not converge whereas $\sum 1/n^2$ converges and in both cases the limsup equals one. This we state as follows.

Fact: If $\limsup \sqrt[n]{|a_n|} = 1$, the series may or may not converge.

At first sight the fact above may appear like a tautology. Afterall, this is true of every series, it may or may not converge; nothing else can happen. But you should keep in mind that we are not talking of a particular series. we are talking of series satisfying some condition, namely, this limsup equals 1. There are examples of series which satisfy the condition and which converge. There are also examples of series which satisfy this condition and do not converge. Thus when limsup equals one we know for sure that no conclusion can be drawn regarding convergence without any further hypothesis.

The above facts go by the name of root test.

You see that the main ingradient in all this discussion of convergence of series is just the high school geometric series. Once the new concept of convergence is understood, you can draw rather non-trivial conclusions using just what you knew already.

In all the above results, we showed actually the convergence of the series $\sum |a_n|$, not just $\sum a_n$. This notion of convergence of the series formed by absolute values is important in applications and has a name.

Definition: A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges. Of course, then $\sum a_n$ converges too. If $\sum a_n$ converges and $\sum |a_n|$ does not converge, we say that the series $\sum a_n$ is conditionally convergent.

Thus the above tests help you in testing for absolute convergence. For example, they fail to tell you if the following series converges.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\cdots.$$

The tests above do say that this series is not absolutely convergent.

Note that if the series $\sum a_n$ is to be convergent, we should at least have $a_n \to 0$. We also know that this condition alone does not imply convergence of the series $\sum a_n$; for example the series $\sum(1/n)$ shows this. Interestingly, as soon as you know that the terms a_n are decreasing, this condition is enough to ensure convergence of the alternating series.

Fact: If $a_n \downarrow 0$, the series $a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ converges.

Such series where the terms are alternatively positive and negative are called 'alternating series'. (Just to emphasize that you should not be deceived by appearance, the series above is alternating not because you see $\pm$ signs, you need to use that the numbers a_n are positive.) Of course, the series is interesting only when all a_n are different from zero. Because, as soon as one a_n equals zero, the sum is actually finite sum and convergence issue is only theoretical. In other words, the series is theoretically infinite series, but after some stage partial sums do not change.

Proof: We show the following;

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq 0 \Rightarrow 0 \leq (a_1 - a_2 + a_3 - \cdots \pm a_k) \leq a_1.$$

Let us see what happens if this is done. Returning to our series, let s_n be its partial sums. If $m > n$, then the above conclusion tells us

$$|s_m - s_n| = |a_{n+1} - a_{n+2} + a_{n+3} - \cdots \pm a_m| \leq a_{n+1}.$$

Since $a_n \downarrow 0$, given $\epsilon > 0$, we can choose p such that $a_p < \epsilon$. The above inequality tells that after p-th stage any two partial sums differ by at most ϵ. In other words, partials sums form Cauchy sequence and hence converge.

We shall now prove the inequality claimed at the beginning of the proof. The left side equals

$$(a_1 - a_2) + (a_3 - a_4) + \cdots \geq 0$$

In fact, each bracketed term is non-negative by decreasing nature of a_n; if k is even there is nothing left over and if k is odd there is a last non-bracketed term which is positive.

To see the other inequality,

$$a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots \leq a_1.$$

Each bracketed term is positive and is being subtracted from a_1. If k is even, there is an unbracketed term a_k at the end which also appears with negative sign. This completes the proof.

There is a useful generalization of this. Suppose that $\sum b_n$ is a series with bounded partial sums and $a_n \downarrow 0$, then the series $\sum a_n b_n$ converges. If we take $\sum b_n$ to be $\sum \pm 1$ we get the special case above. this is an extremely useful generalization.

Cauchy product of series:

We shall now define the concept of product of two series. For certain useful applications, we shall now consider series as $\sum\limits_{n\geq 0} a_n$ rather than $\sum\limits_{n\geq 1} a_n$. In other words we consider series $a_0+a_1+a_2+\cdots$ instead of, $a_1+a_2+a_3+\cdots$ as has been done so far. You should not get confused. Either you can set your starting point a little back and think of zeroth term, first term etc (and zeroth partial sum, first partial sum etc). Or if you have trouble thinking like that, you can think that first term is a_0, second term is a_1 and in general the n-th term is a_{n-1}. But in the long run it will help you if you get used to the first way of thinking. Thus we have partial sums $s_0, s_1, s_2 \cdots$.

So now let $\sum\limits_{n\geq 0} a_n$ and $\sum\limits_{n\geq 0} b_n$ be two series (of real numbers). We define

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0; \qquad n \geq 0.$$

Thus $c_0 = a_0 b_0$; $c_1 = a_0 b_1 + a_1 b_0$. In general, c_n is the sum of all products $a_i b_j$ where $i+j$ adds upto n. These are finite sums and you need not worry. The series $\sum\limits_{n\geq 0} c_n$ is called the Cauchy product of the two series $\sum a_n$ and $\sum b_n$.

At first sight this looks complicated. But think of multiplying two polynomials

$$A(x) = \sum_0^p a_i t^i; \qquad B(x) = \sum_0^q b_j t^j.$$

You know that product of two polynomials is again a polynomial. So let us say the product $A(t)B(t)$ is the polynomial $C(t) = \sum\limits_0^{p+q} c_i t^i$. Then what are the coeficients? You can see $c_0 = a_0 b_0$; $c_1 = a_0 b_1 + a_1 b_0$. In general, c_n is the sum of all products $a_i b_j$ where $i+j$ adds upto n exactly as above. Of course, since a polynomial is a finite sum, when you reach $p+q$ there is only one term $a^p b^q$. In the infinite series case, there is always a_0, a_1 etc and finally a_n — all appearing in c_n.

Pretend that you have two infinite degree polynomials

$$A(t) = \sum_{n\geq 0} a_n t^n; \qquad B(t) = \sum_{n\geq 0} b_n t^n.$$

Just like in the usual polynomial case, suppose you want to multiply these two infinite degree polynomials and write it again as a polynomial by collecting powers of t, then you will exactly get $\sum c_n t^n$ where c_n are as defined above.

Fact: If the series $\sum a_n$ and $\sum b_n$ are absolutely convergent and if $\sum a_n = A$ and $\sum b_n = B$, then the Cauchy product $\sum c_n$ converges and $\sum c_n = AB$.

Proof: Let s_n, t_n and u_n be the partial sums of the series $\sum a_n$, $\sum b_n$ and $\sum c_n$ respectively. Known $s_n \to A$ and $t_n \to B$. Need to show that $u_n \to AB$. Of course, we know that $s_n B \to AB$. Thus if we can show that $u_n - s_n B \to 0$ then

$$u_n = (u_n - s_n B) + s_n B \to 0 + AB$$

as wanted. But

$$u_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0).$$

Collecting terms of a_0, a_1 etc together we get

$$u_n = a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_n t_0.$$

Subtracting from the above $s_n B = a_0 B + a_1 B + a_2 B + \cdots a_n B$, we get

$$u_n - s_n B = a_0(t_n - B) + a_1(t_{n-1} - b) + a_2(t_{n-2} - B) + \cdots + a_{n-1} t_1 + a_n t_0.$$

If n is large, the first few terms on right are small because $(t_n - B)$ is small; remember that $t_n - B \to 0$. The last few terms are small because $a_n \to 0$, remember $\sum a_n$ converges. Not only the individual terms are each small but the entire sum is small, this is where absolute convergence is used.

We wish to show that $|u_n - s_n B|$ can be made small for all large values of n. Here is how. Let $\epsilon > 0$. Let $\sum |a_n| = \alpha > 0$. Note that, if $\alpha = 0$, then each $a_i = 0$ and the conclusion is easy. Choose n_0 so that $|t_n - B| < \epsilon/(2\alpha)$ for $n \geq n_0$. This is possible because $t_n \to B$. Thus as soon as $n > n_0$ we have

$$\begin{aligned}
|u_n - s_n B| &= |\sum_{k=n_0}^{n} a_{n-k}(t_k - B)| + |\sum_{k<n_0} a_{n-k}(t_k - B)| \\
&\leq \sum_{k \geq n_0} |a_{n-k}||t_k - B| + |\sum_{k<n_0} a_{n-k}(t_k - B)| \\
&\leq \frac{\epsilon}{2\alpha} \sum_{k \geq n_0} |a_{n-k}| + |\sum_{k \leq n_0} a_{n-k}(t_k - B)|
\end{aligned}$$

$$\leq \frac{\epsilon}{2} + |\sum_{k \leq n_0} a_{n-k}(t_k - B)|$$

Note that the second sum on the right side consists of n_0 many terms and each of these terms converges to zero because $a_{n-k} \to 0$ for $k = 1, 2, \cdots n_0 - 1$. Remember n_0 is fixed. Hence we can choose $n_1 > n_0$ such that the sum is smaller than $\epsilon/2$ for all $n \geq n_1$. Thus if $n > n_1$ we see that the right side is smaller than ϵ completing the proof.

As you have seen in the proof, we used that $\sum a_n$ is absolutely convergent but did not use that $\sum b_n$ is absolutely convergent. Thus the theorem is true if one of the series is absolutely convergent. If none of them is absolutely convergent, then the Cauchy product may not converge. For example if we take both the series to be the alternating series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + - \cdots$$

then the Cauchy product does not converge. This is because the n-th term c_n $(n \geq 0)$ of the Cauchy product equals $(-1)^n \sum 1/(\sqrt{k}\sqrt{n-k})$. This is sum of $n-1$ terms each at least $2/n$ by using GM $\leq$ AM. Thus c_n does not converge to zero.

This result is very useful. For example let us take two numbers x and y and consider the series for $e(x)$ and $e(y)$. Since these are absolutely convergent, we conclude that their Cauchy product converges to $e(x)e(y)$. But computation shows that the Cauchy product is the series defining the number $e(x + y)$. Thus we conclude that

$$e(x + y) = e(x) \cdot e(y).$$

Since $e(1) = e$ by definition of the number e, we see that for every natural number $e(n) = e^n$. Since $e(0) = 1$ we see that for every integer, positive or negative, $e(n) = e^n$. For an integer $q \geq 1$

$$e(1/q) \cdot e(1/q) \cdots e(1/q) \quad (q \text{ times}) = e(1) = e.$$

By definition of q-th root, it follows that $e(1/q)$ is q-th root of e. That is,

$$e(1/q) = e^{1/q}.$$

It now follows that for every rational number r, $e(r) = e^r$. Thus

$$e(x) = e^x, \qquad x \in Q.$$

We shall show later, using continuity of the functions on both sides, that the equality holds not only for rationals but for all real numbers x.

The sine and cosine functions are also defined by series and are not, at this moment, recognizable as the good old functions of high school. However, using Cauchy product, one can show

$$\sin(x+y) = \sin x \cos y + \cos x \sin y; \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Of course, the very nature of the series shows that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ as well as $\cos 0 = 1$ and $\sin 0 = 0$.

Rearrangements:

When you add finitely many numbers, you can change the order in which you add, but you still get the same answer. How do you make this precise? Let $a_1, a_2, \cdots, a_k$ be numbers. Let π be a permutation of $\{1, 2, \cdots, k\}$. Then $\sum a_i = \sum a_{\pi(i)}$. Prove this. (After you prove this once, you can say it is easy).

Is this true for infinite series? This is the question we now answer. First we need to make precise the concept of changing the order of addition. Let $\sum\limits_{n\geq 1} a_n$ be a series, Let π be a bijection of $\{1, 2, \cdots, \}$. This is called a permutation of the natural numbers. Let $b_n = a_{\pi(n)}$. The series $\sum b_n$ is called a rearrangement of the series $\sum a_n$. Thus π being a permutation, note that each a_n appears exactly once as a b_k and no others appear. If $\pi(1) = 24$ then the first term b_1 of the new series is a_{24}. If $\pi(33) = 1$ then a_1 appears as the 33-rd term of the new series.

Fact: Let $\sum a_n$ be absolutely convergent series. if $\sum a_n = A$ and if $\sum b_n$ is a rearrangement of $\sum a_n$, then the series $\sum b_n$ converges and $\sum b_n = A$.

Proof: Let (s_n) be the sequence of partial sums of $\sum a_n$ and (t_n) be partial sums of $\sum b_n$. We know $s_n \to A$. Need to show $t_n \to A$. Let $\epsilon > 0$. We exhibit n_0 so that $|t_n - A| < \epsilon$ for all $n \geq n_0$.

To do this, first observe that, since the series $\sum |a_n|$ is convergent, its partial sums are Cauchy. So given $\epsilon > 0$, we can choose n_1 so that for $m > n \geq n_1$ we have

$$|a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \epsilon/2.$$

This being true for every $m > n$ we see

$$\sum_{i>n} |a_i| \leq \epsilon/2.$$

This is true, in particular, with $n = n_1$. Thus sum of all the $|a_i|$ with $i \geq n_1$ is at most $\epsilon/2$. In particular, if you add any (not necessarily all) of the $|a_i|$ for $i \geq n_1$ the sum is at most $\epsilon/2$.

Now starting with our $\epsilon > 0$, fix n_1 as above. By taking larger value for n_1 we can assume that $|s_n - A| < \epsilon/2$ for $n \geq n_1$. This is possible because $s_n \to A$. Choose n_0 so that

$$\{\pi(1), \pi(2), \cdots, \pi(n_0)\} \supset \{1, 2, \cdots, n_1\}.$$

You only need to see that if $1 = \pi(k_1)$, $2 = \pi(k_2)$ etc upto n_1, then take n_0 as the maximum of all these finitely many k_i. We claim that for $|t_n - A| < \epsilon$ for $n \geq n_0$. This is simple. Let $n \geq n_0$,

$$t_n = \sum_1^n b_i = \sum_1^n a_{\pi(i)} = \sum_1^{n_1} a_i + \text{ remaining sum.}$$

Last equality is from the fact that among the $\pi(i)$ all the integers upto n_1 appear. Now the first sum on right side differs from A by at most $\epsilon/2$. The second sum is addition of some of the a_i where the indices i are unknown but each index i is larger than n_1. Hence this sum in modulus is at most $\epsilon/2$.

A theorem of Riemann says that if the series is conditionally convergent, then this fails in a drastic way. Whatever number a is given, you can rearrange so that the rearranged series converges to the given value a. This is what we shall do now. The reason for doing this is the following. First, it is spectacular. Second, the idea is beautiful and execution is neat. At the same time you realize that you need good vocabulary to communicate!

So in what follows $\sum a_n$ is a conditionally convergent series.

Fact: Let $\sum a_n$ be conditionally convergent. Then the sum of all the positive terms of the sequence equals $+\infty$ and sum of all negatve terms equals $-\infty$.

Let us put $x_n = a_n$ if $a_n \geq 0$; otherwise put $x_n = 0$. Similarly, put $y_n = a_n$ if $a_n < 0$; otherwise put $y_n = 0$. Clearly we have

$$x_n + y_n = a_n; \qquad x_n - y_n = |a_n|; \qquad n \geq 1.$$

If you say that $\sum x_n$ is convergent, then $\sum y_n = \sum(a_n - x_n)$ is convergent too and then $\sum |a_n| = \sum(x_n - y_n)$ is convergent too, contradicting the hypothesis that $\sum a_n$ is conditionally convergent. Note that $\sum x_n$ is a series of positive terms so that if it is not convergent then partial sums increase to ∞.

Similarly, we argue that $\sum y_n = -\infty$

Fact: If $\sum a_n$ is a conditionally convergent series and if $\alpha \in R$ then there is a rearrangement $\sum b_n$ so that the rearrangement converges and $\sum b_n = \alpha$.

The idea is simple and is as follows. start adding positive terms of the series (keep same order) so that the sum just exceeds α and then stop; now start adding negative terms so that the total sum now just falls below α and stop; now start adding positive terms (begin with where you left off earlier) so that the total sum goes just above α and stop; then start adding negative terms (begin with where you left off earlier) so that the total sum now falls just below α and continue. You can continue forever because the positive and negative terms of the series add to infinites. You will consume all a_n because in each round you are using at least one positive term and at least one negative term.

The question is whether the new series actually converges to α. Clearly partial sums are oscillating around α, but are they converging. Yes. Note that $a_n \to 0$ because the series $\sum a_n$ is convertgent. Thus in the long run whenever you exceed α or fall below α you will do so only by a small amount, not too much, because the numbers you are adding are getting smaller. We shall now make this precise.

Let $f(1)$ be the first $i \geq 1$ such that $a_i \geq 0$. Having defined $f(1) < f(2) < \cdots < f(n-1)$, we define $f(n)$ to be the least $i > f(n-1)$ such that $a_i \geq 0$. Similarly $g(1)$ is the first i such that $a_i < 0$. In general $g(n)$ is the least $i > g(n-1)$ such that $a_i < 0$. Since the positive terms and the negative terms of $\sum a_n$ add up to infinities, we deduce that $f(n)$ and $g(n)$ are defined for all $n \geq 1$.

Each integer $i \geq 1$ appears exactly once — either as an $f(n)$ or as a $g(n)$ but not as both. $(\spadesuit)$

Convergence of $\sum a_n$ combined with the facts $f(n) \uparrow \infty$, $g(n) \uparrow \infty$ gives us the following.

$$a_{f(n)} \to 0; \qquad a_{g(n)} \to 0. \qquad (\clubsuit)$$

Let $s(n) = \sum_1^n a_{f(i)}$ and $t(n) = \sum_1^n a_{g(i)}$.

The fact observed above says

$$s_n \uparrow \infty; \qquad t_n \downarrow -\infty. \tag{$\heartsuit$}$$

We are given α and need to show a rearrangement of the series $\sum a_n$ that converges to α. We assume that $\alpha \geq \mathbf{0}$.

We pick two sequences of integers $n_1 < n_2 < \cdots$, and $m_1 < m_2 < \cdots$ such that the following hold.

(1) n_1 is the least integer with $s(n_1) > \alpha$ and m_1 is the least integer such that $s(n_1) + t(m_1) < \alpha$.

(2) For $i \geq 2$: Having selected $n_1, m_1, n_2, m_2, \cdots n_{i-1}, m_{i-1}$ we select n_i to be the least integer $> n_{i-1}$ such that $s(n_i) + t(m_{i-1}) > \alpha$ and m_i is the least integer $> m_{i-1}$ such that $s(n_i) + t(m_i) < \alpha$.

Existence of such sequences is established by induction. First choose n_1 as stated in (1) and then choose m_1 as stated in (2). Then choose n_2 as stated in (1) and then m_2 as stated in (2). That you can poceed for ever is a consequence of ($\heartsuit$).

Claim 1:

$$0 \leq s(n_1) - \alpha \leq a_{f(n_1)}; \text{ for } \ k \geq 2, \ 0 \leq [s(n_k) + t(m_{k-1})] - \alpha \leq a_{f(n_k)}.$$

Indeed if $n_1 = 1$, then $s(n_1) = a_{f(1)} > \alpha \geq 0$ so that, $0 \leq s(n_1) - \alpha \leq a_{f(n_1)}$ as stated. If $n_1 \geq 2$, then by choice of n_1, $0 \leq s(n_1 - 1) \leq \alpha < s(n_1)$ so that $0 \leq s(n_1) - \alpha \leq s(n_1) - s(n_1 - 1) = a_{f(n_1)}$ as stated. Similarly, by choice of n_k, $s(n_k - 1) + t(m_{k-1}) \leq \alpha < s(n_k) + t(m_{k-1})$, so that $0 \leq s(n_k) + t(m_{k-1}) - \alpha \leq s(n_k) - s(n_k - 1) = a_{f(n_k)}$.

Claim 2: $0 \leq \alpha - [s(n_k) + t(m_k)] \leq |a_{g(m_k)}|$.
This is proved exactly as above.

Claim 3: $|s(n_k) + t(m_{k-1}) - \alpha| \to 0$ and $|s(n_k) + t(m_k) - \alpha| \to 0$.
This follows from claims 1, 2 and ($\clubsuit$).

Here is the permutation (a one-to-one, onto map of natural numbers).

$$\begin{array}{rcll}
\pi(i) & = & f(i) & \text{for} \quad i \leq n_1 \\
& = & g(i - n_1) & \text{for} \quad n_1 + 1 \leq i \leq n_1 + m_1 \\
& = & n_1 + f(i - n_1 - m_1) & \text{for} \quad n_1 + m_1 + 1 \leq i \leq n_2 + m_1 \\
& = & m_1 + g(i - n_2 - m_1) & \text{for} \quad n_2 + m_1 + 1 \leq i \leq n_2 + m_2 \\
& = & n_2 + f(i - n_2 - m_2) & \text{for} \quad n_2 + m_2 + 1 \leq i \leq n_3 + m_2 \\
& = & \cdots &
\end{array}$$

(♠) shows that π is indeed a permutation of natural numbers. Let $u(n)$ be the n-th partial sum of the rearranged series $\sum a_{\pi(n)}$, that is, $u(n) = \sum_{i=1}^{n} a_{\pi(i)}$. Shall now show that $u(n) \to \alpha$.

First note that $u(n_1) = s(n_1)$; $u(n_1+m_1) = s(n_1)+t(m_1)$; and in general $u(n_k + m_{k-1}) = s(n_k) + t(m_{k-1})$ and $u(n_k + m_k) = s(n_k) + t(m_k)$. That these partial sums are as stated follows from the definition of permutaton π. Put $k_1 = n_1, k_2 = n_1 + m_1, k_3 = n_2 + m_1, k_4 = n_2 + m_2, k_5 = n_3 + m_2, k_6 = n_3 + m_3, \cdots$. Then Claim 3 shows that $u_{k_i} \to \alpha$. To show that the entire sequence $u_n \to \alpha$ observe that for $k_i \leq j \leq k_{i+1}$ we have, by construction, $u_{k_i} \leq u_j \leq u_{k_{i+1}}$. This completes the proof when $\alpha \geq 0$.

In case $\alpha \leq 0$, first pick m_1 (instead of n_1) such that $t(m_1) < \alpha$ and then choose n_1 etc. Same proof works. of course, even if $\alpha < 0$, the same construction as above can be used. This completes the proof.

Some of you asked if every rearrangement converges to something. the answer is no. In fact the theorem is more spectacular than the above. Let $-\infty \leq \alpha \leq \beta \leq +\infty$ are given. There is a rearrangement such that if u_n is the n-th partial sum of the rearranged series, then $\liminf u_n = \alpha$ and $\limsup u_n = \beta$.

Exactly the same construction and argument as above works when α and β are finite. You need to go beyond β and below α at each stage. If $\alpha = -\infty$ and $\beta = \infty$ you do the following. You go above n and below $-n$ in the n-th round. If $\alpha = \beta = \infty$, you proceed as follows. at the n-th round go beyond n but then take only one negative term. Other cases are similar.

infinities:

It is time to introduce the objects $+\infty$ and $-\infty$. This is only for convenience. You should keep in mind that every statement made using $\pm\infty$ can also be made, conveying the same meaning, but without using these symbols.

We start with the picture first as to how these objects fit with our picture of the real number system. We put the object $+\infty$ at the right end of the real number line and the object $-\infty$ at the left end of the real number line. So how to operate with these objects and what are the rules to which we agree upon now. First, we make a notational agreement. Just as we write 4 for $+4$ and if we want to say negative 4, we write -4, now also we do the same. Instead of writing $+\infty$, we just write ∞. So when I write ∞, you do not ask me which infinity: plus or minus? (Just as, when I write 4, you do not ask me whether I mean $+4$ or -4).

Rule 1 (order): We agree (as the picture suggests) $x < \infty$ for all $x \in R$ and $-\infty < x$ for all $x \in R$. We agree to say, as is sensible now, $-\infty < +\infty$.

Every set bounded above has a supremum. So far, a set which is not bounded above has no supremum. Now we make a definition. A set which is not bounded above also has a supremum and it is ∞. Similarly, so far a set which not bounded below has no infimum. Now we agree to say that a set which is not bounded below also has a infimum and it is $-\infty$.

Let S be a non-empty set.

(i) For a non-empty set bounded above, its supremum is the least upper bound. In symbols,

$$s = \sup S \ \leftrightarrow \ [\forall x \in S)(x \leq a)]\&[\forall x \in S, x \leq b \to a \leq b].$$

We continue to have this. Remember if S is a bounded set of real numbers the above statement is correct by definition. The symbols express just this fact, namely, supremum is an upper bound and any other upper bound is larger than this. Thus if you take the set S of all numbers which are larger than 5, we now have ∞ to be its supremum. The right side of the above statement is still correct. Unfortunately, we do not say, in words, that ∞ is least upper bound of S, though you are invited to imagine so.

The reason we do not say so is the following: we still reserve our right to say this set S, consisting of all numbers larger than 5, has no upper bound. So the question of least upper bound does not arise at all. You might ask why do we do this. Why not say, if a set is not bounded above then ∞ is its upper bound. Yes, you are invited to imagine and say so. But, even for the interval $[0, 1]$, you agree that ∞ is an upper bound, simply because for every point x of this set we have $x < \infty$. Thus saying that ∞ is an upper bound for a non-empty set is a tautology (what is a tautology?) and conveys no information. On the other hand 'S is not bounded above' conveys information. (Imagine the oxymoron, if S has no upper bound, then ∞ is its least upper bound).

(ii) In a similar manner for sets bounded below, infimum is its greatest lower bound. In symbols,

$$l = \inf S \leftrightarrow \ [\forall x \in S)(a \leq x)]\&[\forall x \in S, b \leq x \to b \leq a].$$

Commenst similar to above apply. For example, $-\infty$ is infimum of the set S of all numbers smaller than 5. Of course we do not say that it is the greatest lower bound.

Also the characterization of supremum and infimum remain correct provided, we formulate carefully. For a bounded set S, the characterization was the following:

$$s = \sup S \leftrightarrow [\forall x \in S, x \leq s] \& [\forall \epsilon > 0, \exists x \in S, x > s - \epsilon].$$

An equivalent formulation, not using ϵ, is the following

$$s = \sup S \leftrightarrow [\forall x \in S, x \leq s] \& [\forall b < s, \exists x \in S, x > b].$$

This formulation is still correct even if the supremum of the set is ∞.

Similarly,

$$l = \inf S \leftrightarrow [\forall x \in S, l \leq x] \& [\forall b > l, \exists x \in S, x < b].$$

Rule 2 (addition): $x + \infty = \infty$ for all $x \in R$ as well as for $x = \infty$. $x - \infty = -\infty$ for all $x \in R$ as well as for $x = -\infty$. We do not talk about $\infty - \infty$. The reason is simple, certain things that we know are true still remain true even with this convention. No meaning of $\infty - \infty$ will validate certain existing statements. Also $-(\infty) = -\infty$.

For example A and B are (non-empty) sets with supremums a and b, then the set $C = \{x + y : x \in A, y \in B\}$ has supremum to be $a + b$. We knew this if the numbers a and b are real. It remains true even if a and b are infinities. The sets being non-empty, the supremum can not be $-\infty$. For example, suppose $a \in R$ and $b = \infty$. This means that the set B is not bounded above. It is easy to see that C is not bounded above. similarly, if $a = \infty = b$, then C is not bounded above. similar remarks apply to infimums.

Let us make a definition. A sequence (x_n) converges to ∞, in symbols, $x_n \to \infty$ if given any number α, there is n_0 such that $x_n \geq \alpha$ for all $n \geq n_0$. This stands to reason. Afterall, ∞ is beyond all numbers. So x_n approaches ∞ should mean that x_n eventually exceeds any given number. Similarly, $x_n \to -\infty$ means that given any number α there is an n_0 such that $x_n < \alpha$ for all $n \geq n_0$.

The statement $x_n \to a$ and $y_n \to b$ implies $x_n + y_n \to a + b$ remains valid even if the limits are infinities, unless they are of infinities of opposite signs.

This is simple to prove. For example if $x_n \to -4$ and $y_n \to \infty$ then given any number α , we can get n_0 such that $x_n > -5$ for $n \geq n_0$ and and an n_1 such that $y_n > \alpha + 5$ holds for $n \geq n_1$. If n is larger than both n_0 and n_1 then, $x_n + y_n > \alpha$ holds.

When $x_n \to \infty$ and $y_n \to -\infty$ holds, you can not say anything about $x_n + y_n$, in general. For example, $x_n = n$ and $y_n = -n$ tells $x_n + y_n \to 0$. $x_n = n$ and $y_n = -n^2$ says that $x_n + y_n \to -\infty$. If $x_n = n^2$ and $y_n = -n$ then $x_n + y_n \to \infty$.

Simailrly, the statement $x_n \to a$ implies $-x_n \to -a$ remains correct even if a is an infinity. Thus $x_n \to a$ and $y_n \to b$ implies $x_n - y_n \to a - b$, unless a and b are the same infinity (both ∞ or both $-\infty$).

Rule 3 (multiplication): $x \times \infty$ equals $+\infty$ if $x > 0$ or $x = \infty$; whereas it equals $-\infty$ if $x < 0$ or $x = -\infty$. Simialrly, $x \times (-\infty)$ equals $-\infty$ if $x > 0$ or $x = \infty$; whereas it equals ∞ if $x < 0$ or $x = -\infty$. What we did not define is $\infty \times 0$ and $(-\infty) \times 0$. Actually there are reasons to define them to be zero, but right now we do not define them.

The fact that $x_n \to a$ and $y_n \to b$ does imply $x_n \times y_n \to a \times b$ in all the cases when $a \times b$ is defined. The reason we did not define product of zero and infinity is the following. $x_n = 1/n$ and $y_n = n$ shows $x_n \cdot y_n \to 1$; whereas $x_n = 1/n^2$ and $y_n = n$ shows that $x_n \cdot y_n \to 0$; $x_n = 1/n$ and $y_n = n^2$ shows that $x_n \cdot y_n \to \infty$.

Finally, let us discuss limit points. For a sequence (x_n) and $a \in R$ we use the same definition as earlier to say a is a limit point of the sequence. Namely, for every $\epsilon > 0$, we have $x_n \in (a-\epsilon, a+\epsilon)$ for in finitely many values of n. We say that ∞ is a limit point if the sequence is not bounded above. This is same as saying that for every α, there are infinitely many n such that $x_n > \alpha$. Similarly, we say $-\infty$ is a limit point if the sequence is not bounded below, equivalently, given any α, there are infinitely many values of n such that $x_n < \alpha$.

Observe that the set L of limit points of a sequence (x_n) is non-empty. If it is not bounded below then $-\infty$ is a limit point. If it is not bounded above then ∞ is a limit point. If it is blounded above and also below (that is, bounded), then we already showed that there is at least one limit point. As a consequence, $\sup L$ and $\inf L$ are well defined and these are called limsup and liminf respectively.

We had the following characterization of limsup.

$$s = \limsup x_n \leftrightarrow [\forall \epsilon > 0 \ \ \exists \text{ only finitely many } \ n; \ x_n > s + \epsilon]\&$$

$$[\forall \epsilon > 0, \exists \text{ infinitely many } n, \ x_n > s - \epsilon].$$

We can restate this in an equivalent manner, without using ϵ, as follows.

$$s = \limsup x_n \leftrightarrow [\forall b > s \ \exists \text{ only finitely many } \ n; \ x_n > b]\&$$

$$[\forall a < s, \exists \text{ infinitely many } n, \ x_n > a].$$

This remains correct, irrespective of whether s is finite or infinity (an expression like a is finite is simply another way of saying $a \in R$). Similarly, the following remains correct, irrespective of whether l is finite or infinite.

$$l = \liminf x_n \leftrightarrow [\forall b < l \ \exists \text{ only finitely many } \ n; \ x_n < b]\&$$

$$[\forall a > l, \exists \text{ infinitely many } n, \ x_n < a].$$

With the concept of convergence as defined, we can say

$$\sum \frac{1}{n} = \infty$$

This is because the sum is limit of partial sums. We know that the partial sums are increasing and are not bounded above, so the sequence of partial syums converges to ∞. So far we only said that the series above does not converge, but now we are saying it is ∞. This is a better information.

Afterall, the series ± 1 also does not converge. But in the latter case the partial sums are, in a sense, oscillating. Whereas, for the sequence $\sum 1/n$, the above equation conveys the extra information that the sums eventually exceed any given value. Of course, in this case, because the series consists of non-negative terms, non-convergence of partial sums is equivalent to saying that they eventually exceed any given number. But for a general series the statement that $\sum a_n = \infty$ conveys more than simply saying that the series does not converge.

The upshot of all this is the following. Introduction of $\pm\infty$ sometimes allows us to express more than what we could do otherwise; sometimes they facilitate in succinctly expressing some statetments. You must understand why we entered this issue at all, why we introduced these objects at all. Remember that a word is invented to convey some information, which could not be conveyed without this word. In the same way we invented $\pm\infty$ to convey certain things. Think.

Some of you are, expectedly, puzzled that while talking supremum and infimum of sets I always considered non-empty sets. In practice we never have to calculate inf and sup of empty set. That is why I did not consider. But if it bothers you, here is the answer. Empty set is bounded. In fact every number is an upper bound as well as lower bound.

Let S be the empty set. If I said that $\forall x \in S, x \leq 5$, I would be correct. I only need to show that given any $x \in S$, then $x \leq 5$. This is true. Or equivalently, if you want to tell me my claim is false, you must produce $x \in S$ with $x > 5$, and you can not do this. Now comes a surprize and I would not like to waste time on it, because you are not going to learn anything by spending time on this (except worrying that infimum is strictly larger than supremum).

Here is the surprize. supremum of empty set is $-\infty$. Reason: every number being an upper bound for S, the infimum of the set of upper bounds is $-\infty$. To put it differently, given anything different from $-\infty$, I can take something smaller than that and say that is also an upper bound. Similar thought process shows that every number is a lower bound for the empty set and hence the greates lower bound is ∞.

It is better not to talk about supremum and infimum of empty set. If you ever need to calculate this, ask yourself if it is necessary at all; whether you have made life unnecessarily complicated.

discussion HA:

Q 27: To show $(n+47)^{589}/2^n \to 0$. The neumerator is a linear combination of powers of n. So if we prove that $n^k/2^n \to)0$ for each k, then we can use theorems on limits to complete the problem. As soon as $n > k$, we see $2^n > nchoosek + 1$ so that

$$\frac{n^k}{2^n} \leq k! \, \frac{n}{n} \, \frac{n}{n-1} \cdots \frac{n}{n-k+1} \, \frac{1}{n-k} \to 0.$$

Note that k is fixed.

Q 32. Note that (x_n) is a sequence where each rational in $(0,1)$ appears and no others. Thus if $x > 1$, then $(x-\epsilon, x+\epsilon)$ where $\epsilon = (x-1)/4$ does not contain any x_n and hence can not be a limit point. Similarly, numbers less than zero are not limit points. Take any x with $0 \leq x \leq 1$. Consider $(x-\epsilon, x+\epsilon)$. Need to show it contains x_n for infinitely many values of n. This follows from the observation that any interval contains infinitely many rationals. Try to think.

Continuous functions:

There are several interesting things one could discuss about sequences and series. But we should stop our discussion on series to proceed to other stories. We shall now discuss continuity of functions. You are already familiar probably. We shall deal with functions defined on R or subsets of R and take values in R.

What is a function. Of course, while discussing cardinality, we did discuss functions. Thus we now deal with functions f that associate a real number with each point where it is defined. The set D of numbers for which f associates a value is called domain of the function. For $x \in D$, the value is denoted by $f(x)$, which is a real number.

Just to make you start thinking (and for nothing else), let us consider the following. For each $x \neq 0$, we associate $f(x) = x^2$. When $x = 0$ we associate $f(0)$ as follows. If there is an earthquake tomorrow, then $f(0) = 35$ and if there is no earthquake tomorrow then $f(0) = 27$. Is this a function? What is the meaning of association? You may say this is meaningless or this depends on time etc etc. But the question remains: is it a function or is it not a function?

(It is not a function today, because I do not know $f(0)$ today. Unless I know the value, I do not accept it as a function. The fact that the value is one of two numbers 27,35 is no consolation. You might then ask: what is meant by knowing. If it were $\sqrt{2}$, do you know? You may think that I do not know $\sqrt{2}$ because I, and no one, knows all its decimal places. But the point is I know that there is exactly one positive number whose square is 2 and this is *the* number we are talking about. Contents of this and previous para are intended to make you think about matters and nothing else. You are free to ignore.)

Returning to examples of funcrions, $f(x) = x^2$ is a function, it is defined on the full R and with a number x it associates the number x^2. The function $f(x) = 1/x$ is defined only for non-zero real numbers and hence this is its domain. For a number x in its domain, this function associates $1/x$. You can think of polynomials. For example

$$f(x) = 5 + 32x^3 + 99x^{55} + \sqrt{2}x^{56}.$$

This function is defined for every number x and the value is as given on the right side above.

We can think of complicated functions. For example,

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is a function. It is defined for every number x. if you take a number x, then consider the series above for that number. We already know that the series of numbers converges. The sum of this series is the value of the function for that number x. This is called exponential function $e(x)$.

Here is another function.

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - + \cdots.$$

We know that for any given real number x, the series of numbers on the right side converges and hence this function is also defined on all of R. This function is denoted $\sin x$. Yes, this is the same function, you know, explained without angles and triangles. It is periodic, but not obvious.

Here is another function.

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - + \cdots.$$

This function is denoted $\cos x$.

In olden days by a function, people understood as something given by an analytical expression, for example the above expressions. Then they realized that the function need not be given by one analytical expression, may involve more than one expression, but you should be able to draw the graph. For example, consider $f(x) = x^2$ when $x < 1$ and $f(x) = x^3$ when $x \geq 1$. Another example is the function defined on the interval $[0, 1)$ by $f(x) = 2x$ when $0 < x < 1/2$and $f(x) = 2x - 1$ when $1/2 \leq x < 1$. It takes values again in the interval $[0, 1)$.

It was realized later that, there need not be analytical expression, and we may not be able to draw the graph also. Such an example is f defined by $f(x) = 1$ when x is irrational and $f(x) = 0$ when x is rational. It is not possible to draw graph of this function in the usual x, y-coordinate system. Remember graph of f is the set G of all pairs (x, y) such that $f(x) = y$.

Whether we can draw the graph or not, the concept of graph is itself well defined and is a subset of $R \times R$. Graph has the interesting property that for every x there is at most one y such that $(x, y) \in G$. In fact if f is not defined for the point x, then there is no point in G whose first coordinate is x. On the other hand, if f is defined at the point x with, say, $f(x) = a$, then (x, a) is the only point of G with first coordinate equal to x.

Interestingly, any subset G of $R \times R$ with the property that for any x there is at most one y with $(x, y) \in G$ defines a function. This is easy. Just take domain D of the function f to be the set of all points x such that there is a y with $(x, y) \in G$. For x in D define $f(x)$ to be the unique y such that $(x, y) \in G$. (Upshot you can ignore: A function is a subset of $R \times R$, you need not use the words 'associate' etc!).

So let us start with a function f defined on all of R. When should we agree to say that it is continuous. Yes, it is continuous if it is continuous at each point a. So when shall we say that the function is continuous at a point a. Idea is the following. If you change a a little bit, then the value also should change only a little bit, not too much. Or equivalently, if x is close to a, then $f(x)$ should be close to $f(a)$. We have to make this precise. This is what we shall do now.