Continuous functions:

Thus we felt that a function is continuous at a point a if, for points near a the values should be close to the value at a. We should now make it precise. First we make an observation.

Fact: Let $f : R \to R$ and $a \in R$. Then the following two statements are equivalent.
(i) $x_n \to a$ implies $f(x_n) \to F(a)$.
(ii) Given $\epsilon > 0$, we can find a $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Proof: Let (ii) hold. Let $x_n \to \delta$. Fix $\epsilon > 0$. We exhibit n_0 so that $|f(x_n) - f(a)| < \epsilon$ for $n \geq n_0$. Go to the hypothesis (ii) and get $\delta > 0$ for this ϵ. Choose n_0 so that $|x_n - a| < \delta$ for $n \geq n_0$. But then, for these n, we have $|f(x_n) - f(a)| < \epsilon$.

Conversely, let (ii) fail. We show (i) fails by producing a sequence $x_n \to a$ but $f(x_n) \not\to f(a)$. Since (ii) is false, there is an $\epsilon > 0$ for which we can not find $\delta > 0$ satisfying the condition of (ii). Fix such an $\epsilon > 0$. Using the fact that $\delta = 1/n$ does not fulfil condition of (ii), we can fix an x_n such that $|x_n - a| < 1/n$ and yet $|f(x_n) - f(a)| \geq \epsilon$. No need to say anything more.

Definition: A function $f : R \to R$ is continuous at a point a if any one of the above conditions holds. We say that f is continuous if it is continuous at every point a.

The two conditions are interesting. Condition (i) allows us to verify discrete instances. That is, to verify this condition we fix any arbitrary sequence $x_n \to a$ and verify $f(x_n) \to f(a)$. Of course there are uncountably many sequences converging to a, but that is a different matter. Whenever you set to verify *one* instance it is a discrete sequence. On the other hand condition (ii) is a non-discrete condition. Even for one instance, you fix $\epsilon > 0$ and produce a δ; but this delta should verify something for every x with $a - \delta < x < a + \delta$, namely, for every such x we must verify $-\epsilon < f(x) - f(a) < \epsilon$. Remember this verification is to be done for every x in an interval. However, both are equivalent. Sometimes (i) and sometimes (ii) would be handy.

When we were making up our mind about continuity, some of you mentioned about limits. Yes, you are right, usually the notion of continuity is defined after defining limits of functions. One defines righthand limit, left-hand limit at a and says that the function is continuous at a if both righthand limit and lefthand limit exist and equal $f(a)$. I find it difficult to begin with these concepts, which demand understanding continuous limits. It is not to say that these concepts are unimportant. They are simple and important. After you feel comfortable with the existing concepts and assimilate them, you would have no problem discovering them yourself.

The collection of all real valued functions on a set S has a nice structure. If f and g are are such functions, you can define the function $h(x) = f(x)+g(x)$ on the set S. This function is simply denoted by $f + g$. Similarly fg denotes the function whose value at $x \in S$ is $f(x)g(x)$. The function $55f$ is the function whose value at $x \in S$ is $55f(x)$. The function $\frac{f}{g}$ or f/g has value $\frac{f(x)}{g(x)}$ at $x \in S$; but now this function may no longer be defined on all of S. It is defined only for those points $x \in S$ such that $g(x) \neq 0$. The domain of this function may not be all of S.

Thus remember sum, product etc of functions is defined by us. On the other hand, the set of real numbers R is a set with operations of addition and multiplication defined on the set satisfying certain conditions. The set of all real valued functions defined on S did not come with such operations. *We* defined those operations.

Fact: If $f : R \to$ and $g : R \to R$ are continuous functions, then so are $f + g$, fg and $29f$. The function f/g is also continuous if $g(x) \neq 0$ for all $x \in R$.

Proof is very simple, follows from definition of continuity and properties of sequences that we know.

The last statement regarding f/g is unsatisfactory, it excludes all functions g which take value zero at one point. It would be more satisfying to allow such functions too and be able to say that f/g is continuous on the set where it is defined. We shall rectify the situation soon, but let us see some examples first, of continuous functions defined ln all of R.

Fact: The following continuous from R to R.

$$f(x) \equiv 49; \quad f(x) = x; \quad f(x) = x^{100}.$$

More generally, every polynomial is a continuous function.

Definition: Let $S \subset R$ be a non-empty subset and $f : S \to R$ and $a \in S$. We say that f is continuous on S at the point a, if any one of the following two equivalent conditions holds.
(i) $x_n \to a$, $x_n \in S$ for all n implies $f(x_n) \to f(a)$.
(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \in S$.
We say f is continuous on S if it is continuous on S at every point of S.

Of course, before making such a definition we should verify that the two conditions are indeed equivalent. This is very easy, if you look at the proof above you realize it. Proof of (ii) implies (i) needs no change. In the proof of '$\neg$ (ii)' implies '$\neg$ (i)' you only need to choose points x_n from S.

Fact: If f, g are real valued continuous functions on $S \subset R$, then so are $f + g$, $29f$ and fg. The function f/g is also continuous on the set $T = \{x \in S : g(x) \neq 0\}$.

Fact: Let $f(x) = (x - 55)$. Then f is continuous on R and $1/f$ is continuous on the set $\{55\}^c = R - \{55\}$.

Here we have used another notation $R - \{55\}$, but what is subtraction. It is not given to us. We define subtraction between two sets as follws: $A - B = \{x \in A : x \notin B\}$ This is same as $A \cap B^c$.

Let us see some functions which are not continuous.
The function $f(x) = 0$ for x rational and $f(x) = 1$ for x irrational is defined on all of R and is not continuous at every $a \in R$.
The function $f(x) = x^2$ for $x \leq 0$ and $f(x) = x + 1$ for $x > 0$ is defined on all of R. It is continuous at every non-zero $a \in R$. It is not continuous at $a = 0$.
The function
$$f(x) = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-12}.$$
is defined on the set $S = \{1, 2, 12\}^c$ and is continuous on S at every point of S.

If S is the set of integers, then every real valued function on S is continuous on S. This is easy to see.
Here is a useful property of continuous functions defined on a closed bounded

interval $[a, b]$.

Fact: Let f be a real valued continuus function defined an an interval $[a, b]$. Then f is bounded, that is there is a number C such that $|f(x)| \leq C$ for all x. In fact, there are points x_0 and x_1 in $[a, b]$ so that $f(x_0) \leq f(x) \leq f(x_1)$ for all x.

Thus the function attains its minimum and maximum values at the points x_0 and x_1 respectively.

Proof: Let us start with an observation. If we proved that every continuous function is bounded, then the second part follows. Indeed, let $\alpha = \inf\{f(x) : a \leq x \leq b\}$. If there is no point where the value of f equals α, then $g = 1/(f - \alpha)$ is continuous and not bounded, because for any $n \geq 1$, there is a point x with $f(x) < \alpha - (1/n)$ so that $g(x) > n$. Thus there is a point x_0 so that $f(x_0) \leq f(x)$ for all x. Similarly, there is an x_1 so that $f(x) \leq f(x_1)$ for all x.

Let us now show that f is bounded. Let $I_1 = [a, b]$. If f is not bounded, then it is not bounded either on the left half $[a, (a+b)/2]$ or on the right half $[(a+b)/2, b]$ of $[a, b]$. Let it be I_1. If f is unbounded on both halfs, take the left half. Since f is unbounded on I_1 let I_2 be the left half of I_1 if f is unbounded there or right half. In this way we get a sequence of closed bounded intervals with lengths decreasing to zero and henec by Cantor's theorem will have exactly one point, say α, in common. Clearly $\alpha \in [a, b]$. Since f is continuous on $[a, b]$ at the point α, get $\delta > 0$ so that $|f(x) - f(\alpha)| \leq 1$ for all $x \in I$ with $|x - \alpha| < \delta$. In particular, on the interval $(\alpha - \delta, \alpha + \delta)$ the function is bounded by $|f(\alpha)| + 1$. Pick k so that length (I_k) is smaller than $\delta/4$. since $\alpha \in I_k$ we see $I_k \subset (\alpha - \delta, \alpha + \delta)$. In other words f is bounded on I_k. This contradiction proves our result.

What if the function is not defined on a closed bounded interval? Clearly, you can not take your set to be unbounded. If S is unbounded, then $f(x) = x$ is a continuous function on S which is not bouinded. Of course, if it is defined on an open interval (a, b), then it need not be bounded. For instance, the function $f(x) = 1/x$ on the interval $(0, 1)$ tells you this. In this example, the function f is defined on the interval $(0, 1)$. There is a sequence in this set converging to a point outside the set, namely $1/n \to 0$. If sequences in your set do not converge to points outside the set, then this will ensure boundedness of the function.

Definition: Say that a set $C \subset R$ is closed under limits, simply **closed** if $x_n \to x$, $x_n \in C$ for all n implies $x \in C$.

For example, as seen above, the interval $(0, 1)$ is not closed but the interval $[0, 1]$ is closed. The set $\{1/n : n \geq 1\}$ is not a closed set but the set consisting of these points along with zero is a closed set.

Fact: Let f be a real valued continuous function on a closed bounded set S. Then f is bounded, in fact, there are numbers $x_0 \in S$ and $x_1 \in S$ so that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in S$.

Proof goes exactly along the same lines as earlier. The first part implies the second part by same argument. First part is proved as follows. Since the set is bounded, get real numbers $a < b$ so that $S \subset [a, b]$ let $I_1 = [a, b]$. If f is unbounded on S, then it is unbounded either on the part of S in the left half or on the part of S in the right half of I_1. Denote it by I_2. Continue always making sure that f is not bounded on the part of S in I_n and lengths are always halved. Get α common to all these intervals. Is $\alpha \in S$? Yes, because f being unbounded on the part of S in I_n, you see that in particular you can pick a point x_n of S from I_n. Obviously, $|x_n - \alpha| \leq$ length $I_n \to 0$. Since S is closed, we conclude $\alpha \in S$. Now use continuity of f at α and proceed as earlier.

Here is another property of continuous functions.

Fact: Suppose f is continuous on S and g is continuous on T. Assume that $f(x) \in T$ for each $x \in S$. Define the composition $h(x) = g(f(x))$ on S. Then h is continuous on S.

In fact, if $x_n \to x$, then $f(x_n) \to f(x)$ and so $g(f(x_n)) \to g(f(x))$.

Disucssion of HA:

Q37 $x_n \to a$ To show $a_n \to a$ where $a_n = \sum\limits_1^n x_i$.

If you are taking average of many many numbers and if most of them are close to a then so should be their average. Here as n becomes large, you are taking average of many many numbers and by hypothesis most of them are close to a.

No loss to assume $a = 0$. This is because, you consider the sequence $y_n = x_n - a$. Then averages of y_i are just averages of x_i minus a. If the result is proved for (y_n), then you can complete the proof.

So let us assume that $x_n \to 0$. Fix $\epsilon > 0$. Shall show n_0 so that

$$|\frac{1}{n}\sum_1^n x_i| < \epsilon, \qquad n \geq n_0.$$

First fix k so that $|x_n| < \epsilon/2$ for $n \geq k$, possible since $x_n \to 0$. Having fixed k like this fix $n_0 > k$ so that

$$\frac{1}{n_0}\sum_1^k |x_i| < \epsilon/2,$$

possible because k is fixed. Now if $n > n_0$

$$\frac{1}{n}|\sum_1^n x_i| \leq \frac{1}{n}\sum_1^k |x_i| \quad + \quad \frac{1}{n}\sum_k^n |x_i| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

as promised.

Q53. To discuss convergence of $\sum \frac{P(n)}{Q(n)}$.

You are forgetting basic rule of life: If there is a complicated problem, you have no clue how to proceed, ask if you can solve a simpler problem.

Why worry about general polynomials? Ask yourself: what if $P(x) = x^l$ and $Q(x) = x^k$? Then the ratio is $\sum n^{-(k-l)}$ and it converges iff $k - l > 1$. Since we are dealing with polynomials, k, l are integers, so this amounts to saying $k \geq l + 2$.

Let now

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_l x^l; \quad Q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k \quad a_l, b_k > 0.$$

There is no loss to assume $a_l > 0$, otherwise argue for $-P(x)$. Similarly, no loss to assume that $b_k > 0$. Observe that

$$\frac{P(n)/Q(n)}{n^l/n^k} \to \frac{a^l}{b^k} > 0$$

and the terms of the given series, namely, $P(n)/Q(n)$ as well as n^l/n_k are positive and hence either both converge or both fail to converge by Q51. But this special case we know already. Thus convergence holds iff $k \geq l + 2$.

Q51 $a_n/b_n \to 55$, all the a's and b's are strictly positive. To argue $\sum a_n$ converges iff $\sum b_n$ converges. Get k so that we have for $n \geq k$,

$$54 < \frac{a_n}{b_n} < 56 \quad (i.e.) \quad 54b_n < a_n < 56b_n.$$

Thus if $\sum a_n$ converges then $\sum\limits_{n>k} b_n \leq \sum_{n \geq k} a_n/54$ converges and hence $\sum b_n$ converges. If $\sum b_n$ converges then $\sum\limits_{n>k} a_n \leq \sum\limits_{n>k} 56b_n$ converges and hence $\sum a_n$ converges.

Return to Q 53. Since $P(n)/n^l \to a_l > 0$ we see $P(n)$ is positive after a stage. similarly $Q(n) > 0$ after some stage. Thus $P(n)/Q(n)$ is positive after some stage. But Q 51 assumes all terms a_n and b_n are positive. Sort it out.

Also hypothesis says that $Q(n)$ is never zero. But they did not say $P(n) \neq 0$. Thus it is quite likely that $P(n)/Q(n) = 0$. But Q 51 assumes all a_n etc are strictly positive. Sort it out.

Unless you sort out these two issues, the proof is incomplete. (you may want to say: but this is easy. *Remeber*, you would know whether it is easy or difficult or even wrong, only after you argue it out once. Also moreover, if you are writing proof, you would not like to wait till someone objects and then modify your proof.)

Q55. Convergence of

$$\sum \frac{1}{n \ \log n \ (\log \log n)^p}.$$

Some of you are again forgetting basic facts of life, nothing to do with maths. I do not know if you are lazy or just afraid of the mathematical expressions and close your brain right away. You *must* attend to this, whatever it be. This is the famous phrase of Paul Erdos: keep your brains open.

Whenever you are in an unknown territory and see an animal like the above one, you should ask the natural question: did I see any similar animal earlier. there pops out the answer: yes I saw the series with terms $1/n^p$ and $1/\{n(\log n)^p\}$. You do not seem to take advantage of the fact that you have seen very few animals and it is easy to go through the list of the animals you have seen very quickly. Once the answer comes out, you should ask: how did I tackle that animal. Of course, similar thing may or may not work, but obviously you should go with your natural instincts and try out.

So coming to the problem at hand: The given series converges iff $\sum 2^n a_{2^n}$ converges, that is, iff the following series converges

$$\sum 2^n \frac{1}{2^n (n\log 2)(\log(n\log 2))^p} = \frac{1}{\log 2}\sum \frac{1}{n(\log n + \log 2)^p}.$$

Ignore the $\log 2$ factor and take the series as $\sum a_n$ and try

$$\sum b_n = \sum \frac{1}{n(\log n)^p}.$$

You see all the terms are positive and

$$\frac{b_n}{a_n} = \left[\frac{\log n + \log 2}{\log n}\right]^p \to 1.$$

Now comnplete the proof.

Remember we started with $n > 1000$ or some such thing to make sure denominator make sense. So the series you are comparing with, let it also start in a similar way. Otherwise, if you blindly compare with $\sum\limits_{n\geq 2} b_n$ then you are not doing correctly.

Q59 Convergence of the series

$$\sum \frac{\log(n+1) - \log n}{(\log n)^2} = \sum \frac{\log(1 + \frac{1}{n})}{(\log n)^2}.$$

This is a series of positive terms and if you use the inequality $\log(1+x) \leq x$ then this series is dominated by the series with terms $1/\{n(\log n)^2\}$ and hence converges.

One of you suggested different argument which is nice, but I forgot it at this moment.

unraveling negations;

We have employed, the method of proof by contradiction, several times. This is how it goes. Need to show $S \Rightarrow T$.
Thus you are granted a hypothesis S.
You want to prove a sentence T.

You are unable to do so directly.
Then, you would say, alright, suppose T is false.
And work hard to show S is false.
Where are we now?
T is false $\Rightarrow$ S is false
Of course, S is true $\Rightarrow$ S is true.
Thus
S is true and T is false $\Rightarrow$ S is true and S is false.
But we can not have S is true and S is false at the same time.
So we can not have S is true and T is false at the same time.
So when S is true, T must also be true.

While running proof by contradiction, it is important to know what is the meaning of: suppose T is false. To take a specific example. in proving the equivalence of the two statements regarding continuity at a point a, we had

$$S: \quad (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon).(\spadesuit)$$

We needed to know its negation. This is not a lecture on logic, we restrict to some common sense aspects, to be able to carry on arguments.

If we have a simple sentence like $(c \leq 5)$ then its negation is easy: $(c > 5)$. If you have compound sentence like $(c < 5) \vee (c > 30)$, it says one of two things happens. So its negation is that none of them happen. Thus its negation is $(c \geq 5) \wedge (c \leq 30)$. Thus negation is $(5 \leq c \leq 30)$.

A sentence $(c < 5) \Rightarrow (d > 7)$ means when $(c < 5)$ holds then $(d > 7)$ must hold. But we always have either $(c < 5)$ or $\neg(c < 5)$. Thus we always have either $(d > 7)$ or $\neg(c < 5)$. In other words $(c < 5) \Rightarrow (d > 7)$ means $\neg(c < 5) \vee (d > 7)$. Thus sentence 1: $A \Rightarrow B$ is same as saying sentence 2: $\neg A \vee B$. What is explained just now tells you why this is so. Of course, in logic you take statement 1 as an abbreviation for statement 2. But let us not bother.

But sentences which involve quantifiers $\forall$ and $\exists$ are to be carefully analyzed for negation. If you follow logical or symbolic method of writing sentences, then making negations is easy.
S: Every student has a pen.
What is its negation: Every student has no pen?
No, it is: There are students without a pen. because even if one student has

no pen, S is negated.
Thus $\forall x A(x)$ would have negation $\exists x \neg A(x)$.

S: there is a student who is sleeping.
What is its negation: there is a student who is not sleeping?
No, it is: every student is not sleeping.
Thus $\exists x A(x)$ would have negation $\forall x \neg A(x)$.

Let us return to our sentence.

$$S: \quad (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon).(\spadesuit)$$

Its negation is what we are interested.

$$\neg S: \quad \neg \Big\{ (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon) \Big\}.(\clubsuit)$$

From what was said above about $\forall$, negation of S is,

$$\neg S: \quad (\exists \epsilon > 0) \neg \Big\{ (\exists \delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon) \Big\}.(\clubsuit)$$

Again from what has been said about $\exists$, we see

$$\neg S: \quad (\exists \epsilon > 0)(\forall \delta > 0) \neg \Big\{ (\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon) \Big\}.(\clubsuit)$$

Again using negation of $\forall$, this is same as

$$\neg S: \quad (\exists \epsilon > 0)(\forall \delta > 0)(\exists x) \neg \Big\{ (|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon) \Big\}.(\clubsuit)$$

Remembering $A \Rightarrow B$ is same as saying $(\neg A) \vee B$, this is same as

$$\neg S: \quad (\exists \epsilon > 0)(\forall \delta > 0)(\exists x) \neg \Big\{ [\neg |x-a| < \delta] \vee [|f(x) - f(a)| < \epsilon] \Big\}.(\clubsuit)$$

remembering negation of $A \vee B$ is $\neg A \wedge \neg B$, we see

$$\neg S: \quad (\exists \epsilon > 0)(\forall \delta > 0)(\exists x) \Big\{ \neg [\neg |x-a| < \delta] \wedge \neg [|f(x) - f(a)| < \epsilon] \Big\}.(\clubsuit)$$

But of course $\neg\neg A$ is same as A. Thus we have,

$$\neg S: \quad (\exists \epsilon > 0)(\forall \delta > 0)(\exists x) \Big\{ (|x-a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon) \Big\}.(\clubsuit)$$

The logical symbols are introduced to convince you how simple it is to understand negations. It is a step by step process. Let us read in words now.

The statement S reads: for every $\epsilon > 0$, there is $\delta > 0$, so that for every x either $|x - a| \geq \delta$ or $|f(x) - f(a)| < \epsilon$.

The statement $\neg S$ reads: There is an $\epsilon > 0$ so that for every $\delta > 0$, there is an x such that $|x - a| < \delta$ and also $|f(x) - f(a)| \geq \epsilon$.

This is fun and you should treat it so. In case you are getting confused, either sort it out or ignore all this and think in your own way. The issue is: you should be able to write negation of sentences without using negation for quantifiers.