When Locality Meets Preservation

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école normale supérieure paris–saclay
LMF Laboratoire Méthodes Formelles
IRIF INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE
Université de Paris

ACT I: Where queries are optimised

Preservation Under Extensions, *a.k.a.*, Łoś (1955); Tarski (1954)

♦ **Query Optimisation**

Input Some FO sentence φ

Promise Upwards closure
for induced substructures ($\subseteq_i$) – *a.k.a.* extensions

Output A simplified query (existential)

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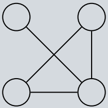

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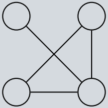

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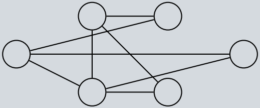

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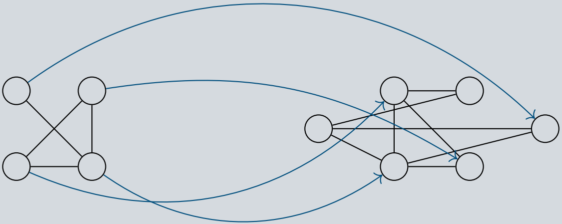

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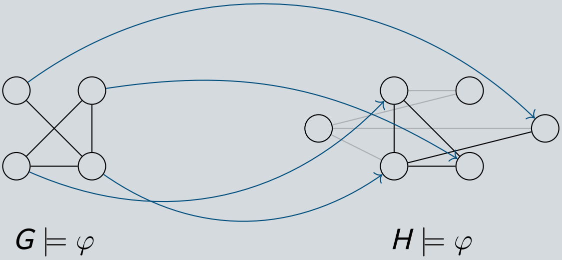

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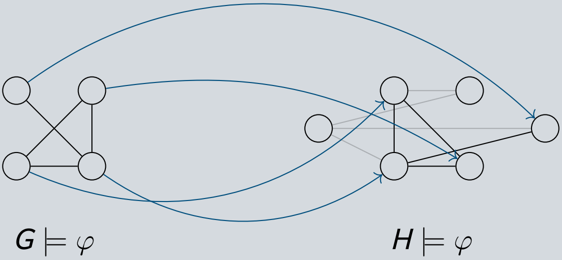

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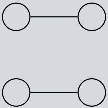


M

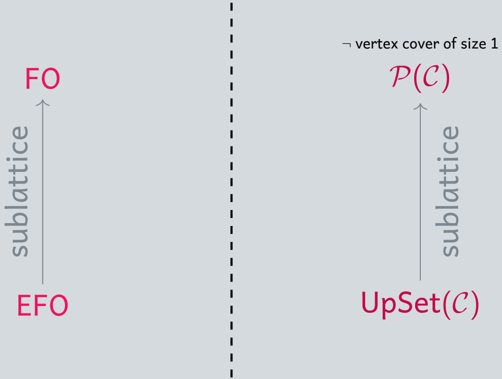
Theorem (Łoś (1955); Tarski (1954))

This algorithm exists.

Proof.

- an equivalent existential sentence exists (heavy use of compactness)
- one can enumerate proofs $\vdash \psi \leftrightarrow \varphi$ with ψ existential. $\square$

Łoś (1955); Tarski (1954) in a diagram?



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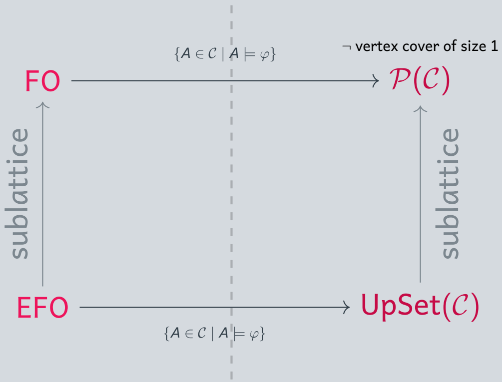

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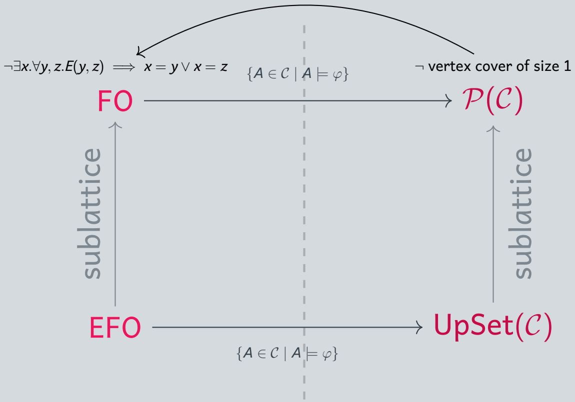

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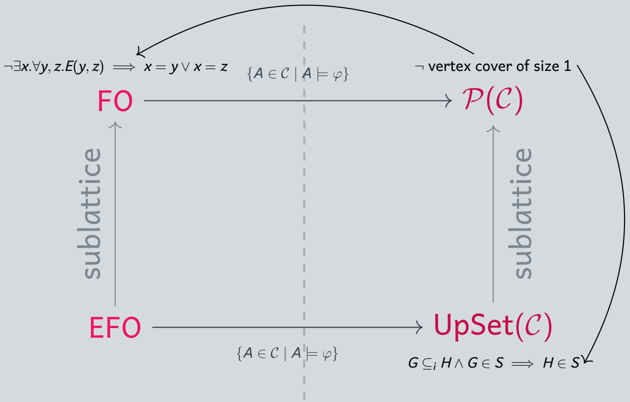

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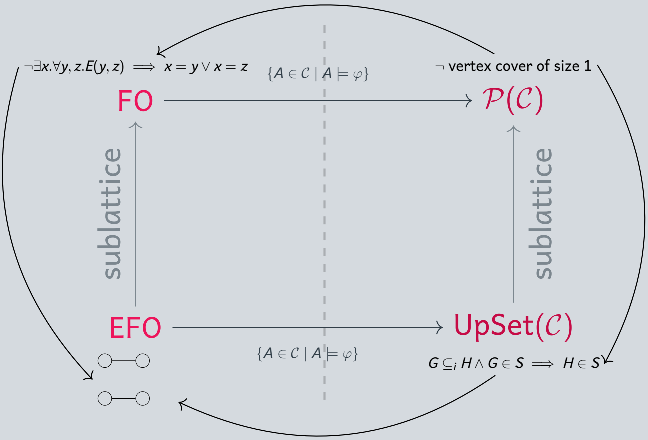

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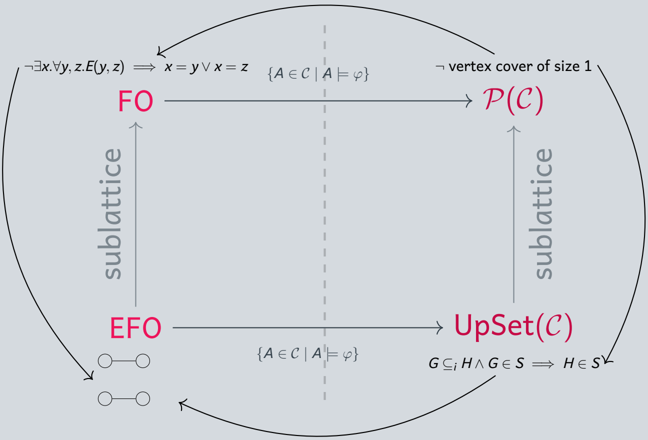


Beware: In computer science $\mathcal{C}$ is a class of finite structures!

ACT II: Where all is lost in a fire

Preservation Under Extensions, a.k.a., Łoś (1955); Tarski (1954)

♦ **Query Optimisation** $\star$ **over** $\mathcal{C}$ $\star$

Input Some FO sentence φ

Promise Upwards closure $\star$ over $\mathcal{C}$ $\star$

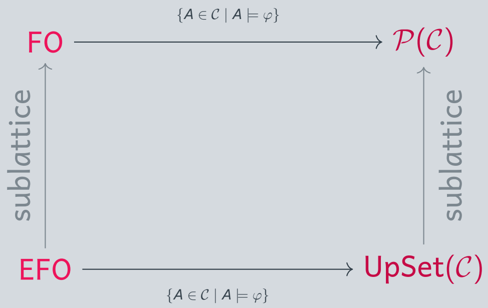
Output An existential sentence ψ equivalent to φ $\star$ over $\mathcal{C}$ $\star$

Over finite structures

- Tait (1959): no such ψ!
- Chen and Flum (2021): no algorithm even if ψ exists.

ACT III: Where the problem is solved

Finding the right nails for our hammer



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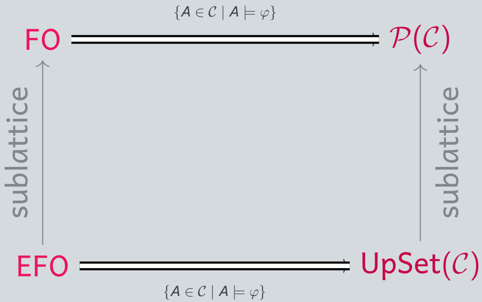
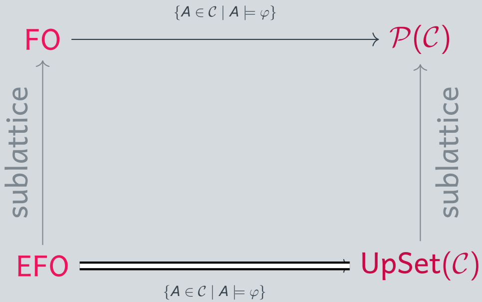


Easy case: $\mathcal{C}$ is a *finite* class of *finite* structures.

Finding the right nails for our hammer



Combinatorics: $\mathcal{C}$ is *well-quasi-ordered* (WQO).

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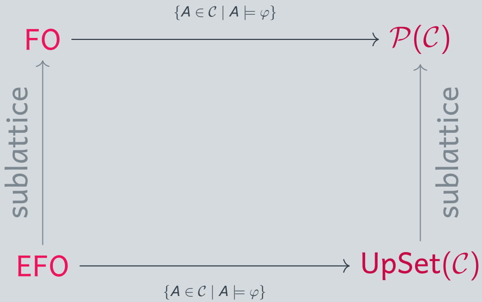


Non Trivial: $\mathcal{C}$ is hereditary (downwards closed), wide, and closed under $\uplus$ (Atserias et al., 2008).

Finding the right nails for our hammer

Preservation under extensions

WQO

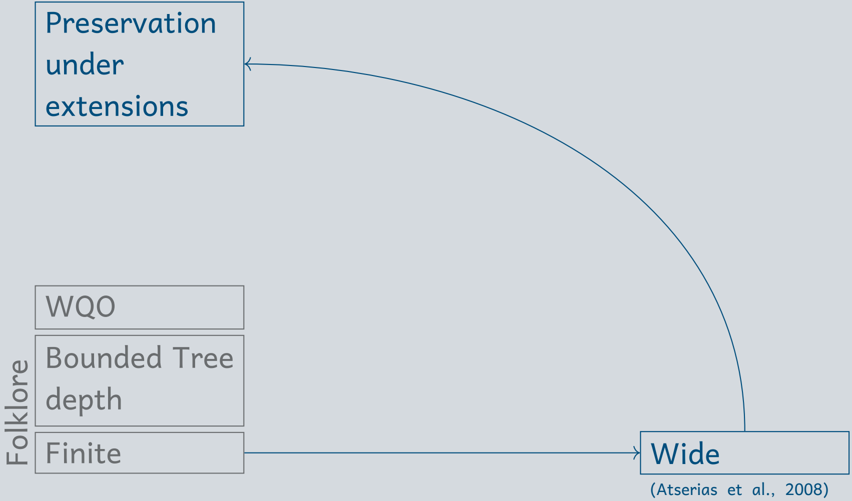
Bounded Tree depth

Finite

Folklore

Property implication over **hereditary** classes of **finite structures** closed under $\uplus$.

Finding the right nails for our hammer



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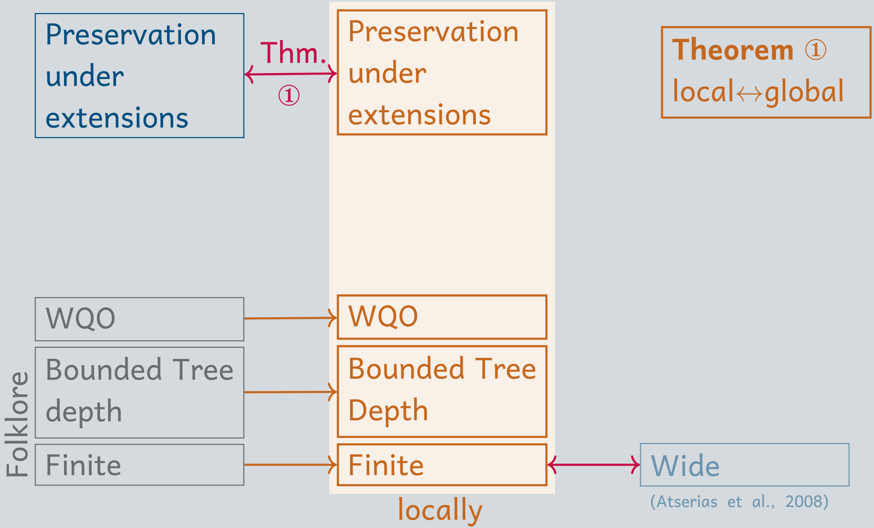
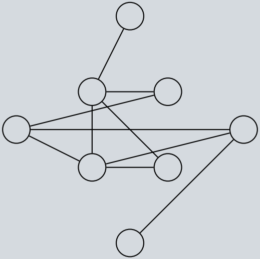


Property implication over **hereditary** classes of **finite structures** closed under $\uplus$.

Locally satisfying a property?

$$\mathrm{Local}(\mathcal{C}, r, k) \triangleq \{\mathcal{N}_A(\vec{a}, r) \mid A \in \mathcal{C}, \vec{a} \in A^k\}$$



A structure A.

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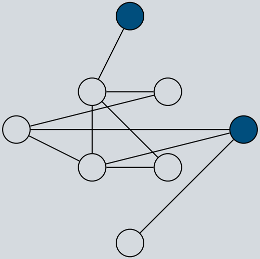

A structure A, with 2 selected nodes.

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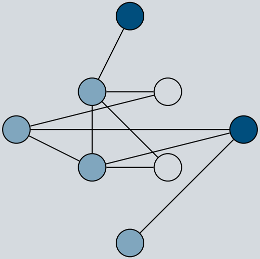

A structure *A*, with 2 selected nodes, *and a* 1-local neighborhood.

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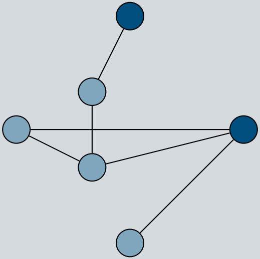

An element of $\mathsf{Local}(\mathcal{C}, 1, 2)$.

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♣ Localise Bounded Degree

$\mathcal{C}$ is of bounded degree if and only if $\mathrm{Local}(\mathcal{C}, r, k)$ is finite for all $k, r \geq 0$, i.e., *locally* finite

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Corollary (of theorem ①; known from Atserias et al. (2008))

Hereditary classes of bounded degree, closed under $\uplus$*, satisfy preservation under extensions.*

ACT IV: Behind the scenes

Snowy mountain tops

♦ **Why assume $\mathcal{C}$ to be hereditary (downwards closed)?**

Let $\varphi \in$ FO be upwards closed, t.f.a.e.:

(i) $\varphi \equiv_{\mathcal{C}} \psi$ with $\psi \in$ EFO.

(ii) φ has finitely many minimal models in $\mathcal{C}$.

(iii) minimal models of φ in $\mathcal{C}$ have bounded size.

Proof scheme of Theorem ①

Assume φ is upwards-closed with respect to $\subseteq_i$ over $\mathcal{C}$.

Step	**Minimal Models**	**Sentence** $\equiv_{\mathcal{C}} \varphi$
✁	bounded radius (r, k)	$\exists x_1, \dots, x_k . \psi(\vec{x})$, ψ r-local
✻	bounded size ℓ	$\exists x_1, \dots, x_\ell . \psi(\vec{x})$, ψ quantifier free

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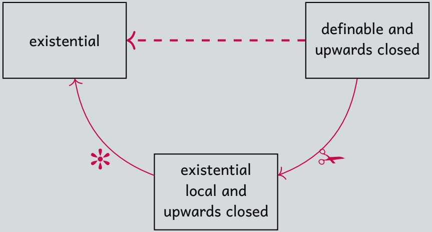
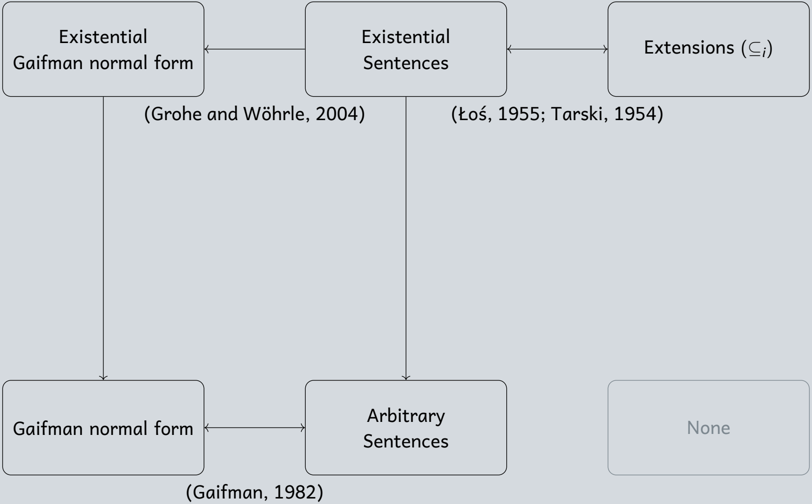


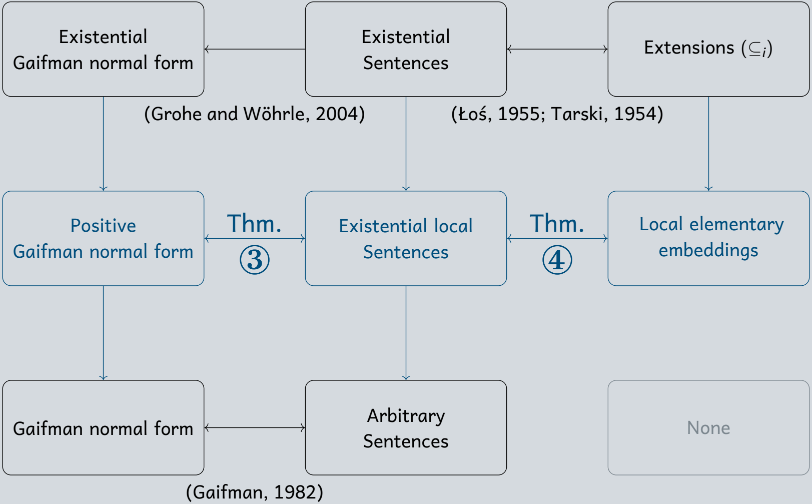
ACT V: The friends made along the way

Locality and preservation under extensions



Over arbitrary structures

Locality and preservation under extensions



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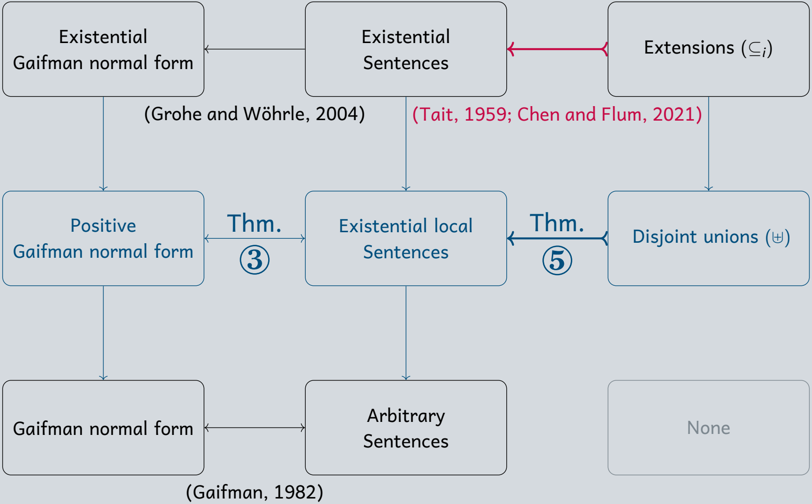

Over finite structures

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