Stochastic Petri Net

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1. Stochastic Petri Net
2. Markov Chain
3. Markovian Stochastic Petri Net
4. Generalized Markovian Stochastic Petri Net (GSPN)
5. Product-form Petri Nets
Plan

1. Stochastic Petri Net

   Markov Chain

   Markovian Stochastic Petri Net

   Generalized Markovian Stochastic Petri Net (GSPN)

   Product-form Petri Nets
Stochastic Petri Net versus Time Petri Net

- In TPN, the delays are non deterministically chosen.
- In Stochastic Petri Net (SPN), the delays are randomly chosen by sampling distributions associated with transitions.

... but these distributions are not sufficient to eliminate non determinism.

Policies for a net

One needs to define:

- The choice policy. What is the next transition to fire?
- The service policy. What is the influence of the enabling degree of a transition on the process?
- The memory policy. What become the samplings of distributions that have not be used?
Stochastic Petri Net versus Time Petri Net

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- The *choice* policy.
  What is the next transition to fire?
- The *service* policy.
  What is the influence of the enabling degree of a transition on the process?
- The *memory* policy.
  What become the samplings of distributions that have not be used?
Choice Policy

In the net, associate a distribution $D_i$ and a weight $w_i$ with every transition $t_i$.

### Preselection w.r.t. a marking $m$ and enabled transitions $T_m$

- Normalize weights $w_i$ of the enabled transitions: $w'_i \equiv \frac{w_i}{\sum_{t_j \in T_m} w_j}$
- Sample the distribution defined by the $w'_i$'s.
- Let $t_i$ be the selected transition, sample $D_i$ giving the value $d_i$.

### Race policy with postselection w.r.t. a marking $m$

- For every $t_i \in T_m$, sample $D_i$ giving the value $d_i$.
- Let $T'$ be the subset of $T_m$ with the smallest delays. Normalize weights $w_i$ of transitions of $T'$: $w'_i \equiv \frac{w_i}{\sum_{t_j \in T'} w_j}$
- Sample the distribution defined by the $w'_i$'s yielding some $t_i$.

Priorities between transitions could be added to refine the selection.
Choice Policy

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Priorities between transitions could be added to refine the selection.
Choice Policy: Illustration

\[
\begin{align*}
& t_1(D_1, w_1) \quad t_2(D_2, w_2) \quad t_3(D_3, w_3) \\
& w_1 = 1 \quad w_2 = 2 \quad w_3 = 2
\end{align*}
\]

Preselection

Sample \((1/5, 2/5, 2/5)\)

Outcome \(t_1\)

Sample \(D_1\)

Outcome \(4.2\)

Race Policy

Sample \((D_1, D_2, D_3)\)

Outcome \((3.2, 6.5, 3.2)\)

Sample \((1/3, -, 2/3)\)

Outcome \(t_3\)
Server Policy

A transition $t$ can be viewed as server for firings:

- A single server $t$ allows a single instance of firings in $m$ if $m[t]$.
- An infinite server $t$ allows $d$ instances of firings in $m$ where $d = \min\left(\frac{m(p)}{Pre(p,t)} \right) | p \in \bullet t$ is the enabling degree.
- A multiple server $t$ with bound $b$ allows $\min(b, d)$ instances of firings in $m$.

This can be generalised by marking-dependent services.
What happens to $d_2$ and $d_3$?

Resampling Memory
Every sampling not used is forgotten.

This could correspond to a “crash” transition.
Memory Policy (2)

What happens to $d_2$ and $d_3$?

**Enabling Memory**

- The samplings associated with still enabled transitions are kept and decremented ($d'_3 = d_3 - d_1$).
- The samplings associated with disabled transitions are forgotten (like $d_2$).

Disabling a transition could correspond to abort a service.
Memory Policy (3)

What happens to $d_2$ and $d_3$?

Age Memory

- All the samplings are kept and decremented ($d'_3 = d_3 - d_1$ and $d'_2 = d_2 - d_1$).
- The sampling associated with a disabled transition is frozen until the transition become again enabled (like $d'_2$).

Disabling a transition could correspond to suspend a service.
Memory Policy (4)

Specification of memory policy
To be fully expressive, it should be defined w.r.t. any pair of transitions.

What happens to \(d_1, d_1', \) and \(d_1''\)?

Interaction between memory policy and service policy
Assume enabling memory for \(t_1\) when firing \(t_2\) and infinite server policy for \(t_1\). Which sample should be forgotten?

- The last sample performed,
- The first sample performed,
- The greatest sample, etc.

Warning: This choice may have a critical impact on the complexity of analysis.
Plan

Stochastic Petri Net

2 Markov Chain

Markovian Stochastic Petri Net

Generalized Markovian Stochastic Petri Net (GSPN)

Product-form Petri Nets
A DTMC is a stochastic process which fulfills:

- For all $n$, $T_n$ is the constant 1
- The process is memoryless

\[ Pr(S_{n+1} = s_j \mid S_0 = s_{i_0}, \ldots, S_{n-1} = s_{i_{n-1}}, S_n = s_i) = Pr(S_{n+1} = s_j \mid S_n = s_i) = P[i, j] \]

A DTMC is defined by $S_0$ and $P$
Analysis: the State Status

The transient analysis is easy and effective in the finite case:
\[ \pi_n = \pi_0 \cdot P^n \text{ with } \pi_n \text{ the distribution of } S_n \]

The steady-state analysis \((\exists \lim_{n \to \infty} \pi_n)\) requires theoretical developments.

Classification of states w.r.t. the asymptotic behaviour of the DTMC

- A state is **transient** if the probability of a return after a visit is less than one. Hence the probability of its occurrence will go to zero. \((p < 1/2)\)

- A state is **recurrent null** if the probability of a return after a visit is one but the mean time of this return is infinite. Hence the probability of its occurrence will go to zero. \((p = 1/2)\)

- A state is **recurrent non null** if the probability of a return after a visit is one and the mean time of this return is finite. \((p > 1/2)\)
State Status in Finite DTMC

In a finite DTMC

- The status of a state only depends on the graph associated with the chain.
- A state is transient iff it belongs to a non terminal *strongly connected component* (scc) of the graph.
- A state is recurrent non null iff it belongs to a terminal scc.

\[
\begin{array}{c}
0.5 & 0.3 & 0.7 & 0.1 & 0.8 & 0.2 & 0.1 \\
0.5 & 0.1 & 0.4 & 0.8 & 0.2 & 0.1 & \\
0.3 & 0.1 & 0.3 & 0.1 & 0.7 & 1 & 0.8 \\
0.2 & 0.8 & & & & & \\
\end{array}
\]

\(T=\{1, 2, 3\}\)

\(C_1=\{4, 5\}\)

\(C_2=\{6, 7, 8\}\)
A chain is *irreducible* if its graph is strongly connected.

The *periodicity* of an irreducible chain is the greatest integer $p$ such that:
the set of states can be partitioned in $p$ subsets $S_0, \ldots, S_{p-1}$ where every transition goes from $S_i$ to $S_{i+1 \% p}$ for some $i$.

**Computation of the periodicity**

\[
\begin{align*}
\text{periodicity} &= \gcd(0, 2, 4) = 2
\end{align*}
\]
Analysis of a DTMC: a Particular Case

A particular case

The chain is irreducible and \textit{aperiodic} (i.e. its periodicity is 1)

\begin{itemize}
\item $\pi_{\infty} \equiv \lim_{n \to \infty} \pi_n$ exists and its value is independent from $\pi_0$.
\item $\pi_{\infty}$ is the unique solution of $X = X \cdot P \wedge X \cdot 1 = 1$
\end{itemize}

where one can omit an arbitrary equation of the first system.

\[ \begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3
\end{pmatrix} =
\begin{pmatrix}
1/8 & 7/16 & 7/16
\end{pmatrix} \]

\[ \pi_1 = 0.3\pi_1 + 0.2\pi_2 \quad \pi_2 = 0.7\pi_1 + 0.8\pi_3 \quad \pi_3 = \pi_2 \]
Analysis of a DTMC: the “General” Case

Almost general case: every terminal scc is aperiodic

- $\pi_\infty$ exists.
- $\pi_\infty = \sum_{s \in S} \pi_0(s) \sum_{i \in I} \text{preach}_i[s] \cdot \pi_i^\infty$ where:
  1. $S$ is the set of states,
  2. $\{C_i\}_{i \in I}$ is the set of terminal scc,
  3. $\pi_i^\infty$ is the steady-state distribution of $C_i$,
  4. and $\text{preach}_i[s]$ is the probability to reach $C_i$ starting from $s$.

Computation of the reachability probability for transient states

- Let $T$ be the set of transient states (i.e. not belonging to a terminal scc)
- Let $P_{T,T}$ be the submatrix of $P$ restricted to transient states
- Let $P_{T,i}$ be the submatrix of $P$ transitions from $T$ to $C_i$
- Then $\text{preach}_i = (\sum_{n \in \mathbb{N}} (P_{T,T})^n) \cdot P_{T,i} \cdot 1 = (\text{Id} - P_{T,T})^{-1} \cdot P_{T,i} \cdot 1$
Analysis of a DTMC: the “General” Case

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Illustration: SCC and Matrices

\[ P_{T, T} = \begin{pmatrix} 0.0 & 0.7 & 0.0 \\ 0.1 & 0.0 & 0.8 \\ 0.0 & 0.2 & 0.0 \end{pmatrix} \]

\[ P_{T, 1} \cdot 1 = \begin{pmatrix} 0.0 & 0.3 \\ 0.0 & 0.0 \\ 0.0 & 0.4 \end{pmatrix} \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.0 \\ 0.4 \end{pmatrix} \]

\[ P_{T, 2} \cdot 1 = \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.1 & 0.0 \\ 0.3 & 0.1 & 0.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 0.0 \\ 0.1 \\ 0.4 \end{pmatrix} \]

\[ T = \{1, 2, 3\}, C_1 = \{4, 5\}, C_2 = \{6, 7, 8\} \]
A CTMC is a stochastic process which fulfills:

▶ Memoryless state change

\[
Pr(S_{n+1} = s_j \mid S_0 = s_{i_0}, ..., S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, ..., T_n < \tau_n, S_n = s_i) \\
\quad = Pr(S_{n+1} = s_j \mid S_n = s_i) \equiv P[i, j]
\]

▶ Memoryless transition delay

\[
Pr(T_n < \tau \mid S_0 = s_{i_0}, ..., S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, ..., T_{n-1} < \tau_{n-1}, S_n = s_i) \\
\quad = Pr(T_n < \tau \mid S_n = s_i) = 1 - e^{-\lambda_i \tau}
\]

Notations and properties

▶ P defines an *embedded* DTMC (the chain of state changes)

▶ Let \( \pi(\tau) \) the distribution de \( X(\tau) \), for \( \delta \) going to 0 it holds that:

\[
\pi(\tau + \delta)(s_i) \approx \pi(\tau)(s_i)(1 - \lambda_i \delta) + \sum_j \pi(\tau)(s_j) \lambda_j \delta P[j, i]
\]

▶ Hence, let \( Q \) the *infinitesimal generator* defined by:

\( Q[i, j] \equiv \lambda_i P[i, j] \) for \( j \neq i \) and \( Q[i, i] \equiv -\sum_{j \neq i} Q[i, j] \)

Then:

\[
\frac{d\pi}{d\tau} = \pi \cdot Q
\]
A CTMC is a stochastic process which fulfills:

- **Memoryless state change**
  \[
  Pr(S_{n+1} = s_j \mid S_0 = s_{i_0}, \ldots, S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, \ldots, T_n < \tau_n, S_n = s_i) = Pr(S_{n+1} = s_j \mid S_n = s_i) \equiv P[i, j]
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- **Memoryless transition delay**
  \[
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  Then:
  \[
  \frac{d\pi}{d\tau} = \pi \cdot Q
  \]
The exponential distribution

Let $F$ be defined by: $F(\tau) = 1 - e^{-\lambda \tau}$

Then $F$ is the exponential distribution with rate $\lambda > 0$.

The exponential distribution is memoryless.

Let $X$ be a random variable with a $\lambda$-exponential distribution.

$$
\Pr(X > \tau' \mid X > \tau) = \frac{\Pr(X > \tau')}{\Pr(X > \tau)} = \frac{e^{-\lambda \tau'}}{e^{-\lambda \tau}} = e^{-\lambda(\tau' - \tau)} = \Pr(X > \tau' - \tau)
$$

The minimum of exponential distributions is an exponential distribution.

Let $Y$ be independent from $X$ with $\mu$-exponential distribution.

$$
\Pr(\min(X, Y) > \tau) = e^{-\lambda \tau} e^{-\mu \tau} = e^{-(\lambda + \mu) \tau}
$$

The minimal variable is selected proportionally to its rate.

$$
\Pr(X < Y) = \int_0^\infty \Pr(Y > \tau) F_X \{d\tau\} = \int_0^\infty e^{-\mu \tau} \lambda e^{-\lambda \tau} d\tau = \frac{\lambda}{\lambda + \mu}
$$
**Convoluting the exponential distribution**

The $n^{th}$ convolution of a distribution $F$ is defined by:

$$F^{n\ast} \overset{\text{def}}{=} F \ast \cdots \ast F \quad (n \text{ times})$$

Let $f_n$ (resp. $F_n$) be the density (resp. distribution) of the $n^{th}$ convolution of the $\lambda$-exponential distribution. Then:

$$f_n(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} \quad \text{and} \quad F_n(x) = 1 - e^{-\lambda x} \sum_{0 \leq m < n} \frac{(\lambda x)^m}{m!}$$

**Sketch of proof**

Recall that: $f_1(x) = \lambda e^{-\lambda x}$.

$$f_{n+1}(x) = \int_0^x f_n(x-u) f_1(u) \, du = \int_0^x \lambda e^{-\lambda(x-u)} \frac{(\lambda(x-u))^{n-1}}{(n-1)!} \lambda e^{-\lambda u} \, du$$

$$= \lambda e^{-\lambda x} \int_0^x \lambda \frac{(\lambda(x-u))^{n-1}}{(n-1)!} \, du = \lambda e^{-\lambda x} \frac{(\lambda x)^n}{n!}$$

Deduce $F_{n+1}$ by:

$$\frac{d}{dx} \left( 1 - e^{-\lambda x} \sum_{0 \leq m \leq n} \frac{(\lambda x)^m}{m!} \right) =$$

$$e^{-\lambda x} \left( \lambda \sum_{0 \leq m \leq n} \frac{(\lambda x)^m}{m!} - \sum_{0 \leq m \leq n-1} \frac{\lambda (\lambda x)^m}{m!} \right) = f_{n+1}(x)$$
A CTMC

\[
\begin{align*}
\lambda & \\
0.3 & \xrightarrow{0.7} 1 \xrightarrow{1} 2 \\
0.2 & \xrightarrow{0.8} 3
\end{align*}
\]

\[
\begin{align*}
P & \\
1 & \xrightarrow{3.5} 2
\end{align*}
\]

\[
\begin{align*}
\lambda' & \\
0.65 & \xrightarrow{0.35} 1 \xrightarrow{0.2} 2 \\
0.02 & \xrightarrow{0.08} 3
\end{align*}
\]

\[
\begin{align*}
P' & \\
0.9 & \xrightarrow{10} 3
\end{align*}
\]

A uniform version of the CTMC (equivalent w.r.t. the states)
Analysis of a CTMC

Transient Analysis

- Construction of a uniform version of the CTMC \((\lambda, P)\) such that \(P[i,i] > 0\) for all \(i\).
- Computation by case decomposition w.r.t. the number of transitions:
  \[
  \pi(\tau) = \pi(0) \sum_{n \in \mathbb{N}} (e^{-\lambda \tau}) \frac{\tau^n}{n!} P^n
  \]

Steady-state analysis

- The steady-state distribution of visits is given by the steady-state distribution of \((\lambda, P)\) (by construction, the terminal scc are aperiodic) ...

- Equal to the steady-state distribution since the sojourn times follow the same distribution.

- A particular case: \(P\) irreducible
  the steady-state distribution \(\pi\) is the unique solution of \(X \cdot Q = 0 \land X \cdot 1 = 1\)
  where one can omit an arbitrary equation of the first system.
Analysis of a CTMC

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Plan

Stochastic Petri Net

Markov Chain

3 Markovian Stochastic Petri Net

Generalized Markovian Stochastic Petri Net (GSPN)

Product-form Petri Nets
Markovian Stochastic Petri Net

Hypotheses

- The distribution of every transition $t_i$ has a density function $e^{-\lambda_i \tau}$ where the parameter $\lambda_i$ is called the rate of the transition.
- For simplicity reasons, the server policy is single server.

First observations

- The weights for choice policy are no more required since equality of two samples has a null probability. (due to continuity of distributions)
- The residual delay $d_j - d_i$ of transition $t_j$ knowing that $t_i$ has fired (i.e. $d_i$ is the shortest delay) has the same distribution as the initial delay. Thus the memory policy is irrelevant.
Markovian Stochastic Petri Net

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First observations

- The weights for choice policy are no more required since equality of two samples has a null probability. *(due to continuity of distributions)*
- The residual delay $d_j - d_i$ of transition $t_j$ knowing that $t_i$ has fired (i.e. $d_i$ is the shortest delay) has the same distribution as the initial delay. Thus the memory policy is irrelevant.
Key observation: given a marking \( m \) with \( T_m = t_1, \ldots, t_k \)

- The sojourn time in \( m \) is an exponential distribution with rate \( \sum_i \lambda_i \).
- The probability that \( t_i \) is the next transition to fire is \( \frac{\lambda_i}{(\sum_j \lambda_j)} \).
- Thus the stochastic process is a CTMC whose states are markings and whose transitions are the transitions of the reachability graph.
Plan

Stochastic Petri Net

Markov Chain

Markovian Stochastic Petri Net

4 Generalized Markovian Stochastic Petri Net (GSPN)

Product-form Petri Nets
Generalizing Distributions for Nets

Modelling delays with exponential distributions is **reasonable** when:

- Only mean value information is known about distributions.
- Exponential distributions (or combination of them) are enough to approximate the “real” distributions.

Modelling delays with exponential distributions is **not reasonable** when:

- The distribution of an event is known and is poorly approximable with exponential distributions: 
  
  *a time-out of 10 time units*

- The delays of the events have different magnitude orders:
  
  *executing an instruction versus performing a database request*

In the last case, the 0-Dirac distribution is required.
Generalized Markovian Stochastic Petri Net (GSPN)

Generalized Markovian Stochastic Petri Nets (GSPN) are nets whose:

- *timed transitions* have exponential distributions,
- and *immediate transitions* have 0-Dirac distributions.

Their analysis is based on Markovian Renewal Process, a generalization of Markov chains.
A Markovian Renewal Process (MRP) fulfills:

- a relative memoryless property

\[ Pr(S_{n+1} = s_j, T_n < \tau \mid S_0 = s_i_0, ..., S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, ..., S_n = s_i) = Pr(S'_{n+1} = s_j, T_n < \tau \mid S_n = s_i) \equiv Q[i, j, \tau] \]

- The embedded chain is defined by: \( P[i, j] = \lim_{\tau \to \infty} Q[i, j, \tau] \)

- The sojourn time \( Soj \) has a distribution defined by:

\[ Pr(Soj[i] < \tau) = \sum_j Q[i, j, \tau] \]

Analysis of a MRP

- The steady-state distribution (if there exists) \( \pi \) is deduced from the steady-state distribution of the embedded chain \( \pi' \) by:

\[ \pi(s_i) = \frac{\pi'(s_i) E(Soj[i])}{\sum_j \pi'(s_j) E(Soj[j])} \]

- Transient analysis is much harder ... but the reachability probabilities only depend on the embedded chain.
A Markovian Renewal Process (MRP) fulfills:

- a *relative* memoryless property

\[ Pr(S_{n+1} = s_j, T_n < \tau \mid S_0 = s_{i_0}, \ldots, S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, \ldots, S_n = s_i) = Pr(S'_n = s_j, T_n < \tau \mid S_n = s_i) \equiv Q[i, j, \tau] \]

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- Transient analysis is much harder ... but the reachability probabilities only depend on the embedded chain.
A GSPN is a Markovian Renewal Process

Observations

- Weights are required for immediate transitions.
- The *restricted* reachability graph corresponds to the embedded DTMC.

![Diagram of GSPN](image-url)
Steady-State Analysis of a GSPN (1)

Standard method for MRP

- Build the restricted reachability graph equivalent to the embedded DTMC
- Deduce the probability matrix $P$
- Compute $\pi^*$ the steady-state distribution of the visits of markings: $\pi^* = \pi^* P$
- Compute $\pi$ the steady-state distribution of the sojourn in tangible markings:

$$\pi(m) = \frac{\pi^*(m)\text{Soj}(m)}{\sum_{m' \text{ tangible}} \pi^*(m')\text{Soj}(m')}$$

How to eliminate the vanishing markings sooner in the computation?
Steady-State Analysis of a GSPN (2)

An alternative method

- As before, compute the transition probability matrix $P$.
- Compute the transition probability matrix $P'$ between tangible markings.
- Compute $\pi'^*\pi$ the (relative) steady-state distribution of the visits of tangible markings: $\pi'^* = \pi'^*P'$.
- Compute $\pi$ the steady-state distribution of the sojourn in tangible markings:

$$\pi(m) = \frac{\pi'^*(m)\text{Soj}(m)}{\sum_{m'\text{ tangible}} \pi'^*(m')\text{Soj}(m')}$$

Computation of $P'$

- Let $P_{X,Y}$ the probability transition matrix from subset $X$ to subset $Y$.
- Let $V$ (resp. $T$) be the set of vanishing (resp. tangible) markings.
- $P' = P_{T,T} + P_{T,V}(\sum_{n\in\mathbb{N}} P_{V,V}^n)P_{V,T} = P_{T,T} + P_{T,V}(\text{Id} - P_{V,V})^{-1}P_{V,T}$
- Iterative (resp. direct) computations uses the first (resp. second) expression.
Steady-State Analysis: Illustration

\[ p_2 = \frac{w_2}{(w_2 + w_3)} \quad p_3 = \frac{w_3}{(w_2 + w_3)} \]

\[ p_4 = \frac{\lambda_4}{(\lambda_4 + \lambda_5)} \quad p_5 = \frac{\lambda_5}{(\lambda_4 + \lambda_5)} \]

\[ p_2 = \frac{w_2}{(w_2 + w_3)} \quad p_3 = \frac{w_3}{(w_2 + w_3)} \]

\[ (p_2)^2 \quad (p_3)^2 \]

\[ 2p_2p_3 \]

\[ c \]

\[ d/\lambda_1 \]

\[ c \]

\[ d/\lambda_6 \]

“c” and “d” are normalizing constants.
Plan

Stochastic Petri Net

Markov Chain

Markovian Stochastic Petri Net

Generalized Markovian Stochastic Petri Net (GSPN)

Product-form Petri Nets
Steady-State Analysis of a Queue

A (Markovian) queue is a CTMC

- Interarrival time: exponential distribution with parameter $\lambda$
- Service time: exponential distribution with parameter $\mu$

Let $\rho = \frac{\lambda}{\mu}$ be the utilization

- The steady-state distribution $\pi_\infty$ exists iff $\rho < 1$
- The probability of $n$ clients in the queue is $\pi_\infty(n) = \rho^n (1 - \rho)$
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**Analysis of Two Queues in Tandem**

![Diagram of two queues in tandem]

**Observation.** The associated Markov chain is more complex than the one corresponding to two isolated queues. However ...

Assume $\rho_1 = \frac{\lambda}{\mu} < 1$ and $\rho_2 = \frac{\lambda}{\delta} < 1$

- The steady-state distribution $\pi_\infty$ exists.
- The probability of $n_1$ clients in queue 1 and $n_2$ clients in queue 2 is
  
  $$
  \pi_\infty(n_1, n_2) = \rho_1^{n_1}(1 - \rho_1)\rho_2^{n_2}(1 - \rho_2)
  $$

- It is the product of the steady-state distributions corresponding to two isolated queues.
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In a steady-state

- Define the (input and output) flow through queue 1 (resp. 2) as $\gamma_1$ (resp. $\gamma_2$).
- Then $\gamma_1 = \lambda + q\gamma_2$ and $\gamma_2 = p\gamma_1$. Thus $\gamma_1 = \frac{\lambda}{1-pq}$ and $\gamma_2 = \frac{p\lambda}{1-pq}$

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- It is the product of the steady-state distributions corresponding to two isolated queues.
Visit ratios (up to a constant)

- Define the visit ratio flow of queue $i$ as $v_i$.
- Then $v_1 = v_3 + qv_2$, $v_2 = pv_1$ and $v_3 = (1 - p)v_1 + (1 - q)v_2$.
  Thus $v_1 = 1$, $v_2 = p$ and $v_3 = 1 - pq$.

Define $\rho_1 = \frac{v_1}{\mu}$, $\rho_2 = \frac{v_2}{\delta}$ and $\rho_3 = \frac{v_3}{\lambda}$

- The steady-state probability of $n_i$ clients in queue $i$ is
  $\pi_\infty(n_1, n_2, n_3) = \frac{1}{G}\rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3}$ (with $n_1 + n_2 + n_3 = n$)
- where $G$ the normalizing constant can be efficiently computed by dynamic programming.
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- where $G$ the normalizing constant can be efficiently computed by dynamic programming.
A (single client class) queuing network can easily be represented by a Petri net.

Such a Petri net is a state machine: every transition has at most a single input and a single output place.

Can we define a more general subclass of Petri nets with a product form for the steady-state distribution?
Product Form Stochastic Petri Nets (PFSPN)

Principles

- Transitions can be partitioned into subsets corresponding to several classes of clients with their specific activities.
- Places model resources shared between the clients.
- Client states are implicitly represented.
The resource graph

- The vertices are the input and the output bags of the transitions.
- Every transition of the net $t$ yields a graph transition $\bullet t \rightarrow t^*$
- Client classes correspond to the connected components of the graph.

First requirement: The connected components of the graph must be strongly connected.
Vector $-p_2-p_3$ is a witness for bag $p_1+p_4$:

$(-p_2-p_3) \cdot W(t_3)=1$

$(-p_2-p_3) \cdot W(t_1)=-1$

$(-p_2-p_3) \cdot W(t)=0$ for every other $t$

where $W$ is the incidence matrix

**Witness for a bag $b$**

- Let $In(b)$ (resp. $Out(b)$) the transitions with input (resp. output) $b$.

- Let $v$ be a place vector, $v$ is a witness for $b$ if:
  - $\forall t \in In(b) \ v \cdot W(t) = -1$ (where $W(t)$ is the incidence of $t$)
  - $\forall t \in Out(b) \ v \cdot W(t) = 1$
  - $\forall t \notin In(b) \cup Out(b) \ v \cdot W(t) = 0$

**Second requirement: Every bag must have a witness.**
Steady-State Distributions of PFSPN

The reachability space:

\[ m(p_1) + m(p_2) + m(p_3) = 2 \]
\[ m(p_4) + m(p_5) + m(p_6) = m(p_1) + 1 \]

Steady-state distribution

- Assume the requirements are fulfilled, with \( w(b) \) the witness for bag \( b \).
- Compute the ratio visit of bags \( v(b) \) on the resource graph.
- The output rate of a bag \( b \) is \( \mu(b) = \sum_{t \mid t \in b} \mu(t) \) with \( \mu(t) \) the rate of \( t \).
- Then: \( \pi_\infty(m) = \frac{1}{G} \prod_b \left( \frac{v(b)}{\mu(b)} \right)^{w(b) \cdot m} \)

Observation. The normalizing constant can be efficiently computed if the reachability space is characterized by linear place invariants.
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