Logique
TD n°6

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We consider the sequent calculus as seen in the course slides, that is the cut-free sequent calculus. Its rules and the axiom rule and all right and left rules (including right and left contractions). We do not (yet) consider the cut rule presented in the course notes.

Exercise 1 : Sequent calculus proofs

Give proofs in sequent calculus of the following formulas:

1. $A \vee (A \Rightarrow B)$
2. $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
3. $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$
4. $\neg\neg A \Rightarrow A$
5. $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$
6. $\neg\forall x.\, R(x) \Rightarrow \exists x.\, \neg R(x)$
7. $\forall x.\, (Q \vee R(x)) \Rightarrow (Q \vee \forall x.\, R(x))$
8. $\exists x.\, [(R(a) \vee R(b)) \Rightarrow R(x)]$
9. Show that if $a \neq b$, there is no proof of 8. that does not use contractions.

> A *unification problem* is a set E of equations of the form $t \stackrel{?}{=} u$.
> A *unifier* (i.e. solution to the unification problem) of a set E is a substitution σ such that for every equation $t \stackrel{?}{=} u$ in E, $t\sigma = u\sigma$.

The unification procedure seen in class are reminded at the end of the exercise sheet (Algorithm 1).

Exercise 2 : Some examples

Apply the procedure to the following unification problems (your answer should either be `fail` or the substitution returned by the procedure):

- $E_1 = f(x, g(a, y)) \stackrel{?}{=} f(h(y), g(y, a))\,;\ g(x, h(y)) \stackrel{?}{=} g(z, z)$
- $E_2 = f(x, x) \stackrel{?}{=} f(g(y), z)\,;\ h(z) \stackrel{?}{=} h(y)$
- $E_3 = f(x, a) \stackrel{?}{=} f(b, y)\,;\ f(x) \stackrel{?}{=} f(y)$

Exercise 3 : Interpolation theorem

If ϕ is a formula, we call $L(\phi)$ the set of free variables and function and predicate symbols appearing in ϕ. By extension, if Γ is a multiset of formulas, we write $L(\Gamma) = \bigcup_{\phi \in \Gamma} L(\phi)$. We want to show that if $L(\Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2)$ does not contain a function symbol and if $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ is provable, then there exists a formula ξ such that:

- $\Gamma_1 \vdash \xi, \Delta_1$ and $\Gamma_2, \xi \vdash \Delta_2$ are provable;
- $L(\xi) \subseteq L(\Gamma_1 \cup \Delta_1) \cap L(\Gamma_2 \cup \Delta_2)$

1. Prove this result. You will consider the following cases in detail : ax ; $\Rightarrow_{right}$; $\Rightarrow_{left}$; $\forall_{left}$; $\forall_{left}$.
2. Prove the *interpolation theorem*: if ϕ and ψ are formulas that do not contain function symbols and if $\vdash \phi \Rightarrow \psi$ is provable, then there exists a formula ξ such that:
   - $\vdash \phi \Rightarrow \xi$ and $\vdash \xi \Rightarrow \psi$ are provable;
   - $L(\xi) \subseteq L(\phi) \cap L(\psi)$.
3. ($\star$) We apply the interpolation theorem to prove the Beth theorem.

   Let P and P be two unary predicates. Let $\Gamma(P)$ be a set of closed formulas that do not contain the symbol P'. We write $\Gamma(P')$ the set of formulas generated by replacing the symbol P by the symbol P' in $\Gamma(P)$.

   We say that $\Gamma(P)$ *implicitely defines* P if $\Gamma(P), \Gamma(P') \vdash \forall x. (P(x) \Leftrightarrow P'(x))$ is provable; we say that $\Gamma(P)$ *explicitely defines* P if there exists a formula $\phi(x)$ using neither P nor P' such that $\Gamma(P) \vdash \forall x. (A(x) \Leftrightarrow P'(x))$.

   Prove that $\Gamma(P)$ implicitely defines P iff $\Gamma(P)$ explicitely defines P.
4. ($\star\star$) In fact, the interpolation theorem holds even when ϕ and ψ contain function symbols. Can you see how to use the previous result to treat this case?

Exercise 4 : Studying the unification algorithm

We study the properties of the unification algorithm `Unif`.

1. Is the procedure deterministic?
2. Show that it always terminates (in failure or by returning a substitution).
3. We now want to prove that it indeeds calculates a unifier of the set of equations E given as entry. We will in fact prove a more general result.
   (a) Show that if E contains $f(u_1, ..., u_n) \stackrel{?}{=} g(t_1, ..., t_m)$, where $f \neq g$, then it is not unifiable.
   (b) Show that if E contains $x \stackrel{?}{=} t$, where $x \in \mathcal{X}$, $v \notin \mathcal{X}$ and $x \in Var(t)$, then it is not unifiable.

   > A *most general unifier* (mgu) σ of a unification problem E is a unifier of E such that for every unifier τ of E, there exists η such that $\tau = \eta \circ \sigma$.

   (c) Show that the unification problem $x \stackrel{?}{=} f(y)$ has an infinity of unifiers (you can use a constant, i.e. 0-ary, function symbol a).
   Is there a most general one? When it exists, is there unicity of the mgu?
   (d) Show that if $E = E' \cup \{ x \stackrel{?}{=} x \}$, then σ unifies E iff it unifies E'.
   Observe that this implies that σ is a mgu of E iff it is a mgu of E'.
   (e) Show that if $E = E' \cup \{ f(u_1, ..., u_n) \stackrel{?}{=} (t_1, ..., t_n) \}$, then σ unifies E iff it unifies $E' \cup \{u_1 \stackrel{?}{=} t_1, \ldots, u_n \stackrel{?}{=} t_n\}$.
   Observe that this implies that σ is a mgu of E iff it is a mgu of E'.
   (f) Show that for every substitution σ, variable x and term t, if $x\sigma = t\sigma$ then $\sigma = \sigma \circ \{x \mapsto t\}$.
   (g) Show that if $E = E' \cup \{ x \stackrel{?}{=} t \}$ where $x \notin var(t)$, then σ unifies $E'\{x \to t\}$ iff $\sigma \circ \{x \mapsto t\}$ unifies E.
   Prove using the previous question that: σ is a mgu of $E'\{x \to t\}$ implies $\sigma \circ \{x \mapsto t\}$ is a mgu of E.
   (h) Show that if on input E the algorithm
   - returns a substitution σ, then σ is a mgu for E;
   - fails, then the unification problem E has no solution.

1 Additional exercises

Exercise 5 : A new rule

In this exercise we introduce the cut rule to the sequent calculus:

$$\frac{\Gamma \vdash \psi \qquad \Gamma, \psi \vdash \phi}{\Gamma \vdash \phi}\ \text{cut}$$

Give two proofs in sequent of $A \Rightarrow B, A \Rightarrow C, B \wedge C \Rightarrow D \vdash A \Rightarrow D$: a first with cuts, a second without cuts.

Exercise 6 : Return to natural deduction

This exercise is a course reminder on the equivalence between natural deduction and sequent calculus.

1. Show that a proof in natural deduction of a sequent $\Gamma, \phi \wedge \psi \vdash \xi$ can be transformed in a proof in natural deduction of the sequent $\Gamma, \phi, \psi \vdash \xi$.
2. And for other left rules?

The unification algorithm

The unification procedure is the following:

Algorithm 1: `Unif`

Input : a unification problem E

if $E = E' \cup \{f(u_1, ..., u_n) \stackrel{?}{=} f(t_1, ..., t_n)\}$ **then**
  `Unif`$(E' \cup \{u_1 \stackrel{?}{=} t_1, \ldots, u_n \stackrel{?}{=} t_n\})$
else if $E = E' \cup \{f(u_1, ..., u_n) \stackrel{?}{=} g(t_1, ..., t_m)\}$ *where* $f \neq g$ **then**
  `fail`
else if $E = E' \cup \{x \stackrel{?}{=} x\}$ **then**
  `Unif`(E')
else if $E = E' \cup \{x \stackrel{?}{=} t\}$ *where* $x \in \mathcal{X}$ *and* $x \notin Var(t)$ **then**
  `Unif`$(E'\{x \to t\}) \circ \{x \mapsto t\}$
else if $E = E' \cup \{x \stackrel{?}{=} t\}$ *where* $x \in \mathcal{X}$, $t \notin \mathcal{X}$ *and* $x \in Var(t)$ **then**
  `fail`
else if $E = \emptyset$ **then**
  id