A Tale of Two Dualities

Jean Goubault-Larrecq

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Outline

1 Introduction

2 De Groot Duality

3 Angelic and Demonic Nondeterminism

4 Probabilistic Choice

5 Belief Functions, Plausibilities

6 Concluding Remarks

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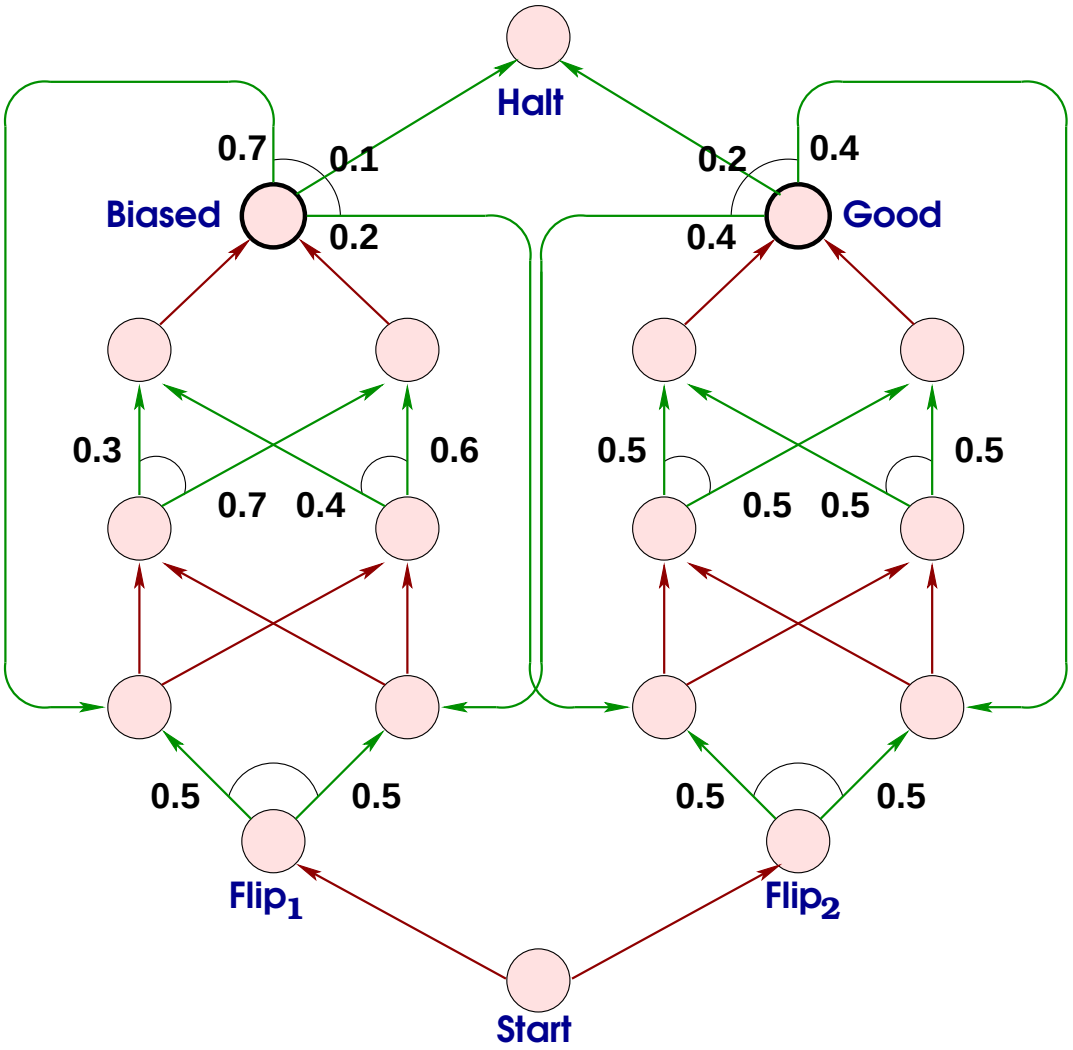
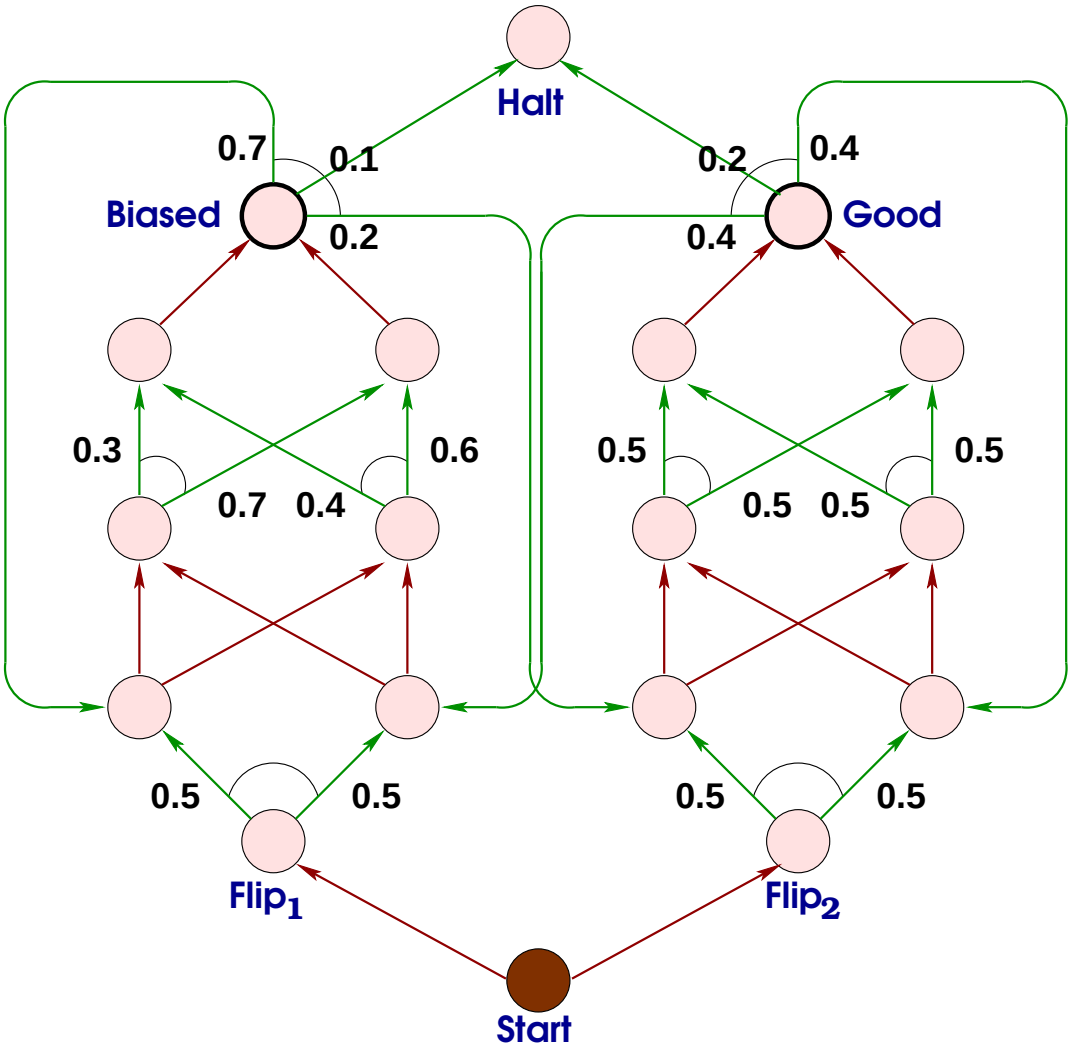
A $1\frac{1}{2}$ Game (Demonic Case)



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

Start



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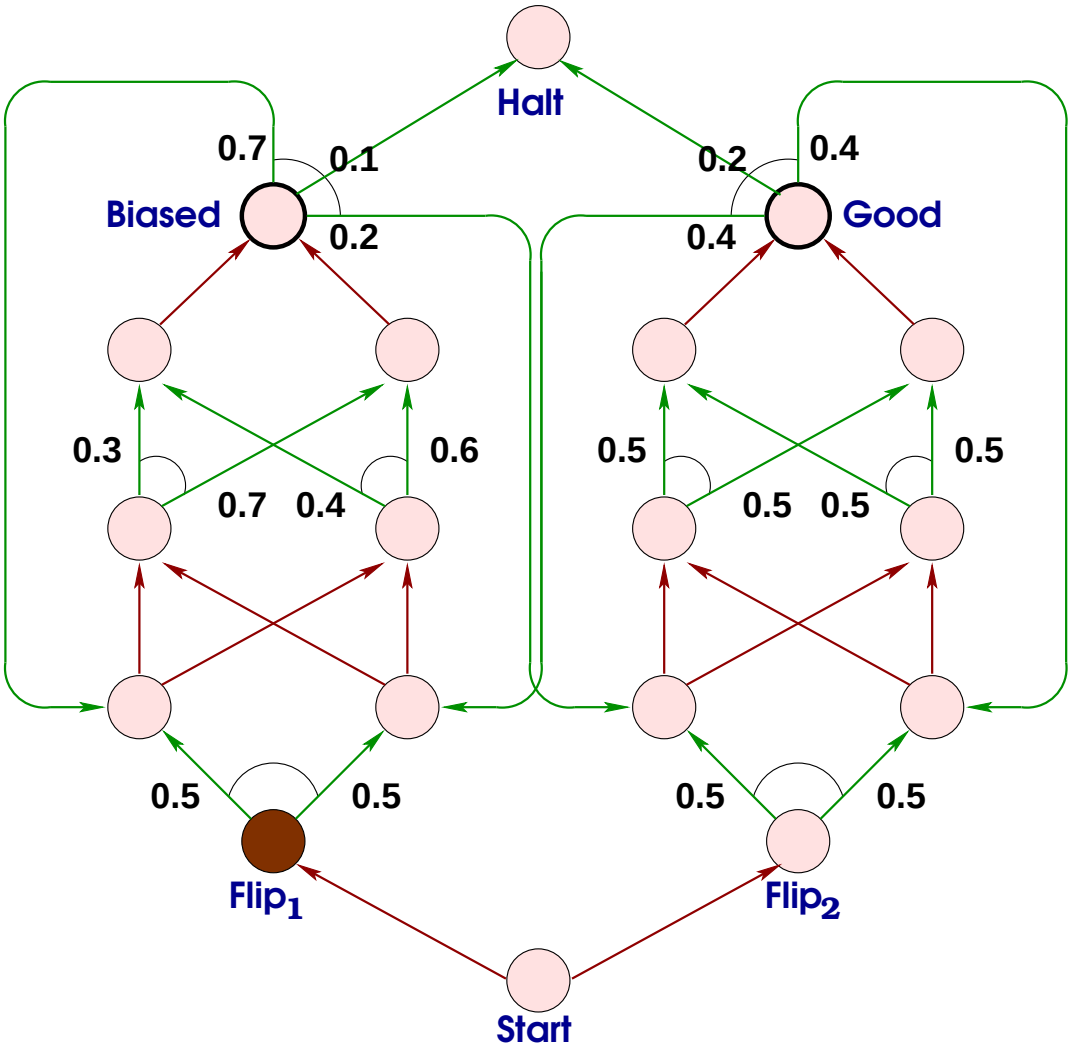
Probabilistic choice (by program)

C's Turn: Malevolently Chooses Biased Side



Non−deterministic (demonic) choice (by adversary)

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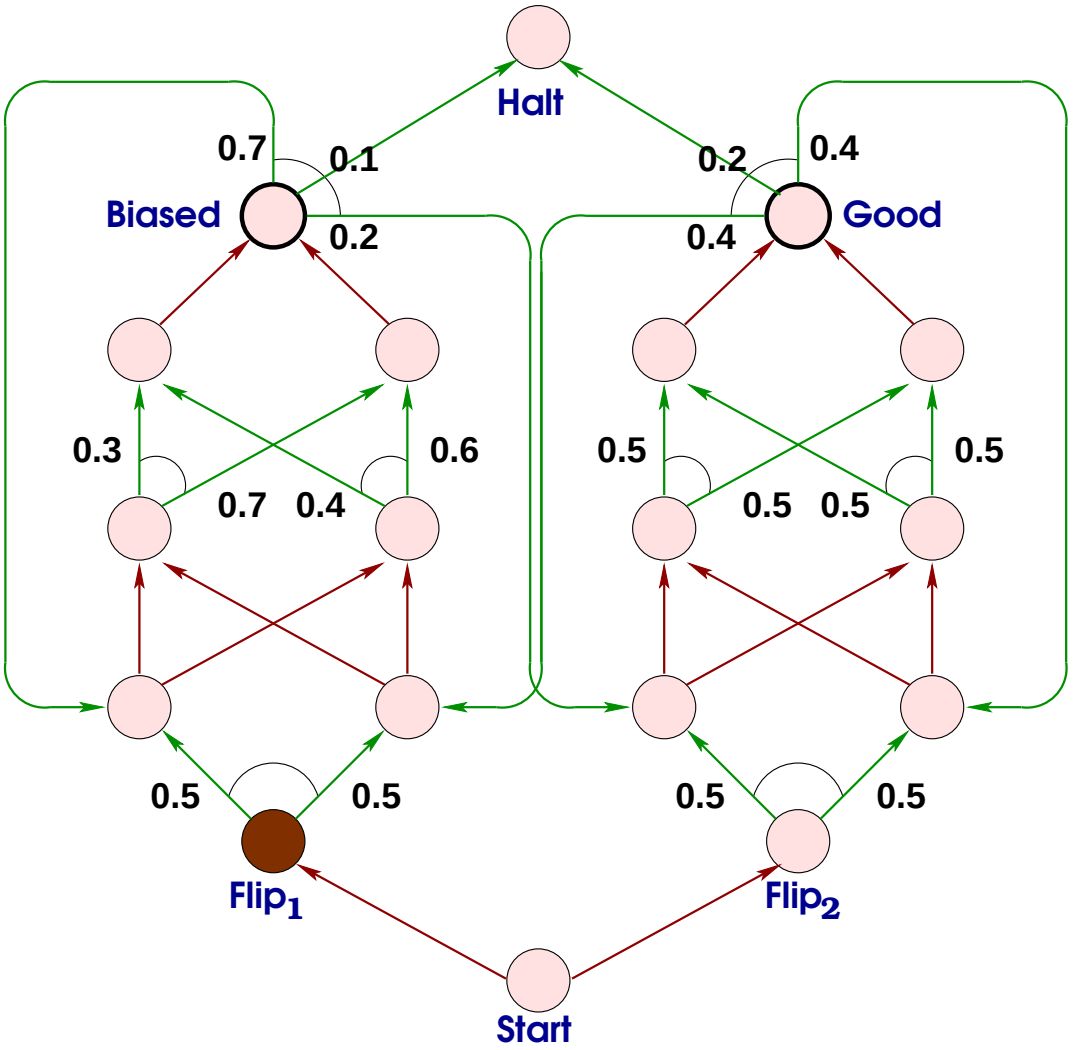
P's Turn: Flipping a Coin



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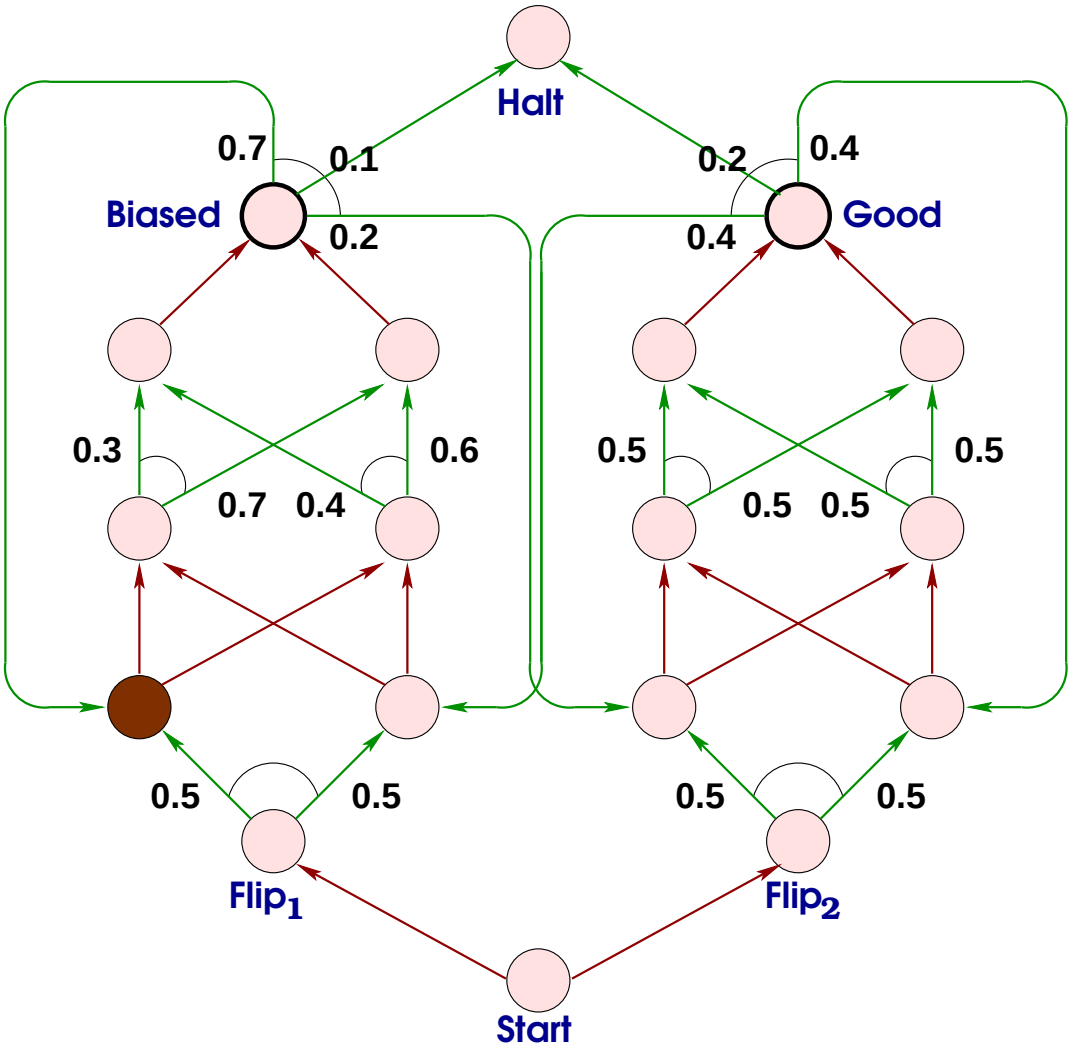
P's Turn: Advancing



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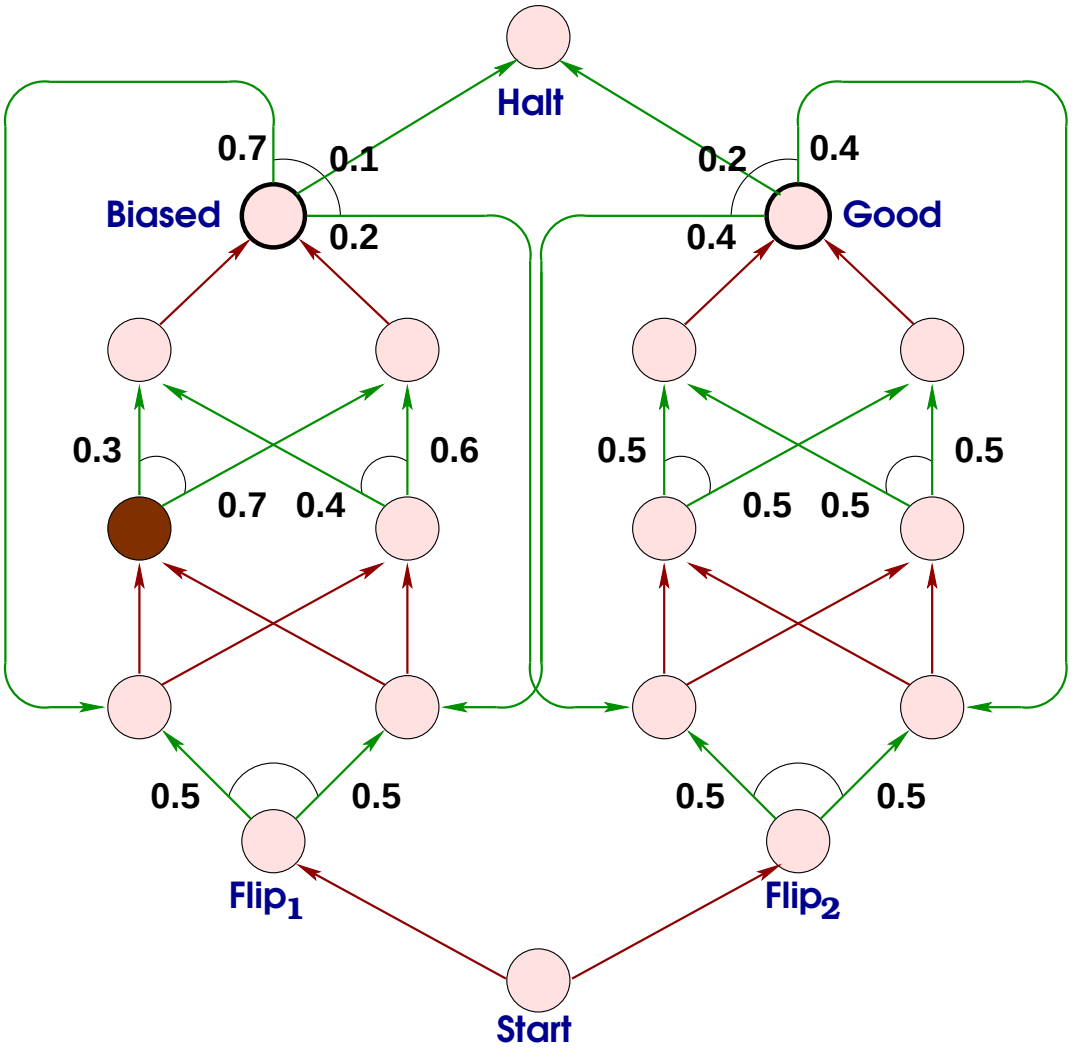
C's Turn: Picking Most Biased Side



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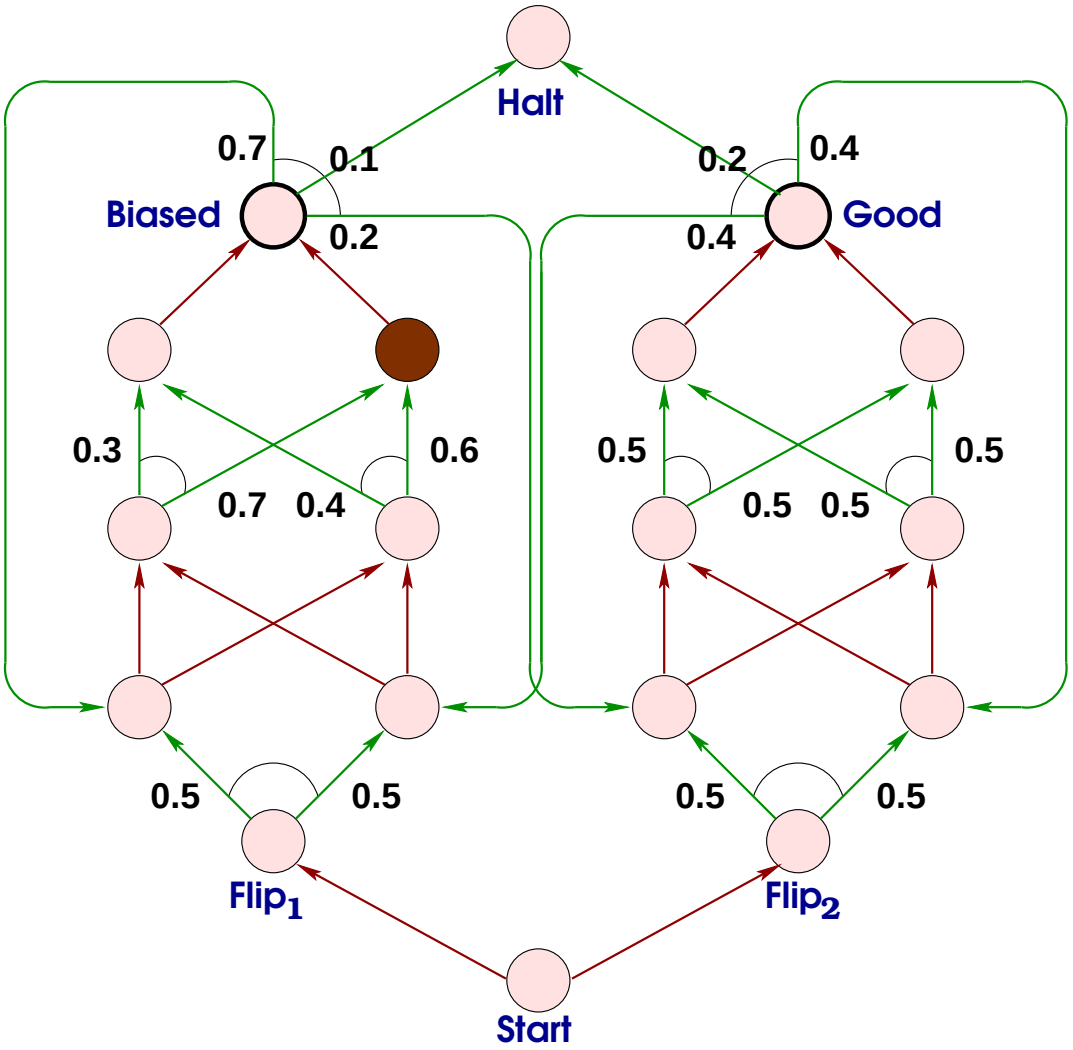
P's Turn



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Stably Compact Spaces

Note: we use the letter U for opens, Q for saturated compacts.

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Definition (Stably Compact)

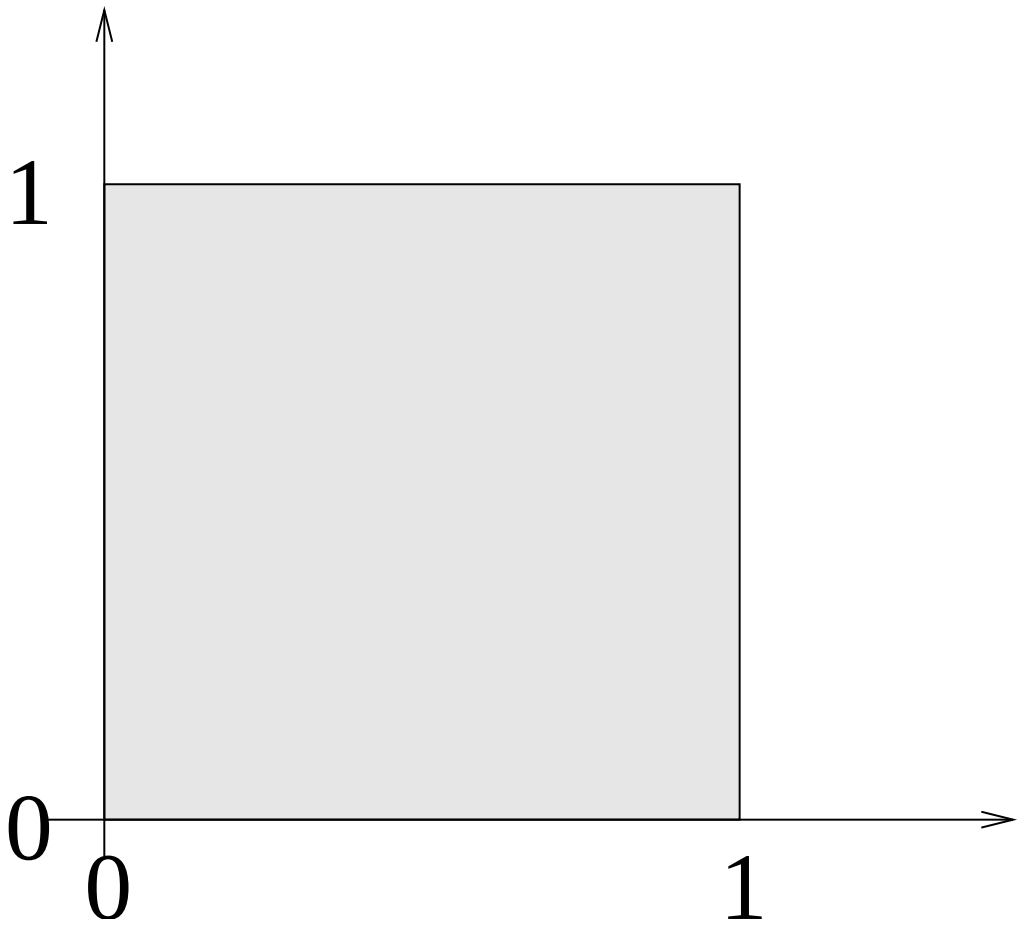
A T_0 topological space X is *stably compact* iff:

- it is *well-filtered*: if $\bigcap^{\downarrow}_{i \in I} Q_i \subseteq U$, then $Q_i \subseteq U$ for some $i \in I$ already;

  Compare: in any space, if $\bigcup^{\uparrow}_{i \in I} U_i \subseteq Q$, then $U_i \supseteq Q$ for some $i \in I$;
- it is *locally compact*: if $x \in U$, then $x \in U \subseteq \overset{\circ}{Q} \subseteq Q \subseteq U$ for some Q;
- it is *coherent*: $Q \cap Q'$ is compact for any Q, Q';
- it is *compact*.

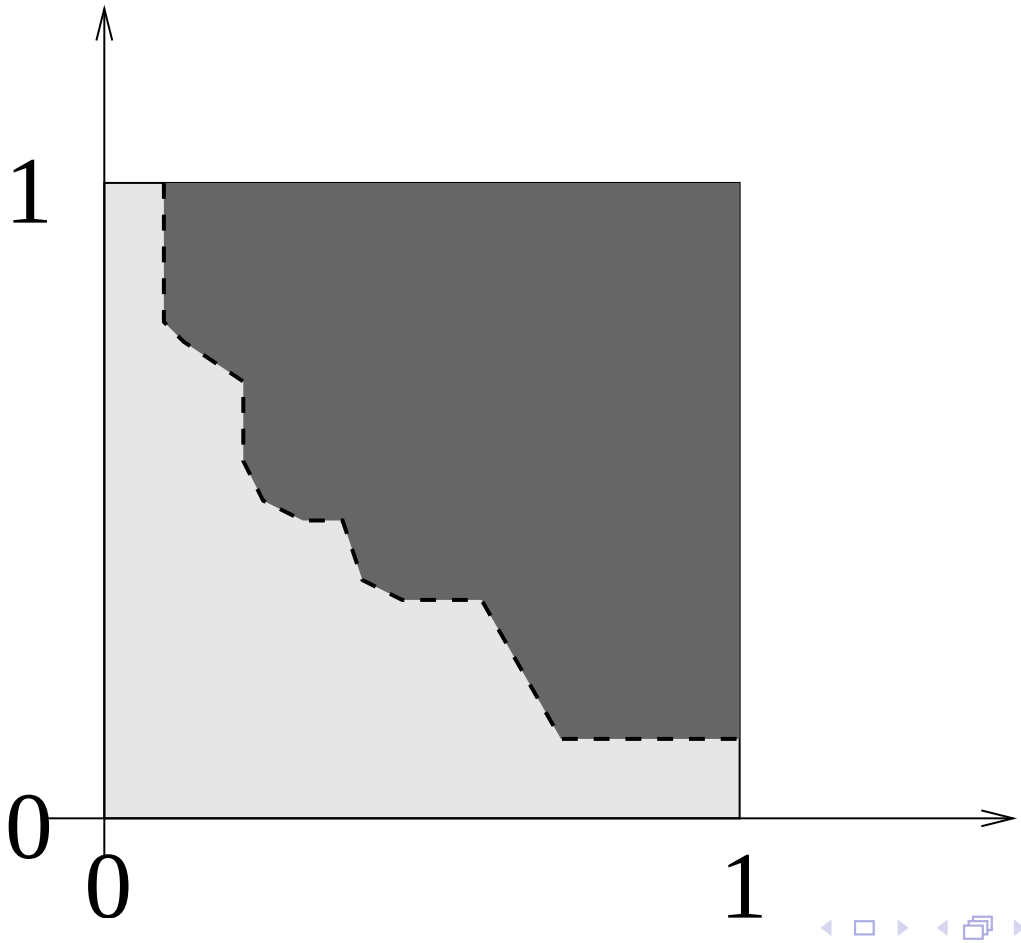
Example: $[0,1]^2_\sigma$

Consider $[0,1]^2_\sigma$, with its Scott topology:



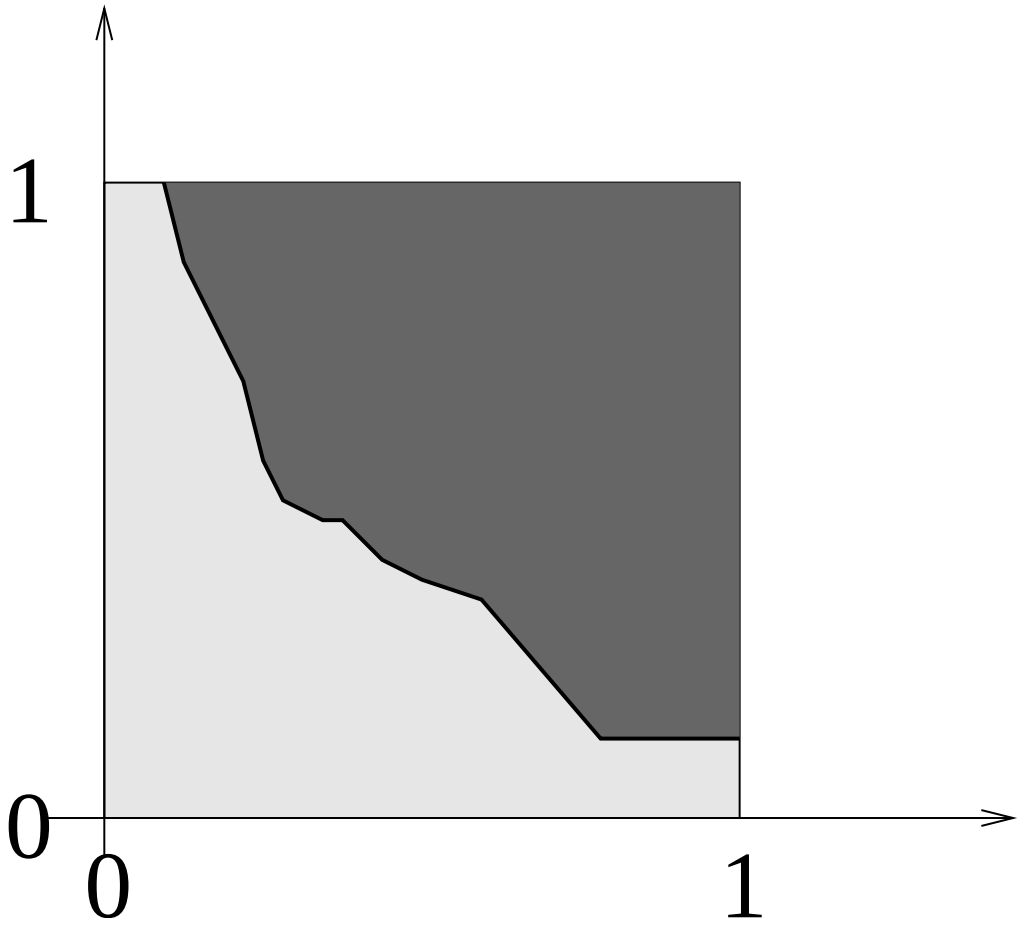
(Scott-)Opens in $[0,1]^2_\sigma$

(Scott-)Opens U are *upward-closed*, and *inaccesible from below*: if $\sup^{\uparrow}_{i \in I} x_i \in U$, then $x_i \in U$ for some i:



Saturated Compacts in $[0,1]^2_\sigma$

(In a nutshell, saturated=upward-closed.)



Compact Pospaces

Definition (Nachbin 1948)

A *compact pospace* is:

- a compact, Hausdorff space X,
- with an ordering $\leq$ whose graph is closed in $X \times X$.

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Note: $[0,1]^2$, with its usual topology, and the componentwise ordering $\leq$, is a compact pospace.

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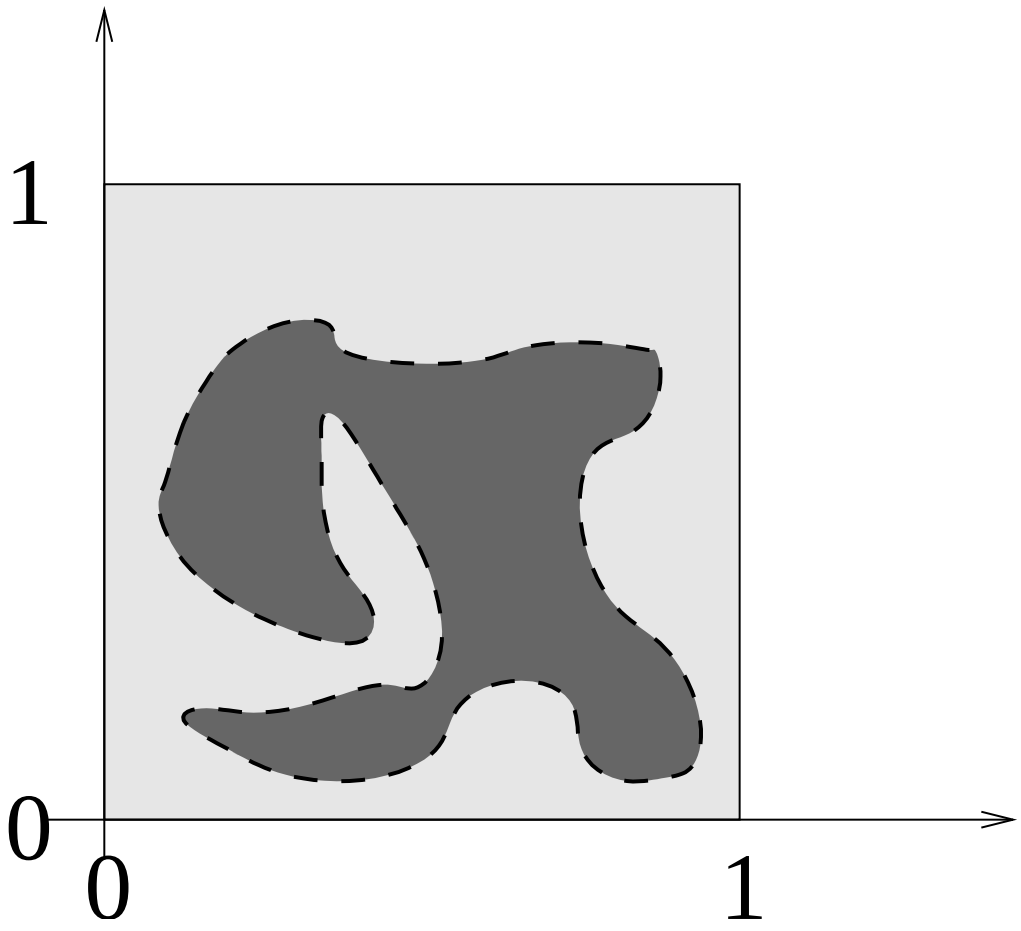
Note: $[0,1]^2$, with its usual topology, and the componentwise ordering $\leq$, is a compact pospace.
Let the *upwards topology* $\mathcal{U}$ of X consists of all *upward-closed opens* U of X: if $x \leq y$ and $x \in U$ then $y \in U$.

Proposition (de Groot duality, part 1)

Let $(X, \leq)$ be a compact pospace.
Then $(X, \mathcal{U})$ is stably compact.

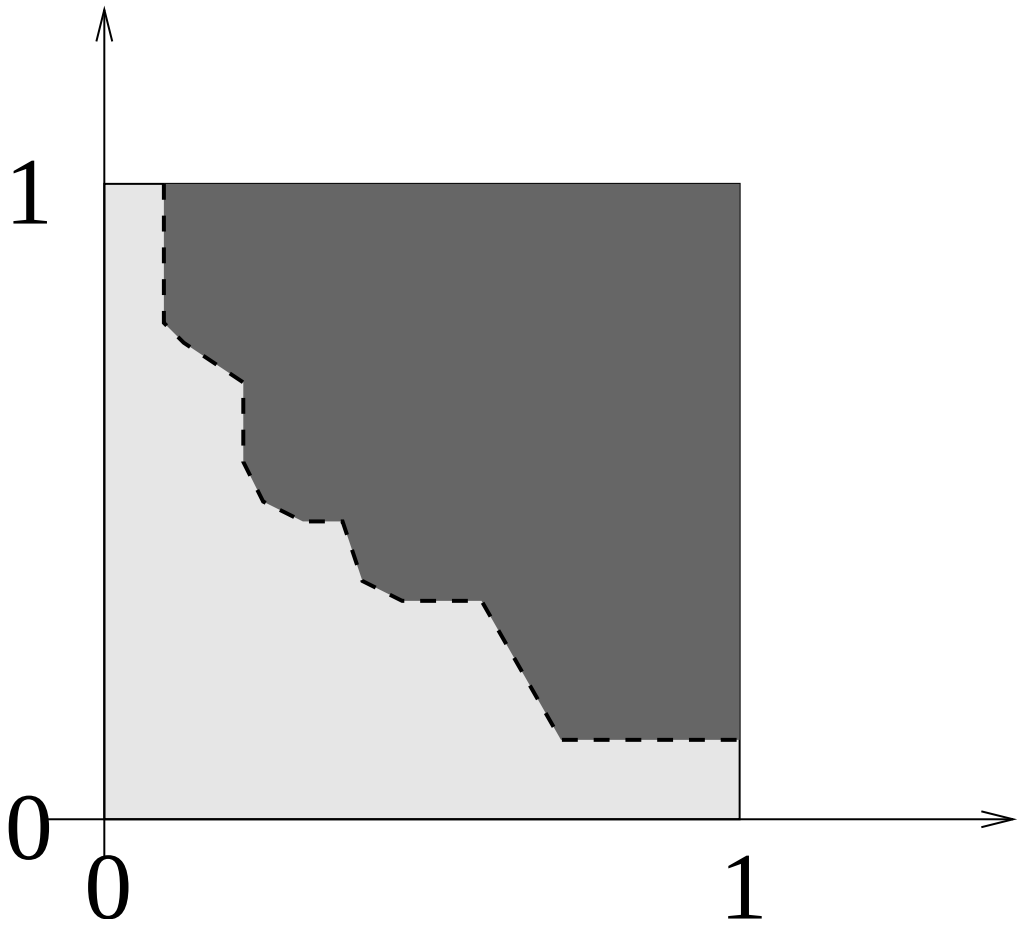
$[0,1]^2$, as a Compact Pospace

Here is an open of $[0,1]^2$:



$[0,1]^2$, as a Compact Pospace

Now here is an open of $[0,1]^2$ in the upwards topology:



Fact: upwards topology of $[0,1]^2 =$ Scott topology (of $[0,1]^2_\sigma$).

Conversely (1): the Cocompact Topology

In a stably compact space X, the *cocompacts* $\overline{Q}$, a.k.a., the complements of compact saturated subsets Q form another topology, the *cocompact topology*:

Proposition

Let X be stably compact, X^{d} is X with the cocompact topology, and is called the de Groot dual of X.

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Let X be stably compact, X^{d} is X with the cocompact topology, and is called the de Groot dual of X.

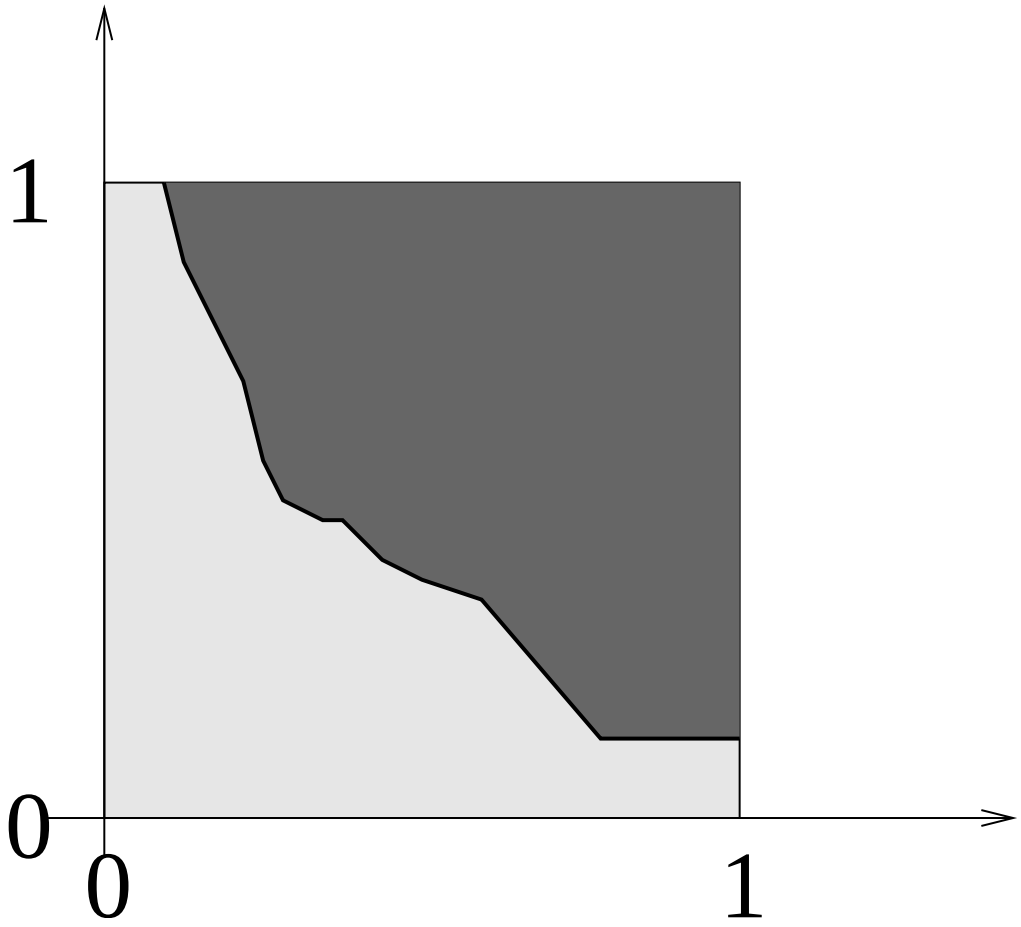
Note: This is a real duality:

Proposition

Let X be stably compact. Then X^{d} is stably compact, and $X^{\mathrm{dd}} = X$.

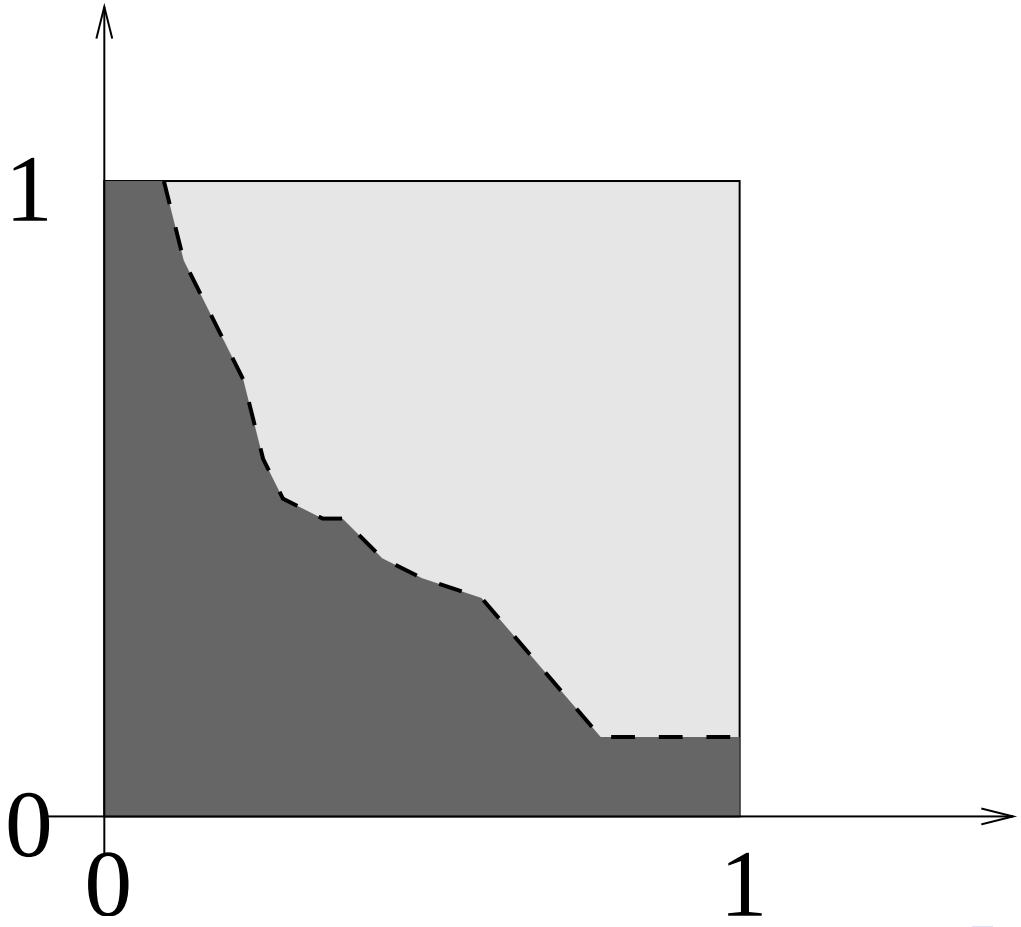
Cocompacts in $[0,1]^2_\sigma$

Take a compact saturated subset Q:



Cocompacts in $[0,1]^2_\sigma$

Then look at its complement — this is open, and downward-closed:



The Specialization Ordering

Any T_0 space X has an ordering $\leq$, defined as:

Definition (Specialization Ordering)

Let $x \leq y$ iff every open containing x also contains y.

Note: Think of opens U as tests (does $x \in U$?), and $\leq$ as a *simulation relation*: y passes all the tests that x passes.

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Example: The specialization ordering of $[0,1]^2_\sigma$ is the usual, pointwise ordering.
Fact: Well, for any dcpo $(X, \leq)$, the specialization ordering *is* $\leq$.

The Patch Topology

Definition

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Let X^{patch} be X with the patch topology.

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Let X^{patch} be X with the patch topology.

Looping the loop:

Theorem

Let X be stably compact. Then X^{patch}, together with the specialization ordering $\leq$ of X, is a compact pospace.
The upward topology of $(X^{\mathrm{patch}}, \leq)$ is just that of X.
The downward topology of $(X^{\mathrm{patch}}, \leq)$ is just that of X^{d}.

How Special is $[0,1]^2_\sigma$?

$[0,1]^2_\sigma$ is both a dcpo and stably compact.

Lemma

Every continuous dcpo with a $\perp$ is a compact pospace, as soon as it is coherent.

Note: for continuous dcpos, stably compact$\Leftrightarrow$Lawson-compact.

Scott domains	$\subset$	continuous bounded-complete dcpos **"Domains"** (e.g., $[0,1]^2_\sigma$)	$\subset$	Retracts of bifinite domains [G. Plotkin]	$\subseteq$	***FS***-domains [A. Jung]
						$\subset$ continuous, coherent, dcpos with $\perp$
						$\cap$
						stably compact spaces

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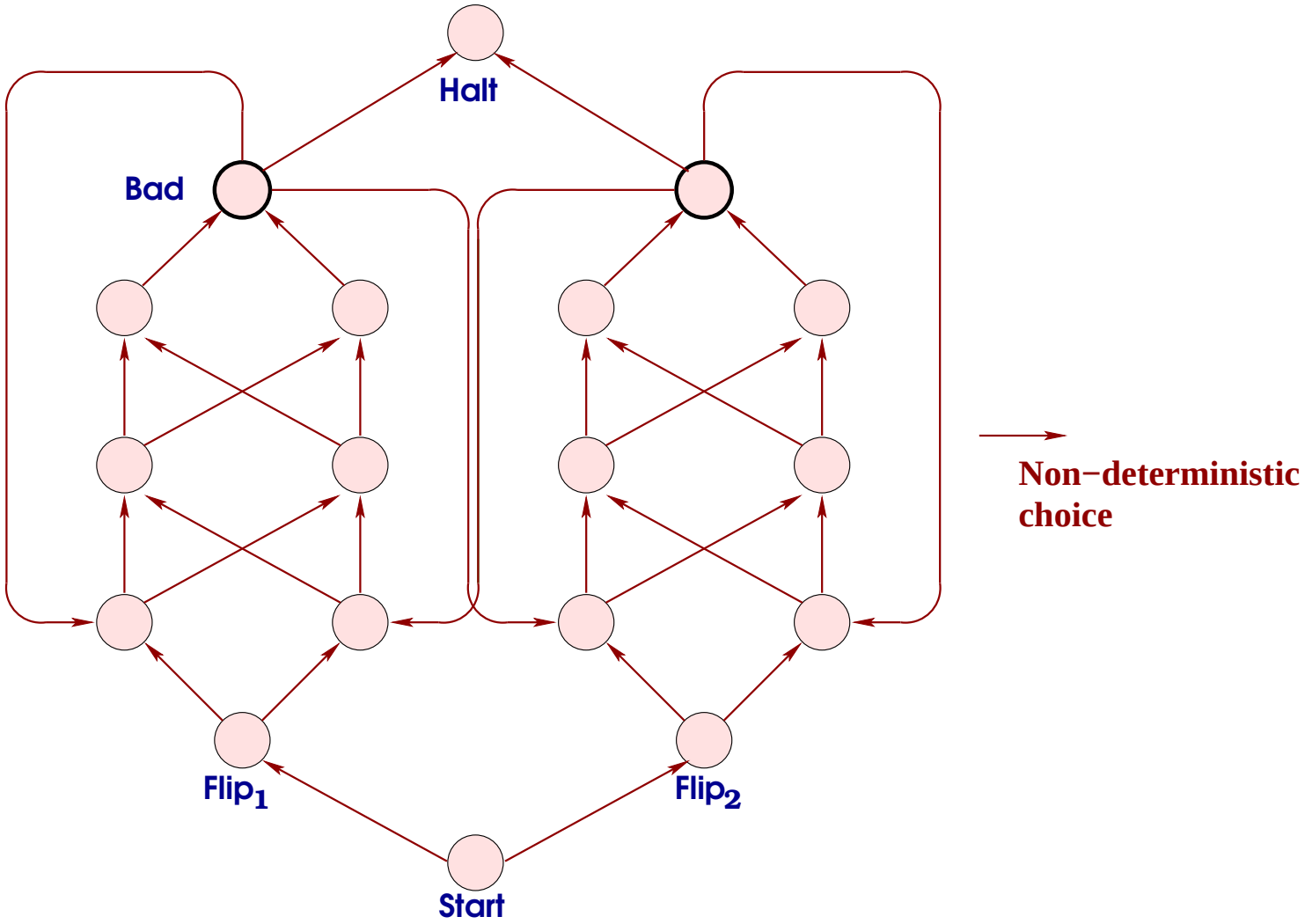
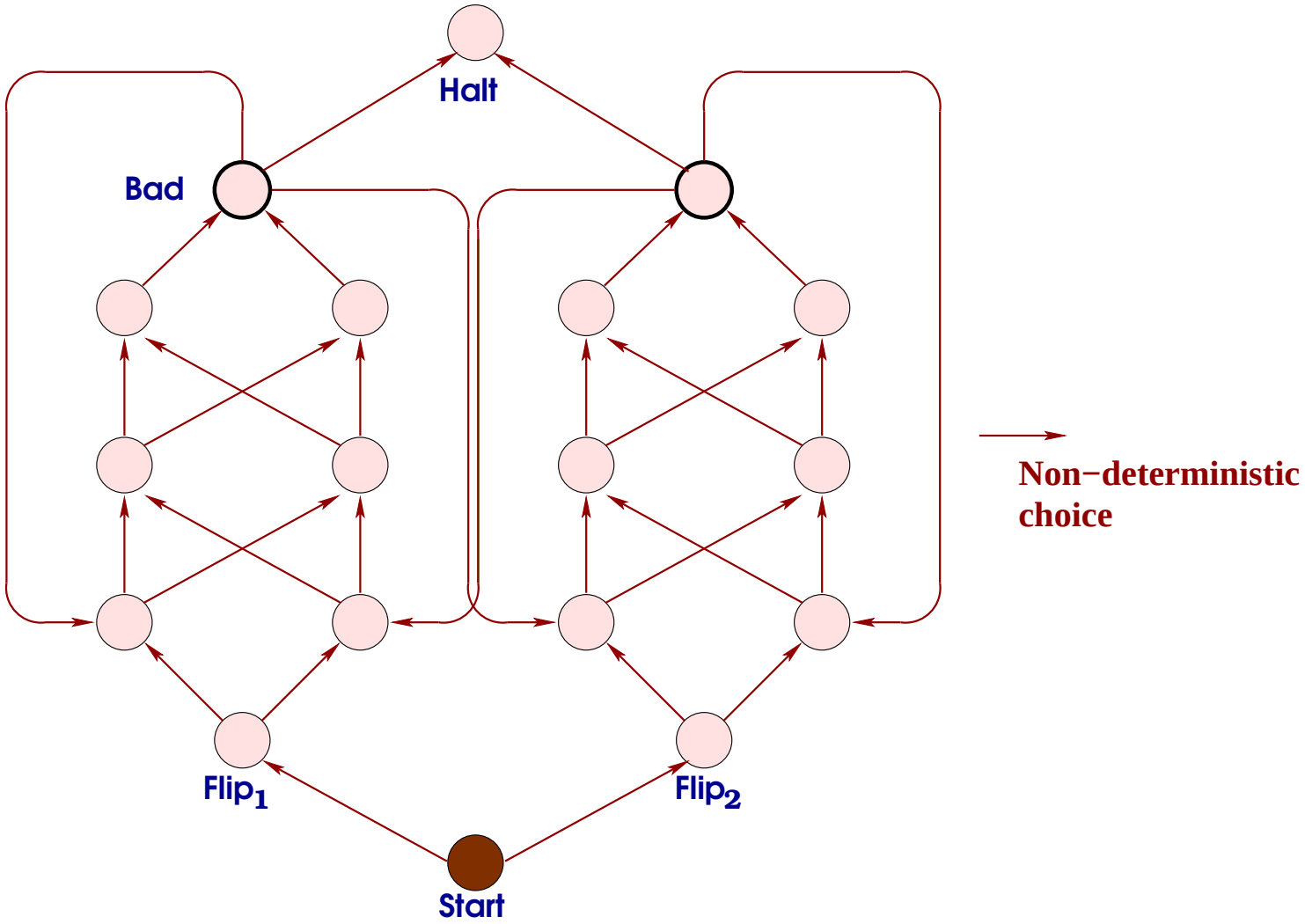
Non-Deterministic Choice Only: Automata



Trying to Reach a Bad State



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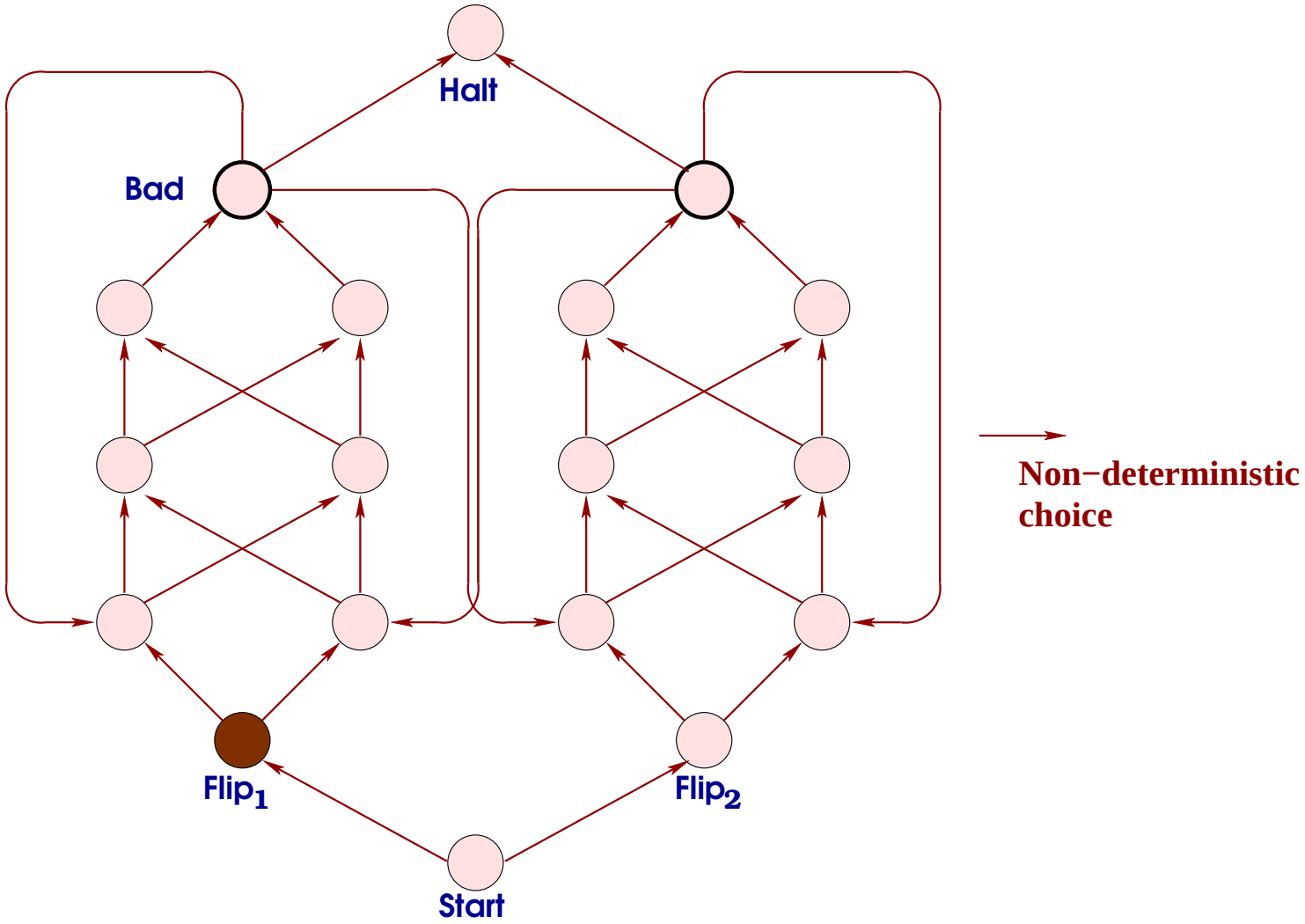

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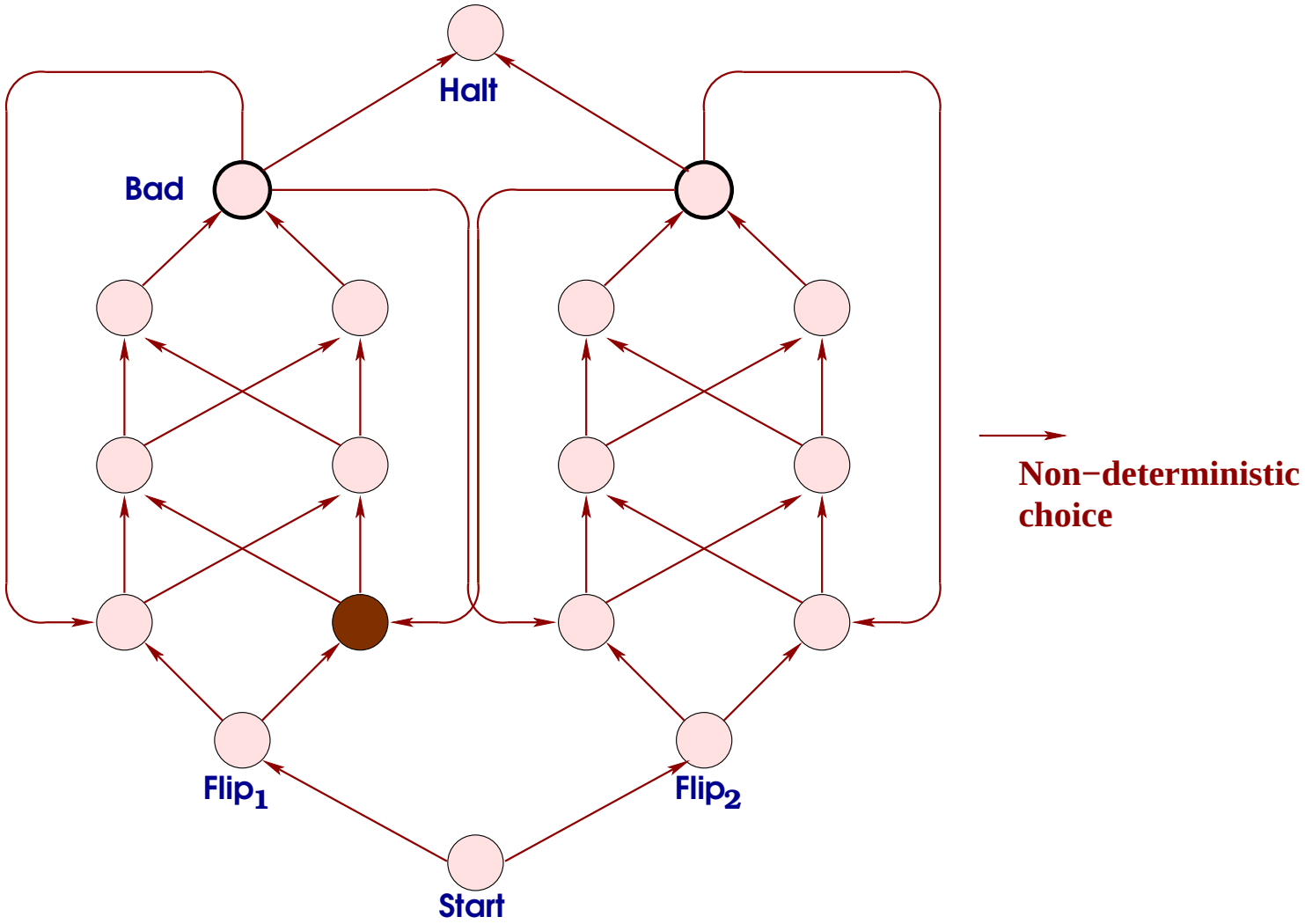

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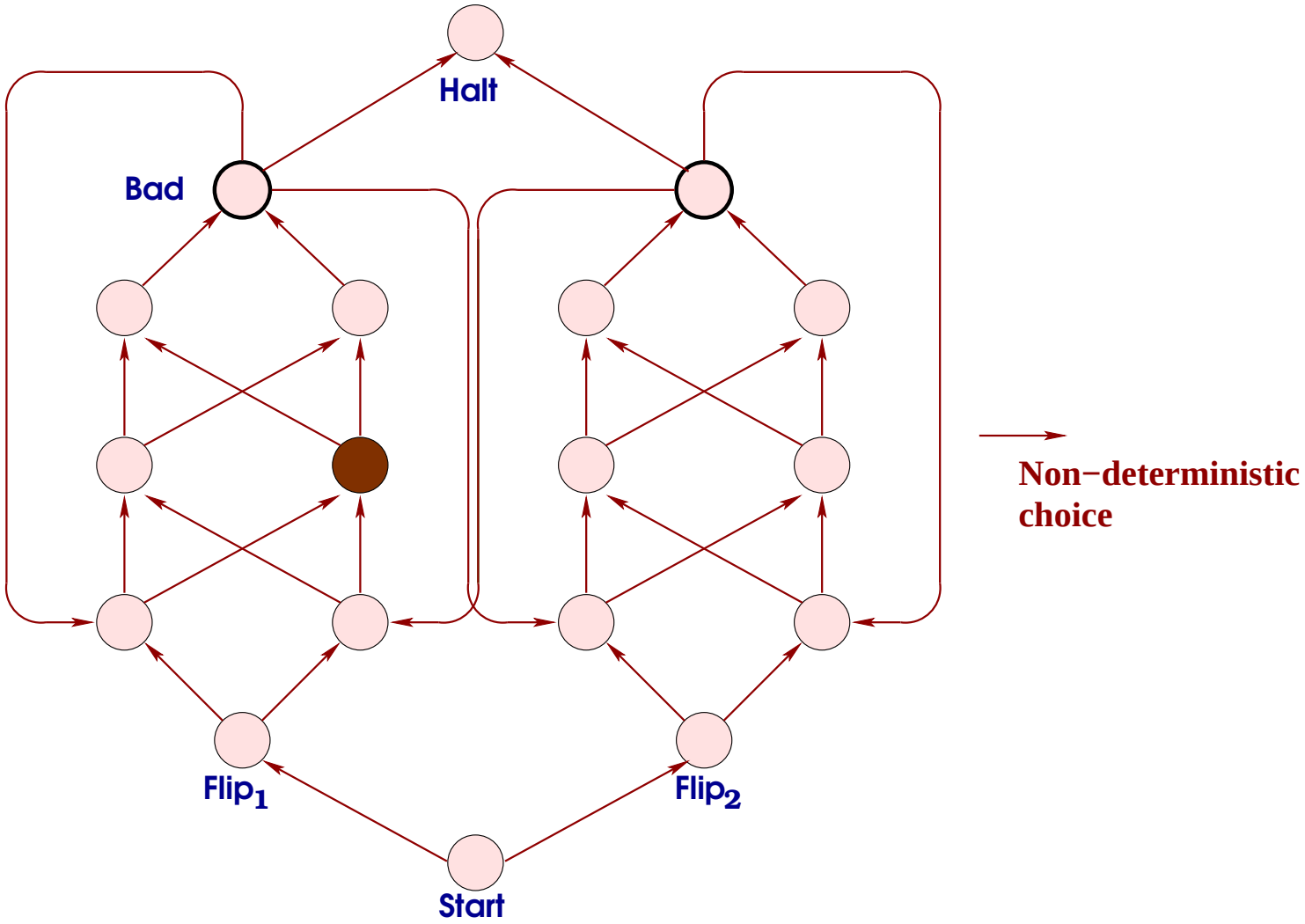

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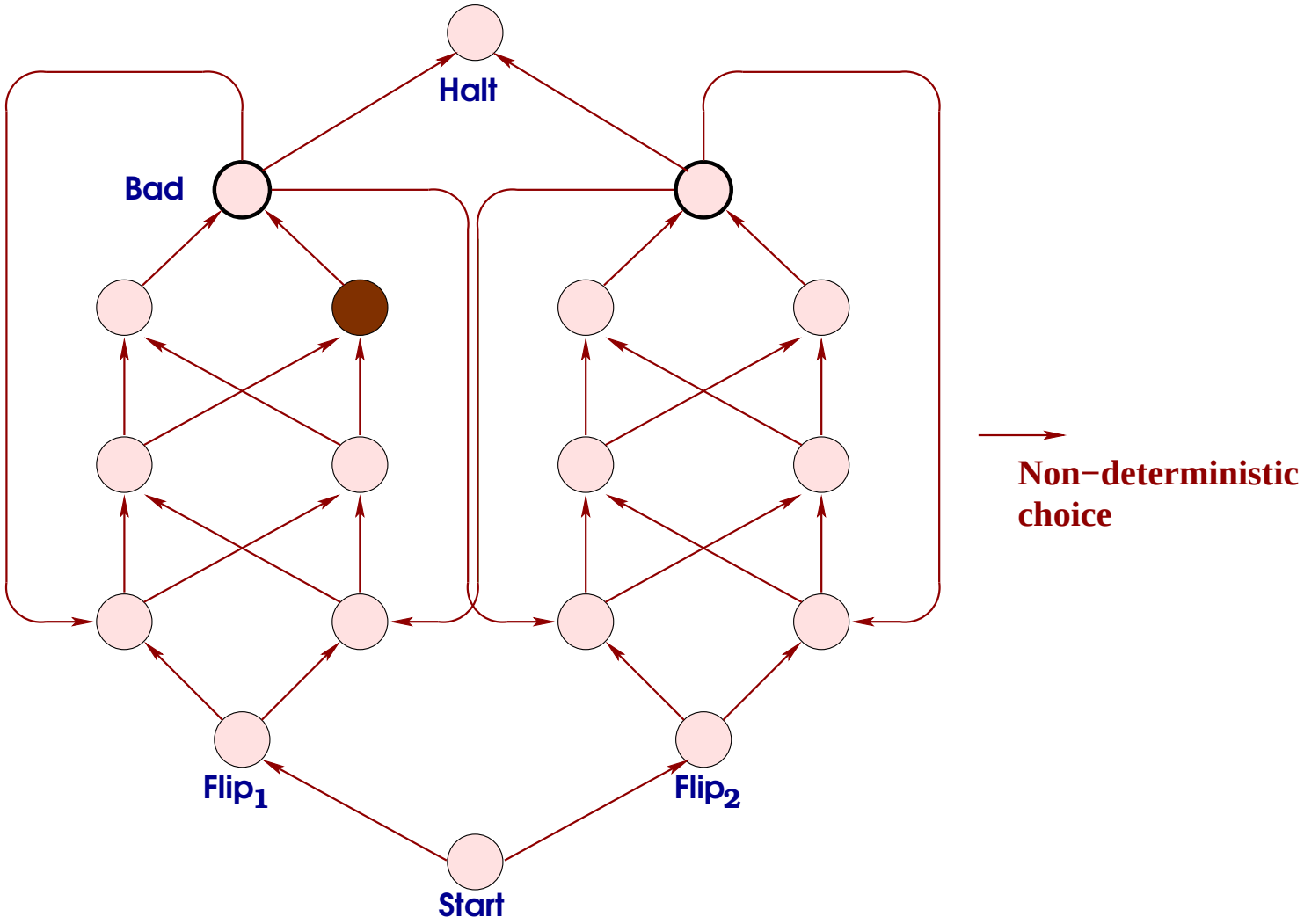

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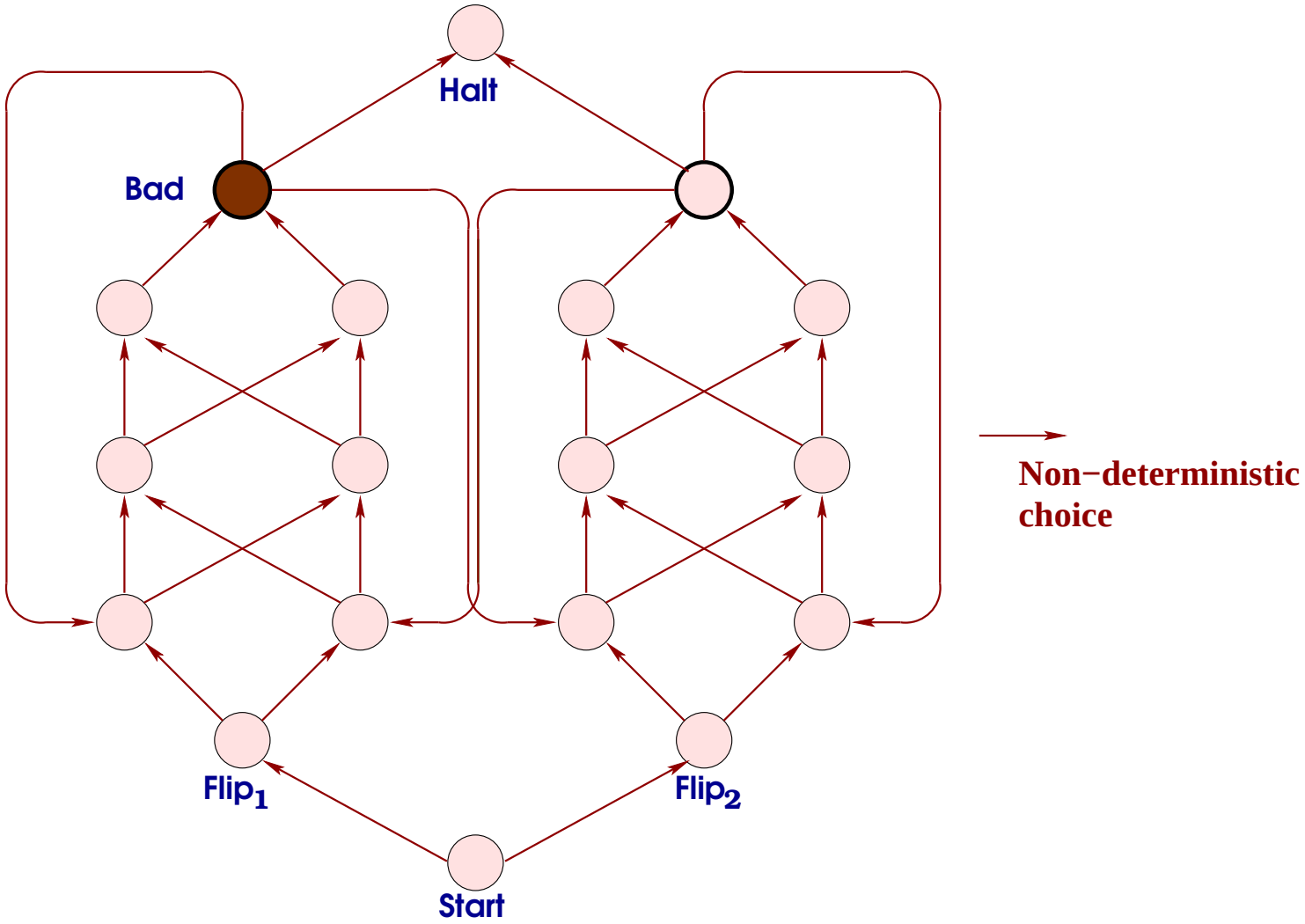

Modelling Non-Determinism

Transition System

State space X (possibly infinite).
Transition relation $\theta : X \to P(X)$, for some notion of "powerset" $P(X)$.

In standard transition systems, $P(X) = \mathbb{P}(X)$.

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Transition System

State space X (possibly infinite).
Transition relation $\theta : X \to P(X)$, for some notion of "powerset" $P(X)$.

In standard transition systems, $P(X) = \mathbb{P}(X)$.
When X is a T_0 space (including dcpos), there are three kinds:

- $P(X) = \mathcal{Q}(X)$ [Smyth] demonic nondeterminism;
- $P(X) = \mathcal{H}_u(X)$ [Hoare] angelic nondeterminism;
- $P(X) = \mathcal{P}\ell_{\mathcal{V}}(X)$ [Plotkin] chaotic nondeterminism.

The Smyth Powerdomain

Definition

$\mathcal{Q}(X)$ is the set of all non-empty compact saturated subsets Q of X, with:

- either the Scott topology of $\supseteq$;
- or the *upper Vietoris* topology, generated by:

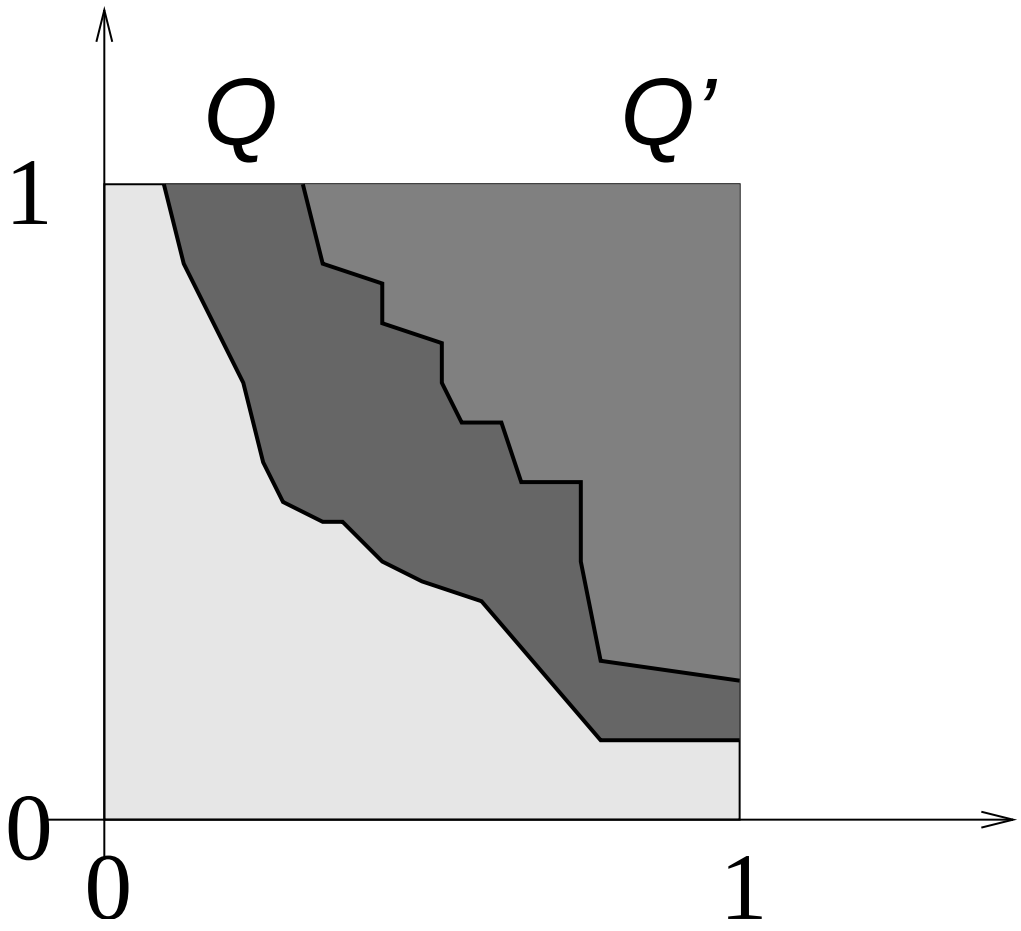
$$\Box U \quad = \quad \{Q \in \mathcal{Q}(X) \mid Q \subseteq U\}$$

where U ranges over the opens of X.

(This is the same topology under mild conditions; in particular, certainly, when X is stably compact.)

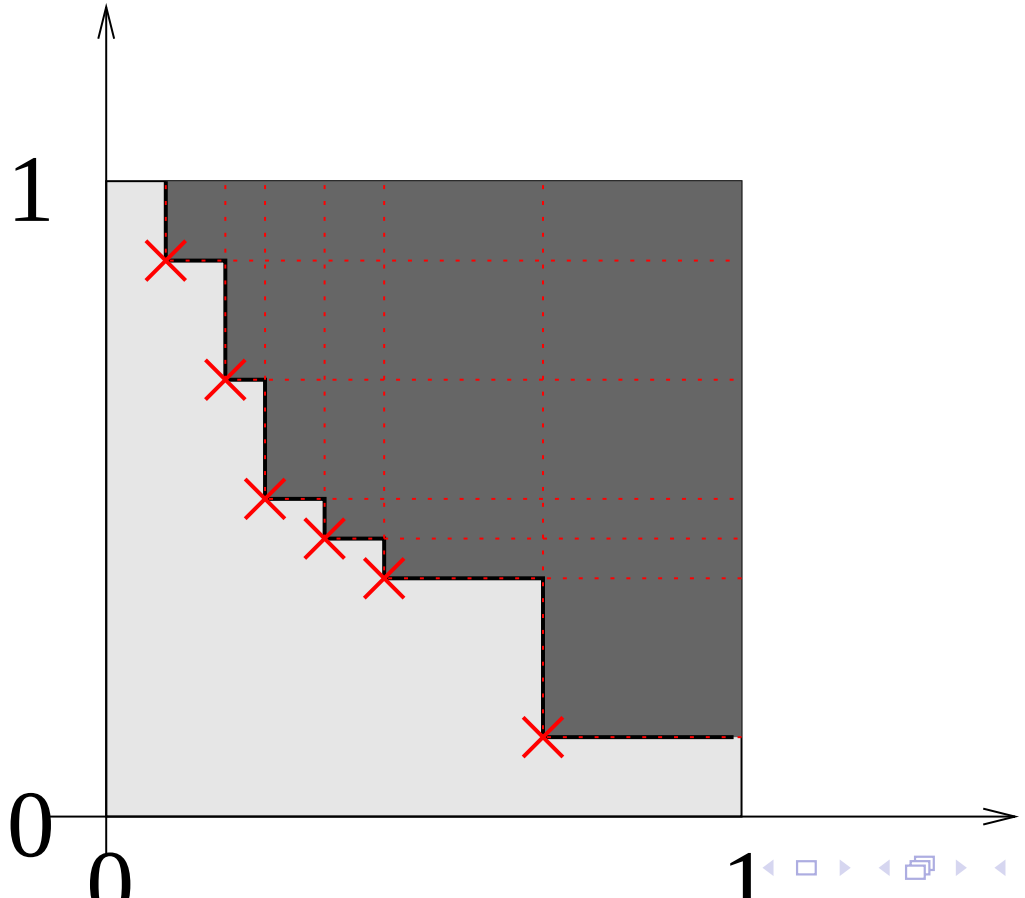
Illustration

$Q \leq Q'$ (i.e., $Q \supseteq Q'$) in $\mathcal{Q}([0,1]^2_\sigma)$:



Why is $\mathcal{Q}$ "Demonic"?

- When X is a continuous dcpo, every $Q \in \mathcal{Q}(X)$ is the directed limit of *finitary compacts* $\uparrow E$, $E \neq \emptyset$ finite $\subseteq X$.
- Then E is the collection of worst (least) possible choices.



Why is $\mathcal{Q}$ "Demonic"?

- Better: for any continuous payoff function $h : X \to \mathbb{R}^+_\sigma$, let (risk):

$$\begin{aligned} h_* \quad : \quad & \mathcal{Q}(X) \to \mathbb{R}^+_\sigma \\ & Q \mapsto \min_{x \in Q} h(x) \end{aligned}$$

Then h_* is again continuous: least possible payoff.

Taking sups would not even be monotonic, much less continuous...

The Hoare Powerdomain (Topological version)

Definition

$\mathcal{H}_u(X)$ is the set of all non-empty closed subsets F of X, with:

- either the upper topology of $\subseteq$;
- or the *lower Vietoris* topology, generated by:

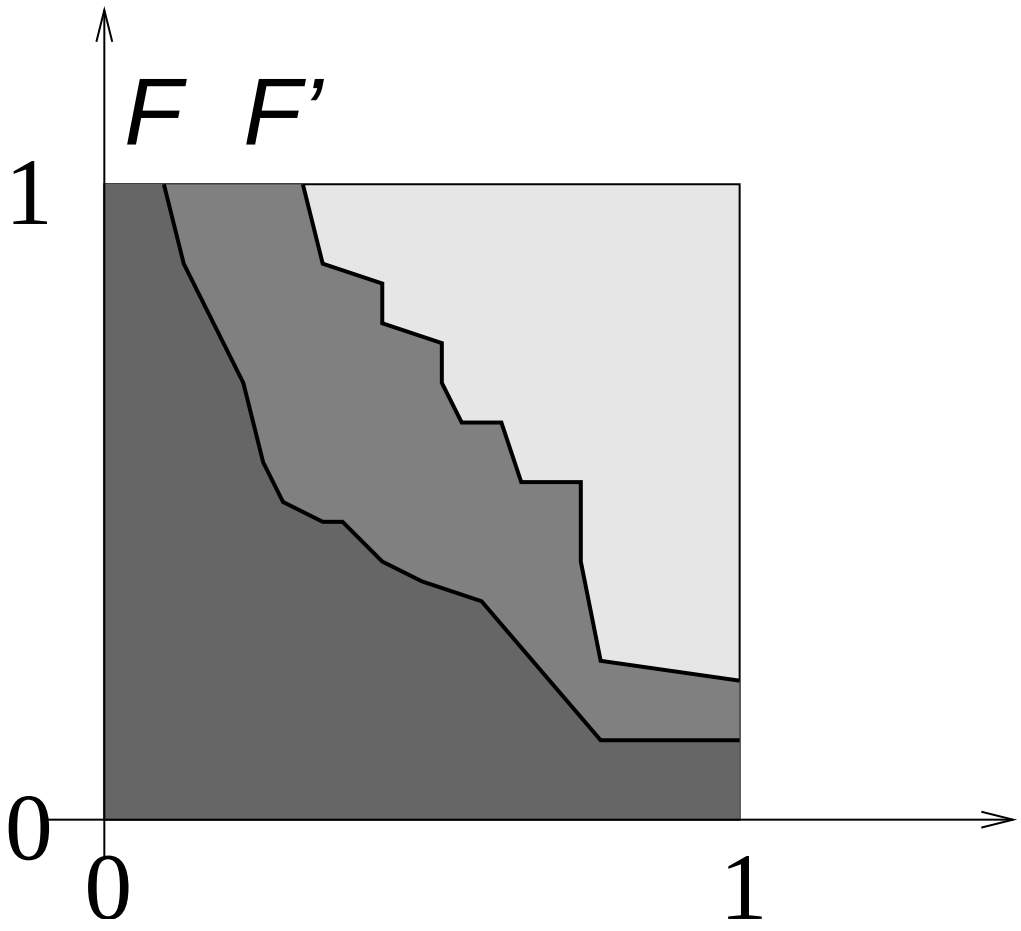
$$\Diamond U \quad = \quad \{F \in \mathcal{H}_u(X) \mid F \cap U \neq \emptyset\}$$

where U ranges over the opens of X.

(This is the same topology; it coincides with the Scott topology when X is a continuous dcpo.)

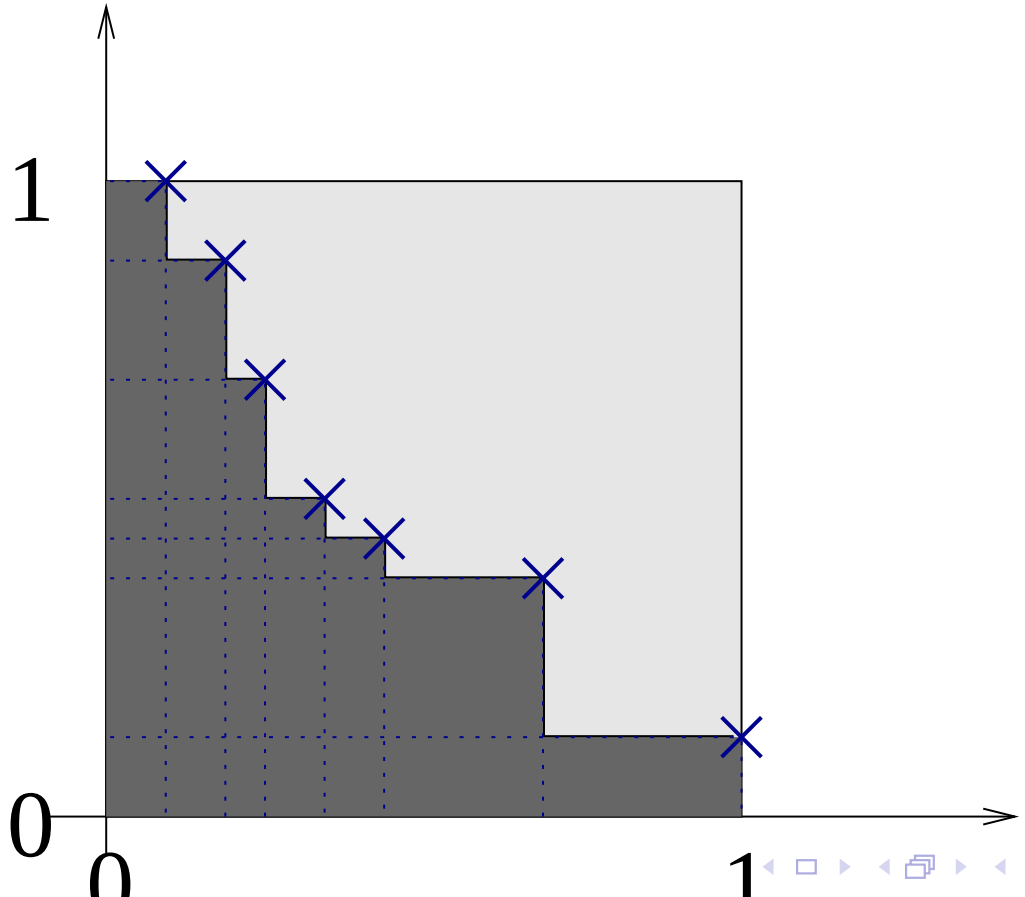
Illustration

$F \leq F'$ (i.e., $F \subseteq F'$) in $\mathcal{H}_u([0,1]^2_\sigma)$:



Why is $\mathcal{H}_u$ "Angelic"?

- When X continuous dcpo, every $F \in \mathcal{H}_u(X)$ is the directed limit of *finitary closed* subsets $\downarrow E$, $E \neq \emptyset$ finite $\subseteq X$.
- Then E is the collection of best (highest) possible choices.



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Then h^* is again continuous: best possible payoff.

Taking mins would not even be monotonic, much less continuous...

De Groot Duality and Non-Determinism

Here is (at last) the second, and main, duality of this talk:

Theorem

Let X be stably compact. Then $\mathcal{Q}(X)^{\mathsf{d}} = \mathcal{H}_u(X^{\mathsf{d}})$.

In particular:

- Both $\mathcal{Q}(X)$ and $\mathcal{H}_u(X)$ are stably compact, whenever X is;
- De Groot Duality exchanges demons with angels.

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Proof.

- $\mathcal{Q}(X)$ is a domain, hence is stably compact;
- $\mathcal{Q}(X)^{\mathrm{d}}$ and $\mathcal{H}_u(X^{\mathrm{d}})$ have the same elements Q;
- Every closed subset of $\mathcal{H}_u(X^{\mathrm{d}})$ is closed in $\mathcal{Q}(X)^{\mathrm{d}}$, i.e., compact saturated in $\mathcal{Q}(X)$ (use Alexander Subbase Lemma);
- As in any continuous dcpo, every compact saturated subset of $\mathcal{Q}(X)$ can be written $\bigcap^{\downarrow}_{i\in I} \uparrow E_i$. This equals $\bigcap^{\downarrow}_{i\in I} \bigcup_{Q\in E_i} \uparrow Q = \bigcap^{\downarrow}_{i\in I} \bigcup_{Q\in E_i} \overline{\Diamond\overline{Q}}$, hence is closed in $\mathcal{H}_u(X^{\mathrm{d}})$.

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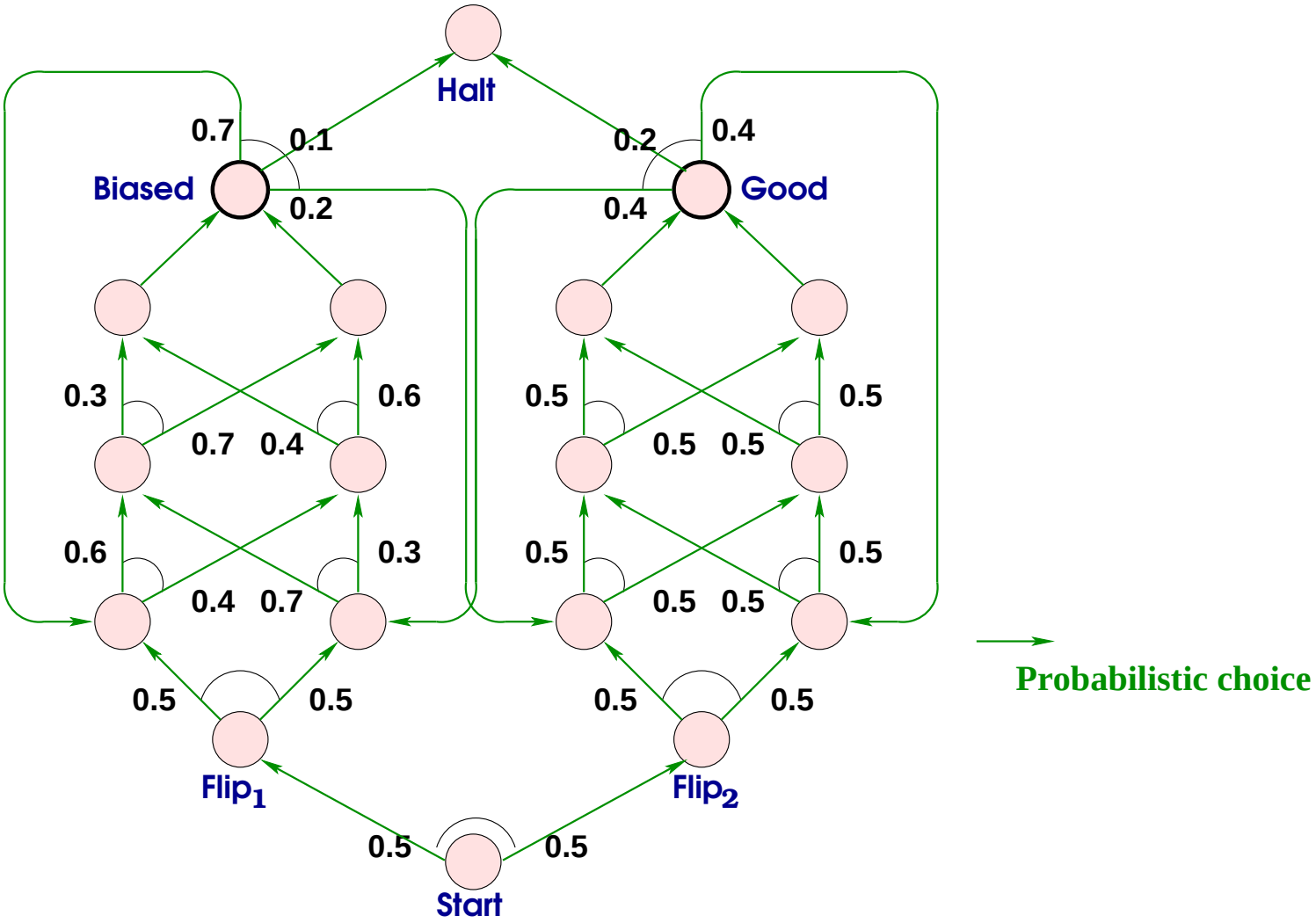
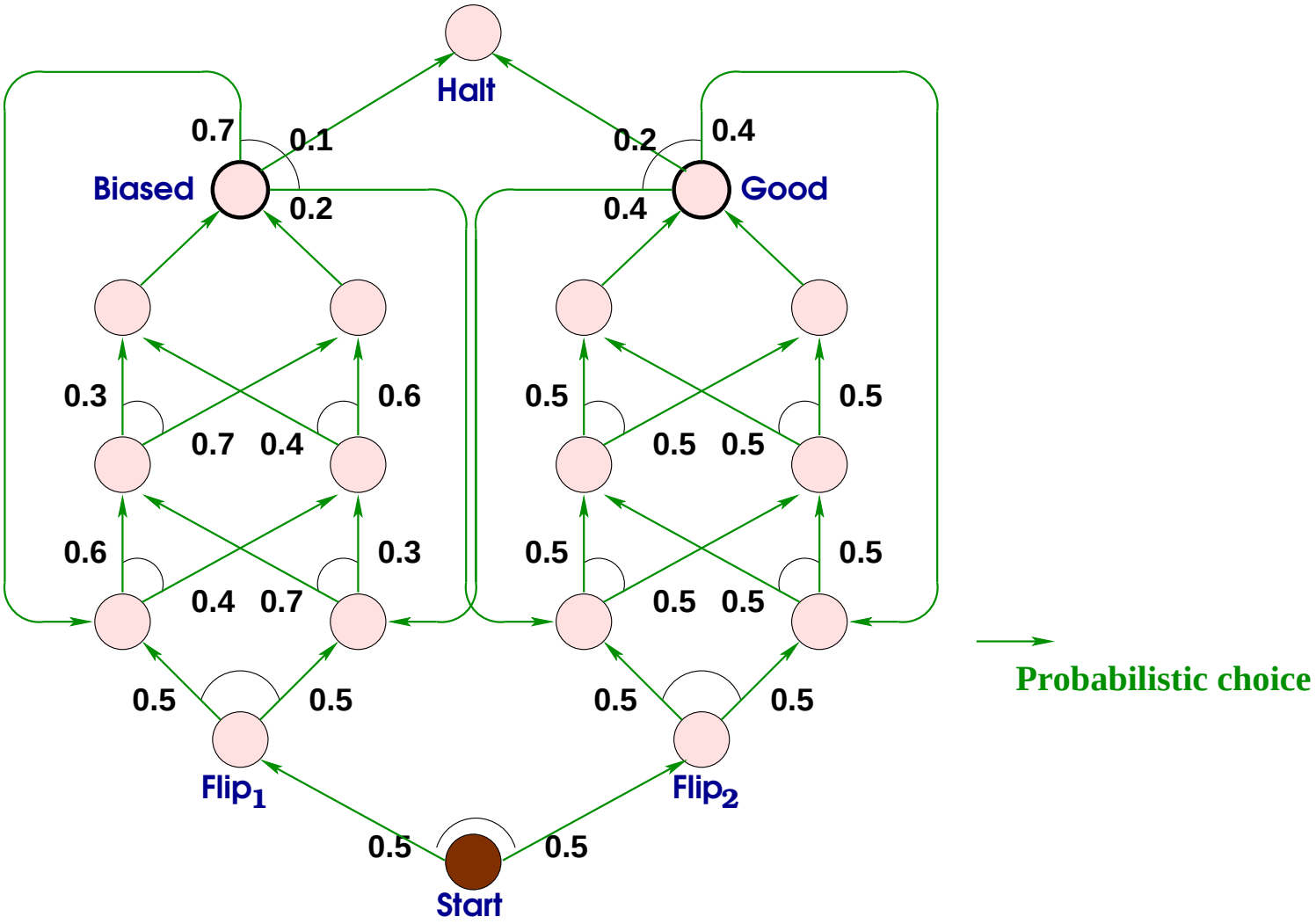
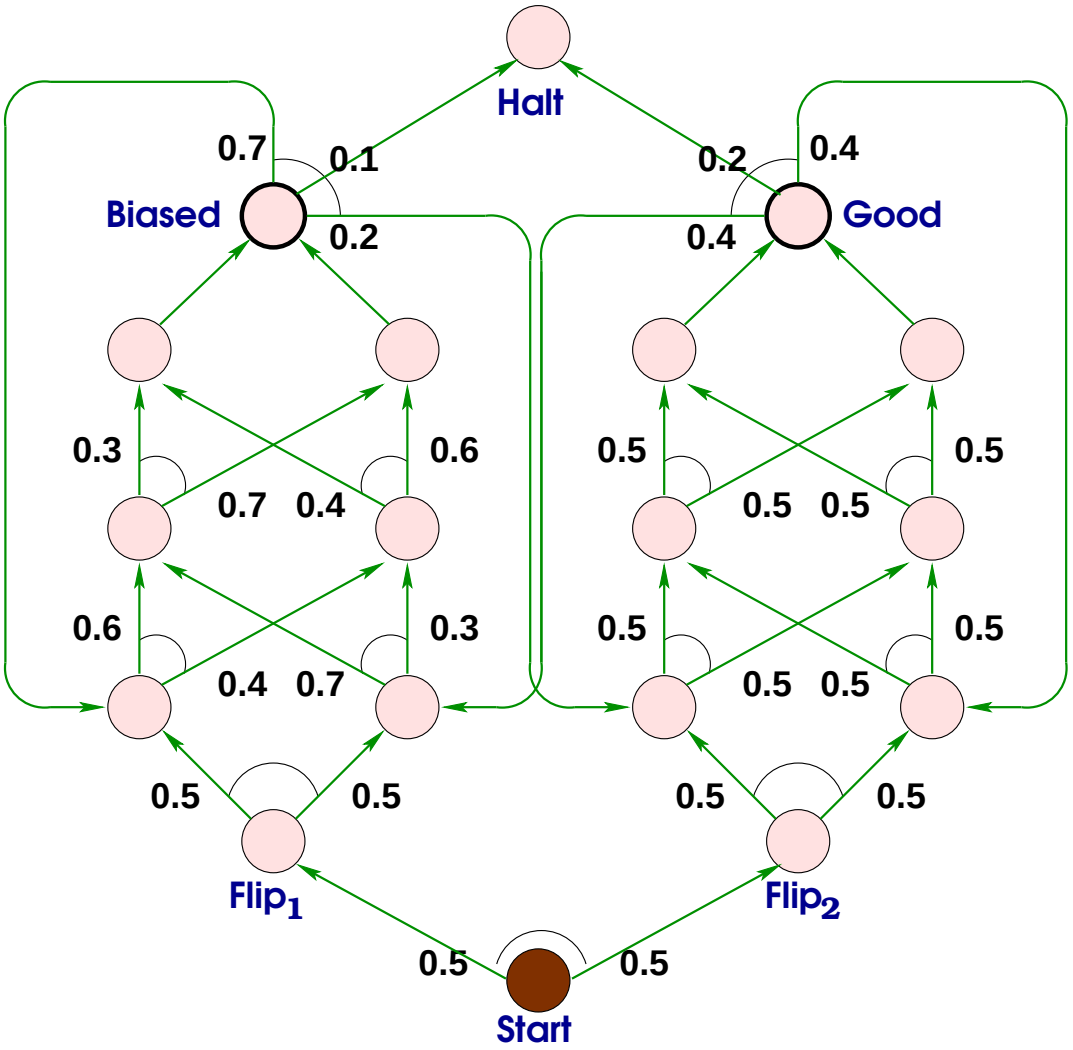
A (Finite) Markov Chain



Start

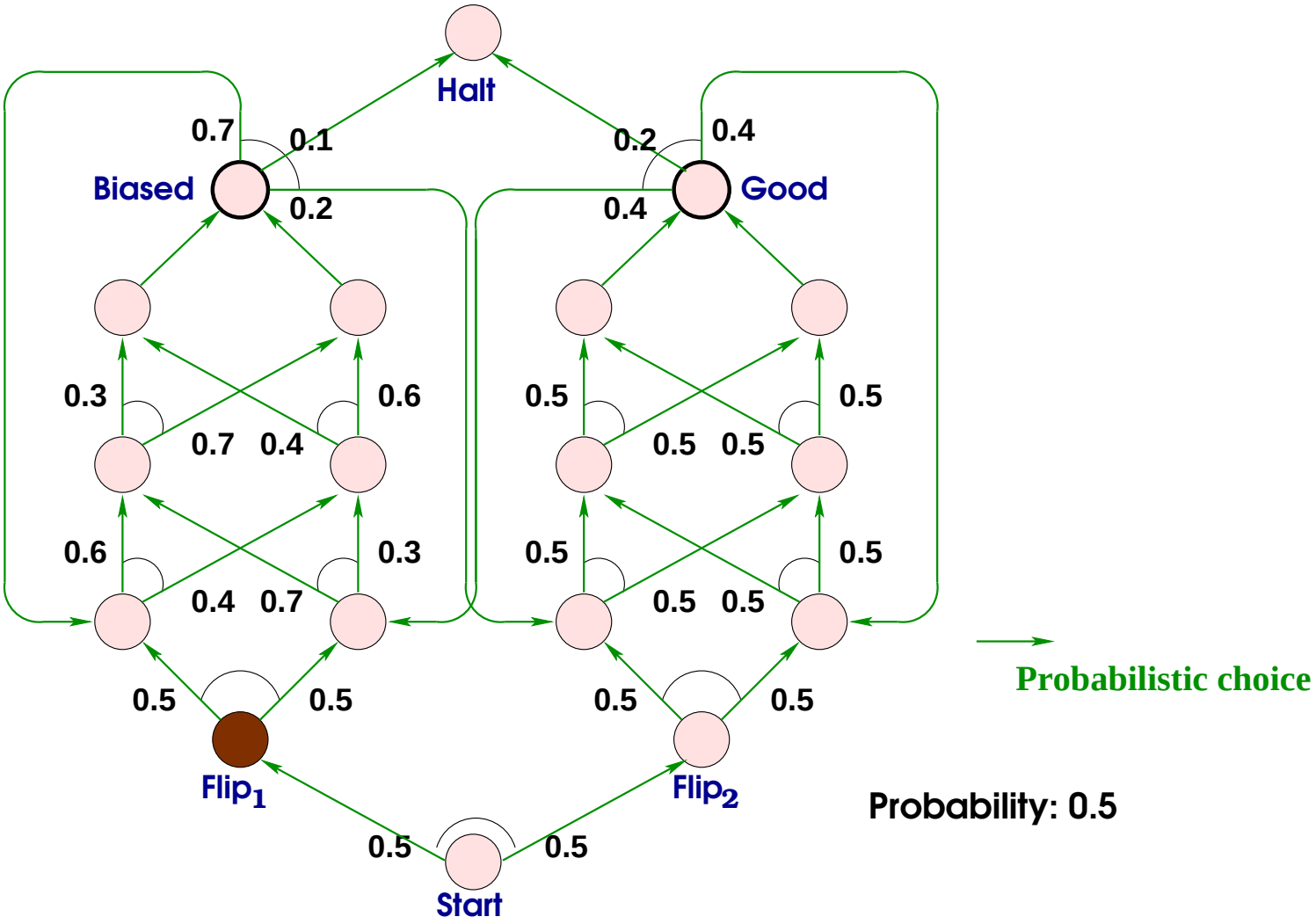

Halt
0.7
0.1
0.2
0.4
Biased
Good
0.2
0.4
0.3
0.6
0.7
0.4
0.5
0.5
0.5
0.5
0.6
0.3
0.4
0.7
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
Flip1
Flip2
0.5
0.5
Start
Probabilistic choice

Flip a Coin

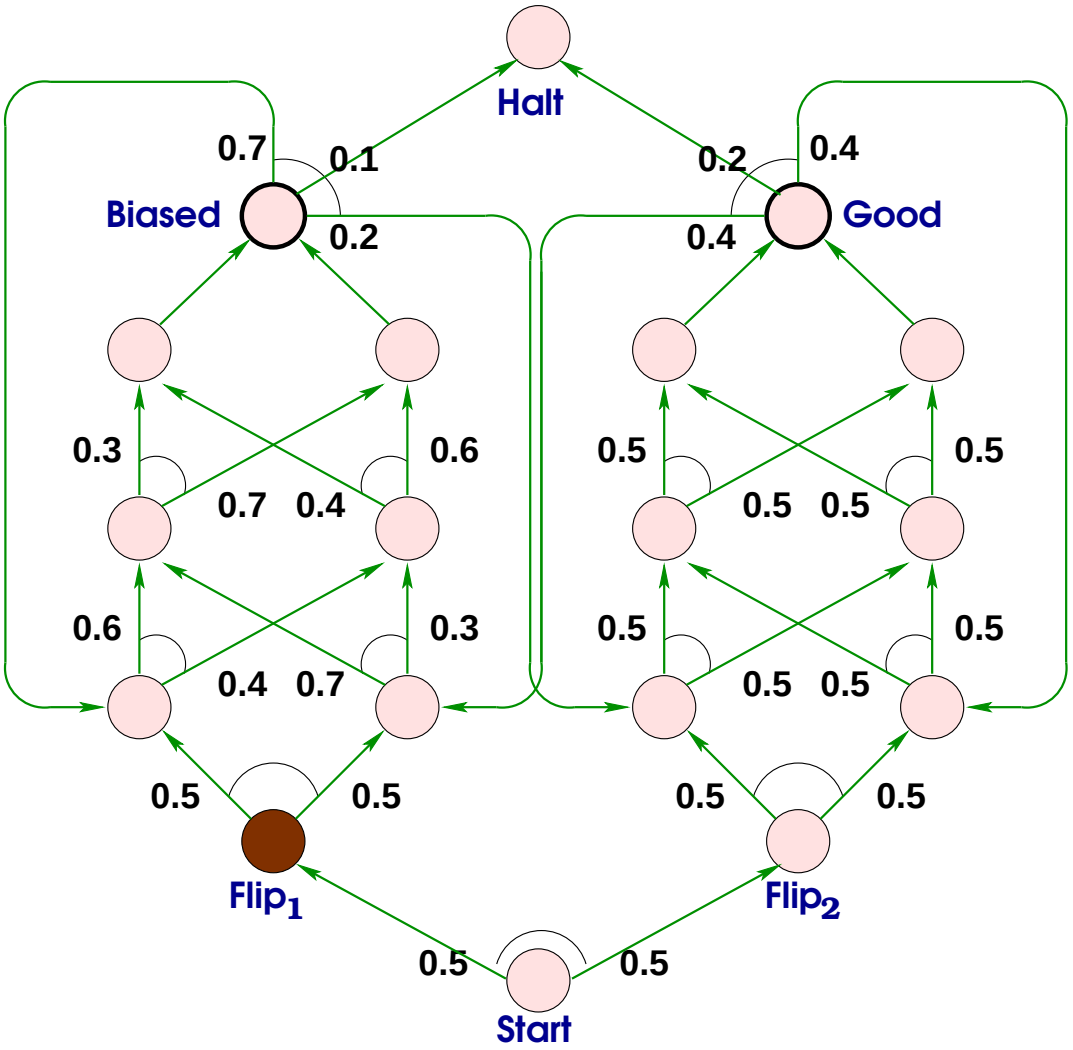


Probabilistic choice

Advance

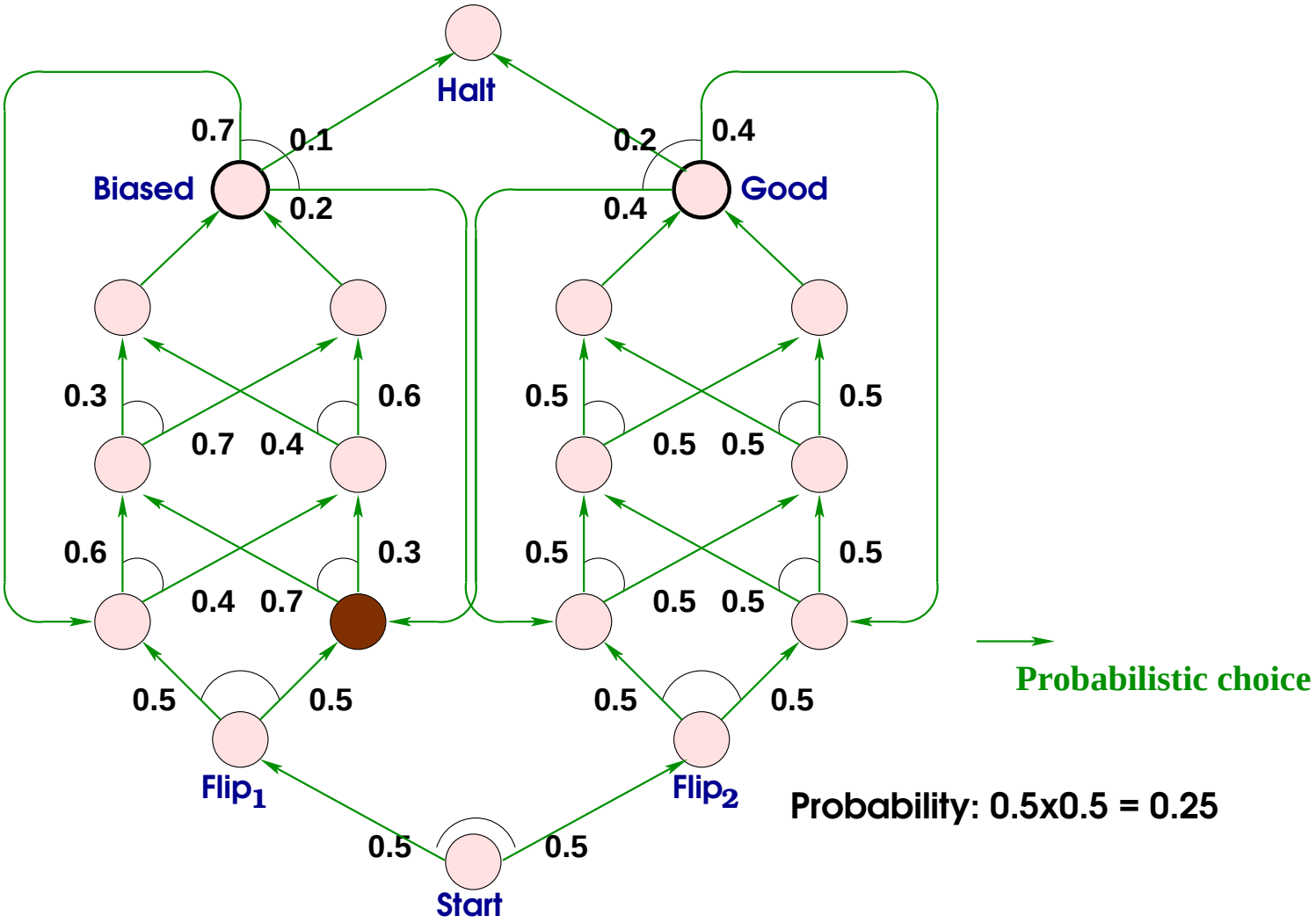

Halt
0.7
0.1
0.2
0.4
Biased
0.2
0.4
Good
0.3
0.6
0.7
0.4
0.5
0.5
0.5
0.5
0.6
0.3
0.4
0.7
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
Flip1
Flip2
0.5
0.5
Start
Probabilistic choice
Probability: 0.5

Flip a Coin

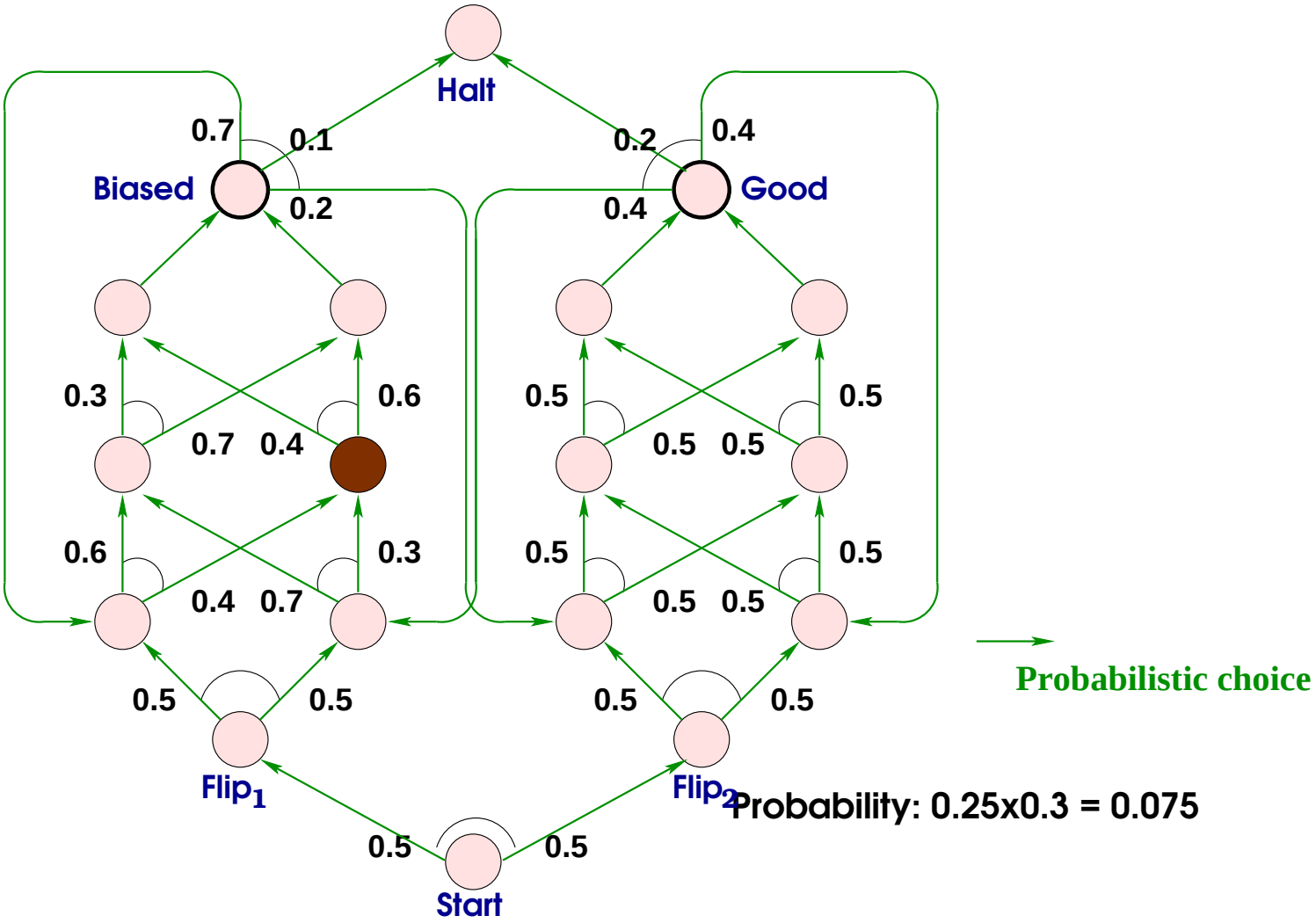


Probabilistic choice

Advance


Halt
0.7
0.1
0.2
0.4
Biased
0.2
0.4
Good
0.3
0.6
0.7
0.4
0.5
0.5
0.5
0.5
0.6
0.3
0.4
0.7
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
Flip1
Flip2
0.5
0.5
Start
Probabilistic choice
Probability: 0.5x0.5 = 0.25

Advance


Halt
Biased
Good
0.7
0.1
0.2
0.2
0.4
0.4
0.3
0.6
0.7
0.4
0.6
0.3
0.4
0.7
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
Flip1
Flip2
Start
Probabilistic choice
Probability: 0.25x0.3 = 0.075

Modelling Probabilistic Choice

(Continuous) Markov Transition System

State space X (possibly infinite).
Transition relation $\theta : X \to \mathbf{V}_1(X)$.

$\mathbf{V}_1(X)$ is the space of continuous probability valuations ν on X ($\sim$ probability measures, but only measure *opens*), i.e., $\nu : \mathcal{O}(X) \to \mathbb{R}^+$ with:

- **Strict** $\nu(\emptyset) = \emptyset$;
- **Mod.** for any two opens U, V, $\nu(U \cup V) = \nu(U) + \nu(V) - \nu(U \cap V)$;
- **Mono.** for any two opens $U \subseteq V$, $\nu(U) \leq \nu(V)$;
- **Cont.** $\nu(\bigcup^{\uparrow}_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$;
- **Proba.** $\nu(X) = 1$.

Studied by many, e.g., Jones, Plotkin, Jung, Tix, Keimel, Lawson, ...

Also, continuous valuations=measures in many cases [KeimelLawson].

De Groot Duality and Probabilities

One may equip $\mathbf{V}_1(X)$ with:

- the Scott topology of $\leq$ ($\nu \leq \nu'$ iff $\nu(U) \leq \nu'(U)$ for all U),
- or with the weak topology:

Definition (Weak Topology, $\mathbf{V}_{1\,wk}(X)$)

The *weak topology* on $\mathbf{V}_1(X)$ is generated by:

$$[U > r] \quad = \quad \{\nu \in \mathbf{V}_1(X) \mid \nu(U) > r\}$$

with U open in X, $r \in \mathbb{R}$. Write the resulting space $\mathbf{V}_{1\,wk}(X)$.

Note: Think of $[U > r]$ as the set of ν's passing the *test* $\nu(U) > r$.

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Note: Think of $[U > r]$ as the set of ν's passing the *test* $\nu(U) > r$.

- $\mathbf{V}_1(X)$ is a continuous dcpo with $\perp$ if X is [Jones, Edalat];
- In this case, Scott topology = weak topology [Jung];
- $\mathbf{V}_{1\,wk}(X)$ is stably compact whenever X is [JungTix, Jung].

Dualizing Valuations

Instead of measuring opens U, one may measure saturated compacts Q [Tix99]:

$$\nu^{\dagger}(Q) \quad = \quad \inf_{U \supseteq Q} \nu(U)$$

Dualizing Valuations

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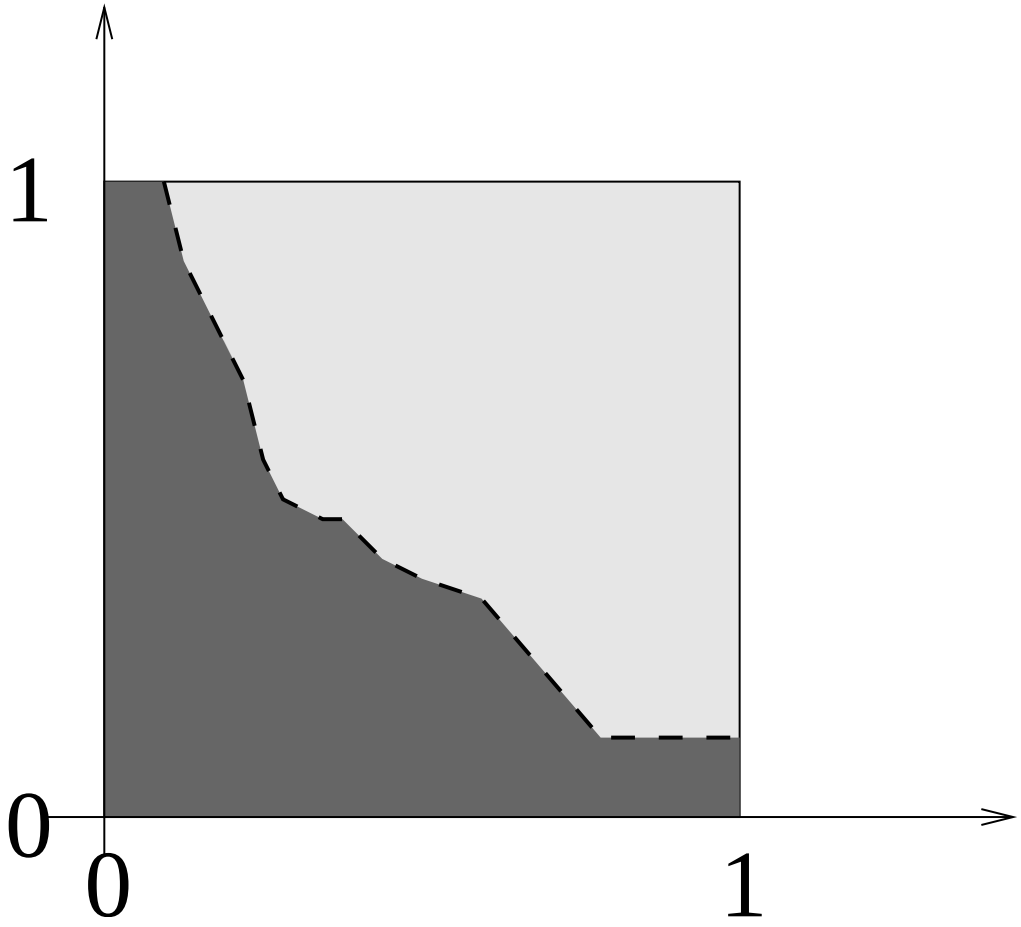
… or their complements $\overline{Q}$:

Definition ($\nu^{\perp}$)

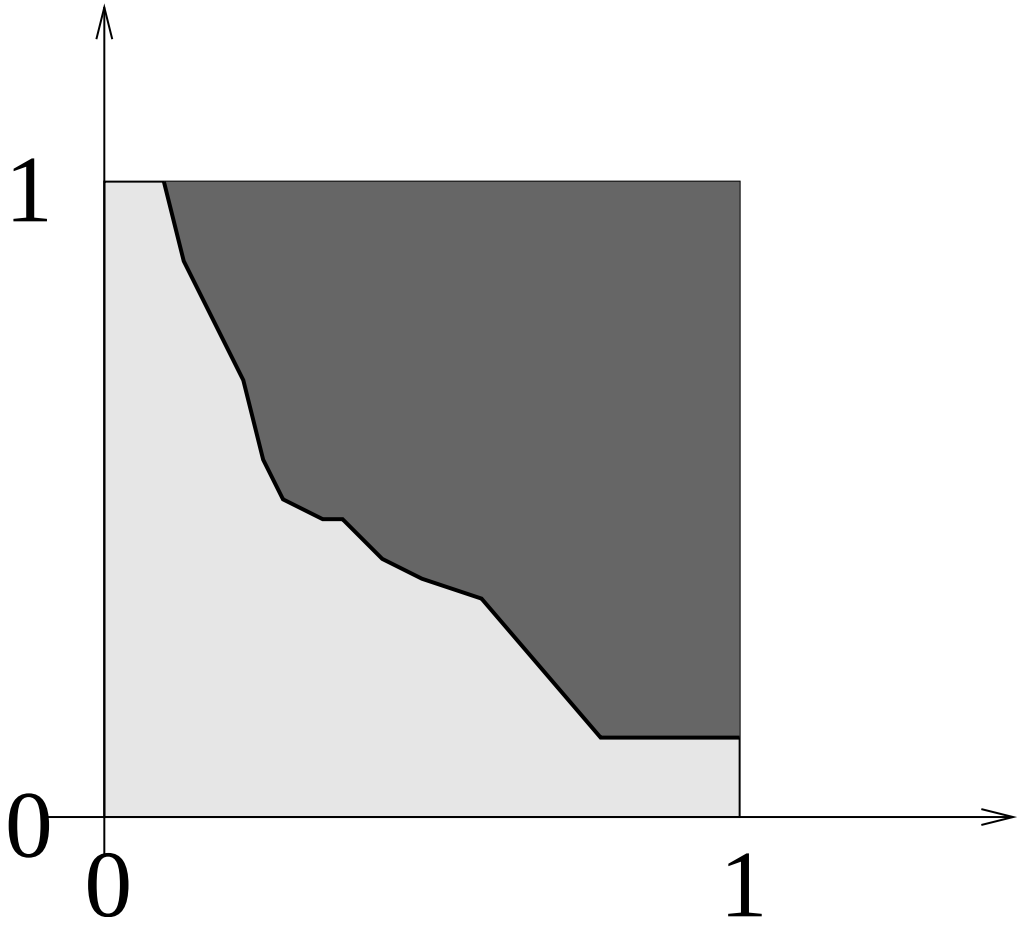
Let X be stably compact.
For any continuous probability valuation ν, let:

$$\nu^{\perp}(\overline{Q}) \quad = \quad 1 - \nu^{\dagger}(Q)$$

Measuring Cocompacts



Measuring Cocompacts



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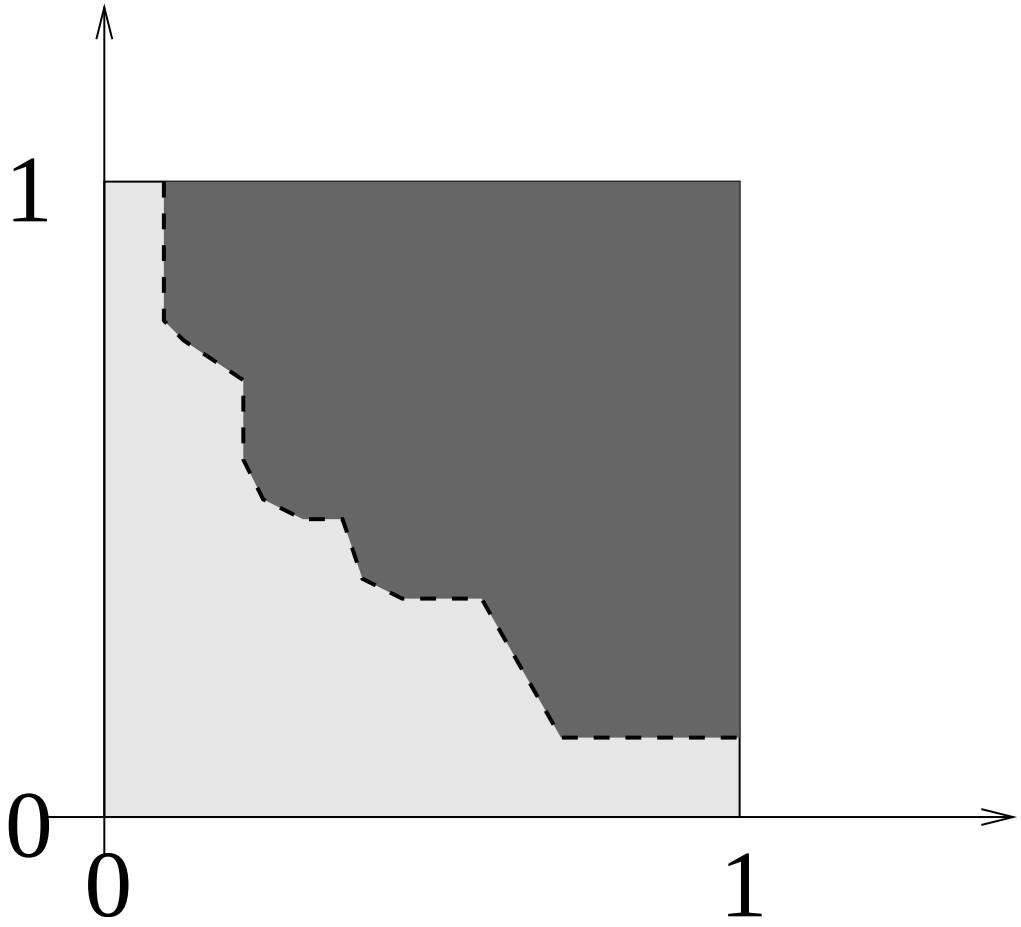

The Dual of a Continuous Probability Valuation

Proposition

For any continous probability valuation ν on X,

- $\nu^{\perp}$ *is a continuous probability valuation on* X^{d}*;*
- $\nu^{\perp\perp} = \nu$*.*

Proof.

Essentially a rephrasing of Tix's Diplomarbeit (1999), Satz 3.4. □

De Groot Duality and Probabilities

Theorem (Duality)

Let X be stably compact. The map:

$$\begin{aligned} {}_{-}{}^{\perp} \quad : \quad & \mathbf{V}_{1\,wk}(X)^{\mathrm{d}} \to \mathbf{V}_{1\,wk}(X^{\mathrm{d}}) \\ & \nu \mapsto \nu^{\perp} \end{aligned}$$

is an isomorphism.

- I.e., Nature does not take sides: probabilistic choice (nature) is impervious to exchanging demons and angels.

De Groot Duality and Probabilities

Theorem (Duality)

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Proof.

Amounts to characterizing the compact saturated subsets of $\mathbf{V}_{1\,wk}(X)$ as generated by $\langle Q \geq r \rangle = \{\nu \in \mathbf{V}_1(X) \mid \nu^{\dagger}(Q) \geq r\}$ — mentioned as fundamental to [JungKegelmannMoshier] by [Jung04]. □

Outline

1 Introduction

2 De Groot Duality

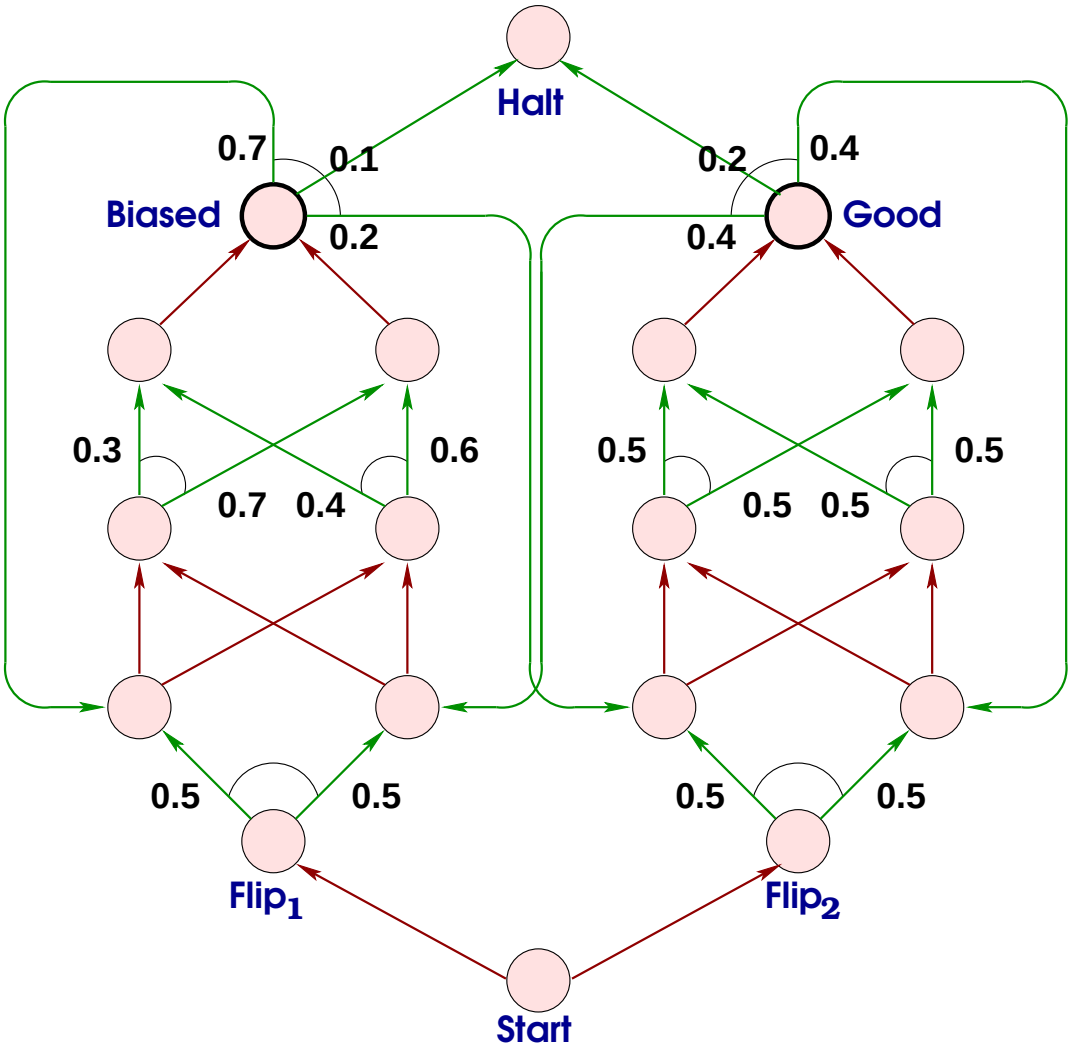
3 Angelic and Demonic Nondeterminism

4 Probabilistic Choice

5 Belief Functions, Plausibilities

6 Concluding Remarks

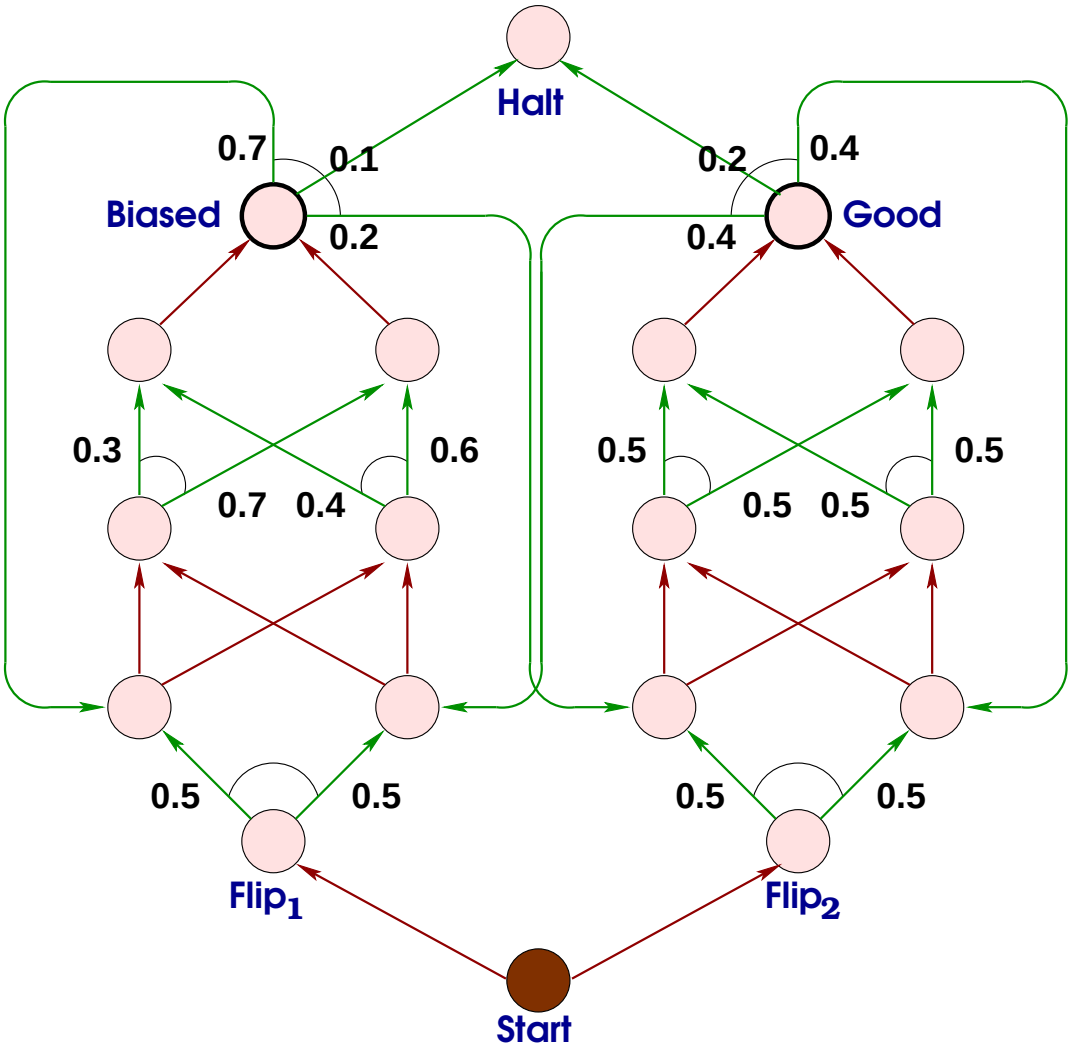
Reminder: $1\frac{1}{2}$-Player Stochastic Games



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

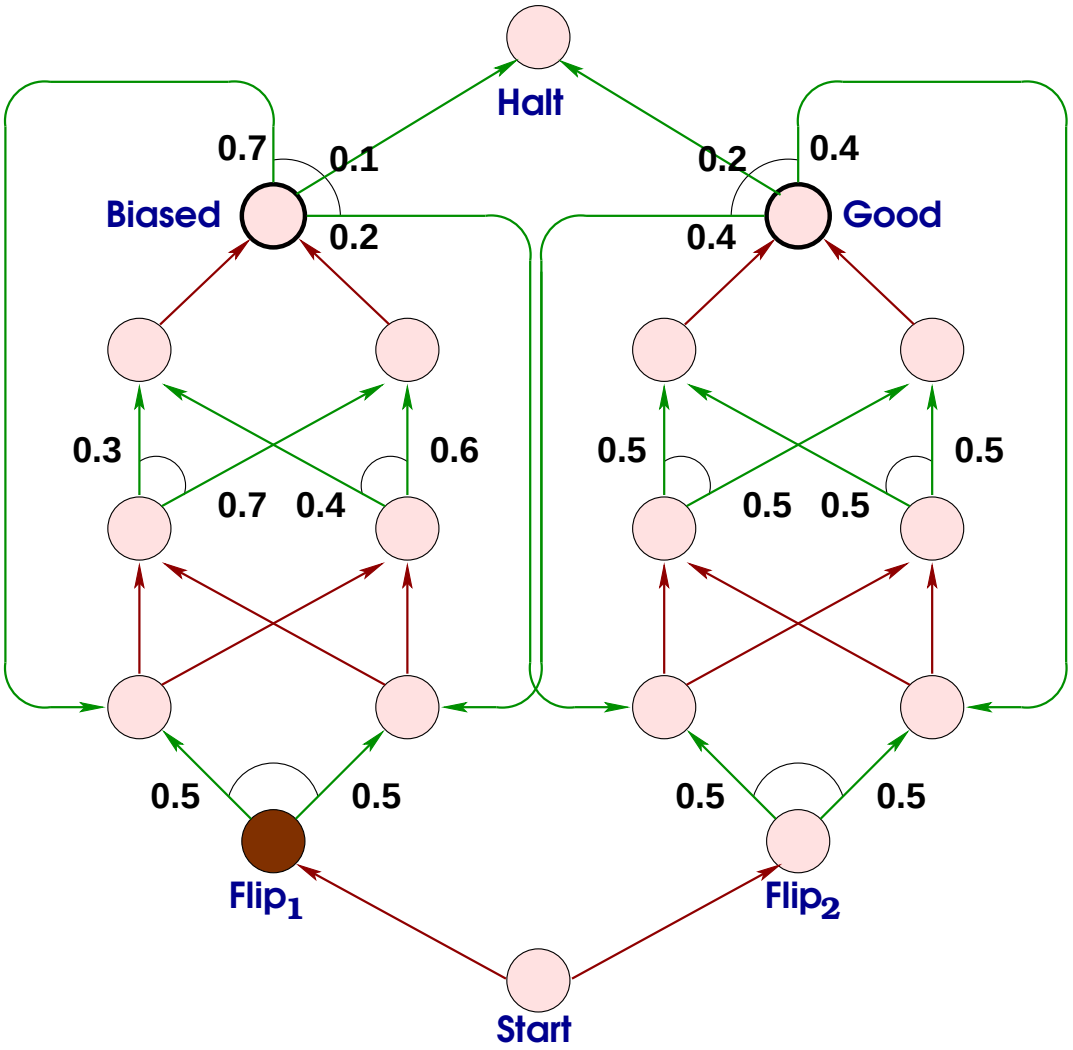
Start



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

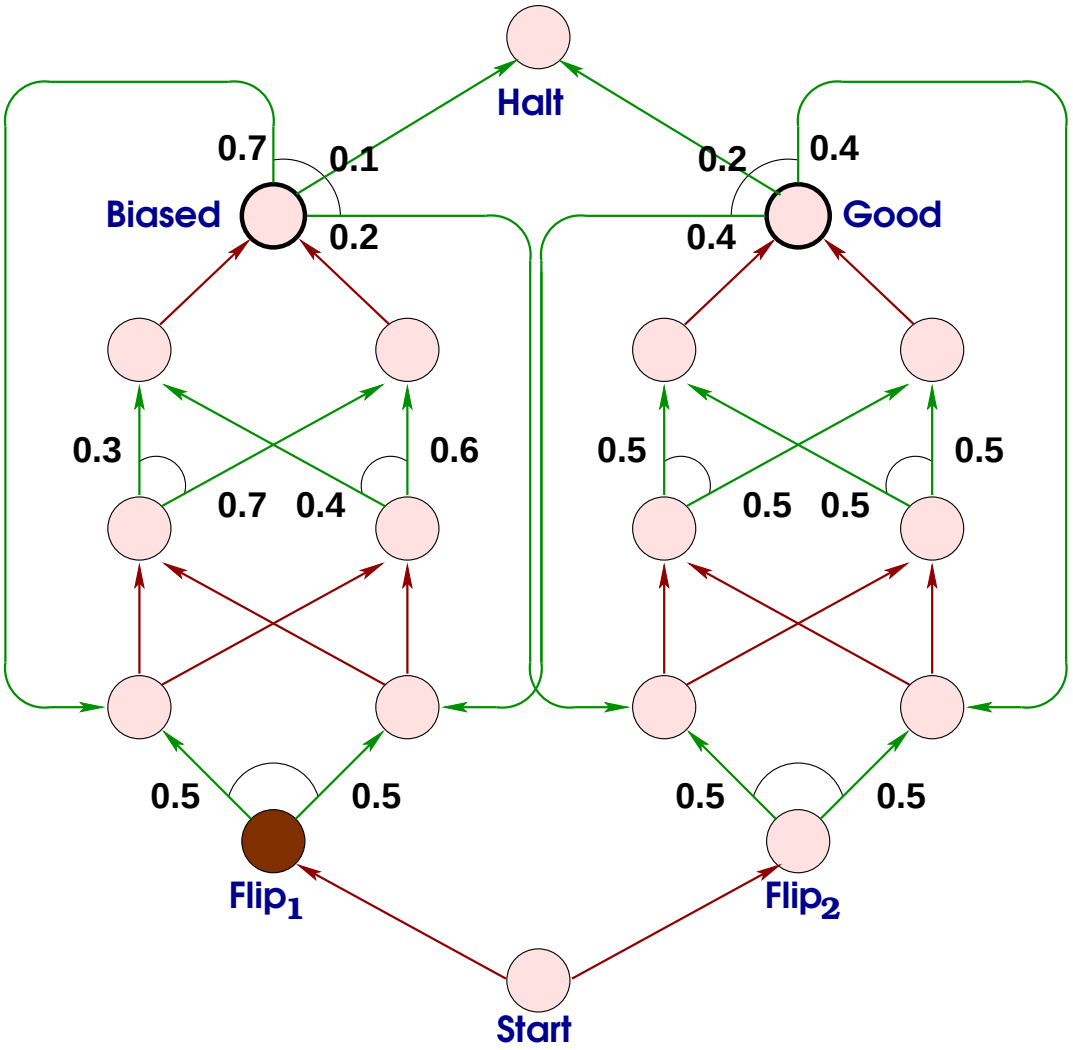
C's Turn: Malevolently Chooses Biased Side



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

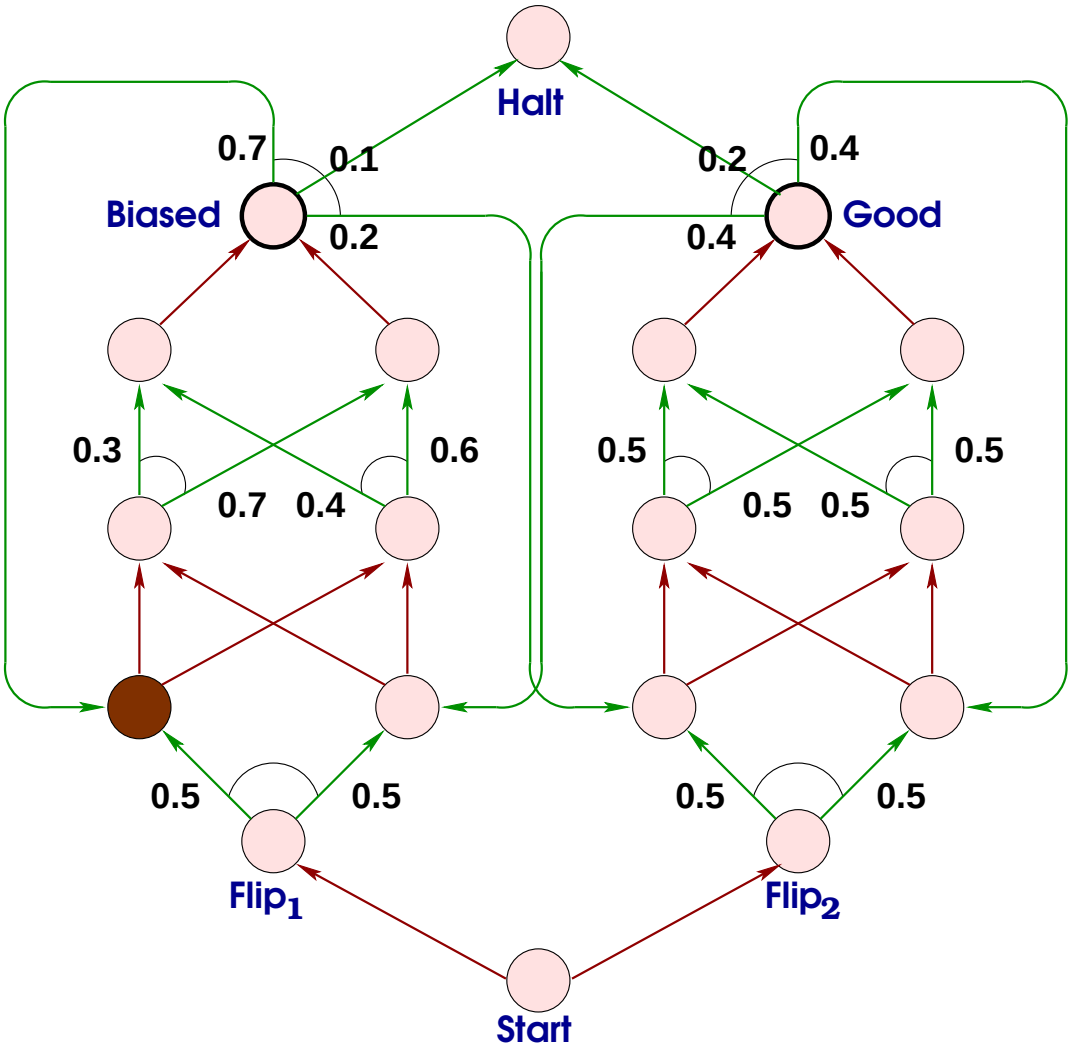
P's Turn: Flipping a Coin



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

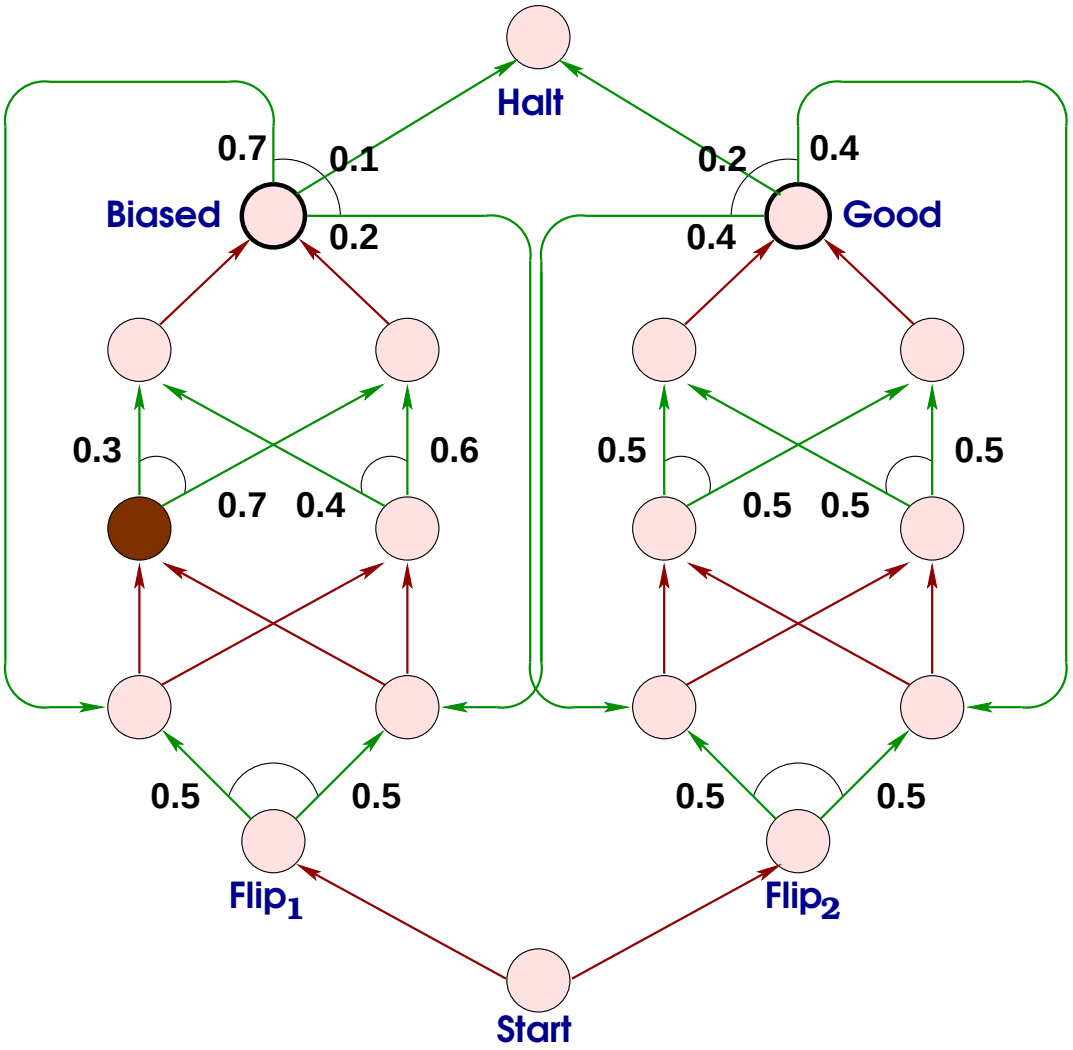
P's Turn: Advancing



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

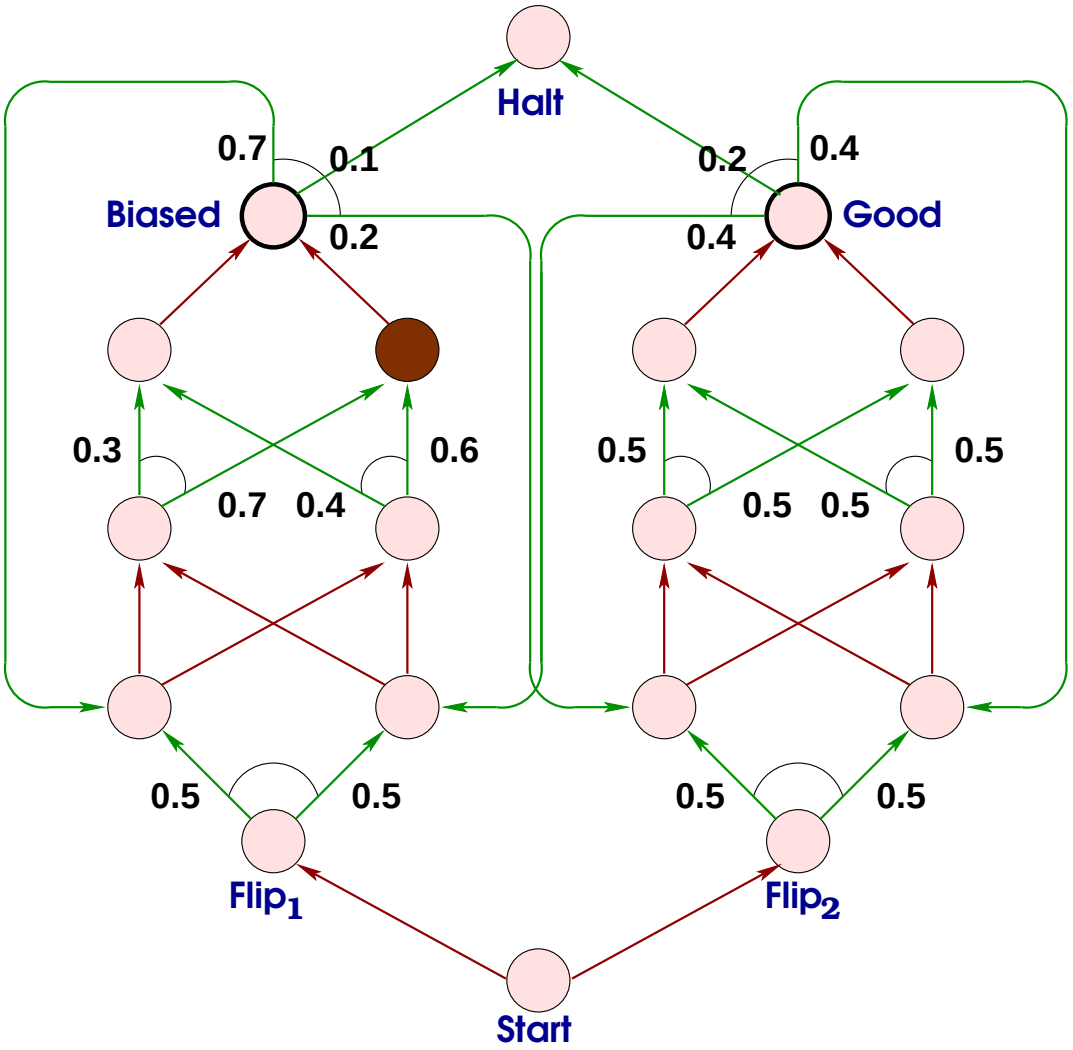
C's Turn: Picking Most Biased Side



Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

P's Turn



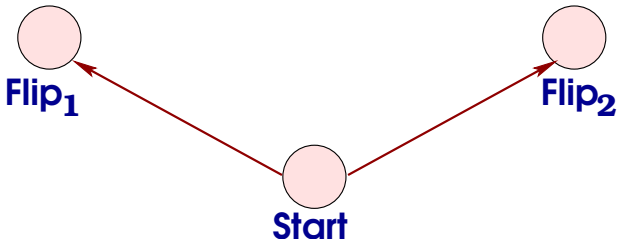
Non−deterministic (demonic) choice (by adversary)

Probabilistic choice (by program)

“Preprobabilities”

An idea by F. Laviolette and J. Desharnais: simulate non-deterministic choice by some form of non-additive probabilistic choice: “Preprobabilities”.

E.g., demonic choice:



Preprobability that, from **Start**, we jump into some set U:

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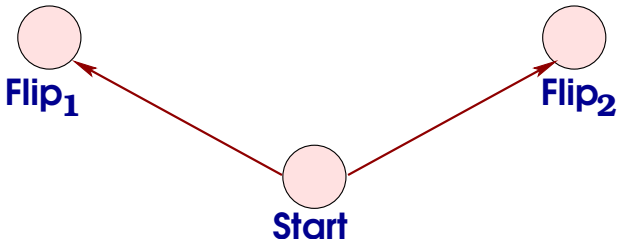

Preprobability that, from **Start**, we jump into some set U:

U	proba.	
$\emptyset$	0	
$\{\mathbf{Flip}_1\}$		

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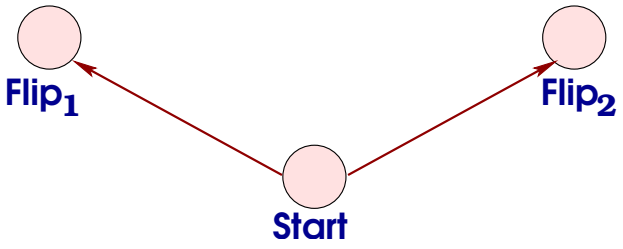

Preprobability that, from **Start**, we jump into some set U:

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$\{\mathbf{Flip}_1\}$	0	C can always pick $\mathbf{Flip}_2$

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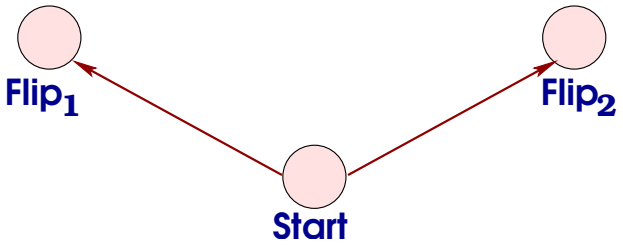

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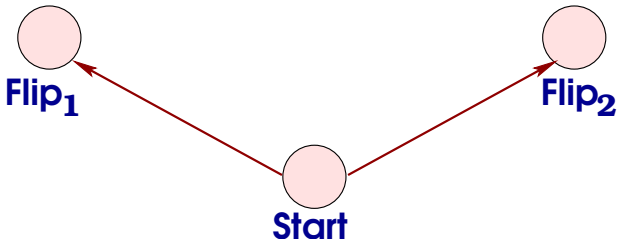

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$\{\mathbf{Flip}_1, \mathbf{Flip}_2\}$		

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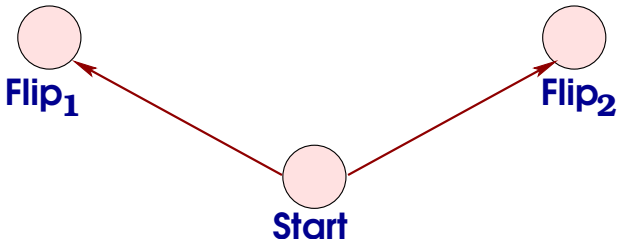

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$\{\mathbf{Flip}_1\}$	0	C can always pick $\mathbf{Flip}_2$
$\{\mathbf{Flip}_2\}$	0	C can always pick $\mathbf{Flip}_1$
$\{\mathbf{Flip}_1, \mathbf{Flip}_2\}$	1	C cannot escape it!

Unanimity Games

Definition

The unanimity game $\mathfrak{u}_Q$ is the set function such that:

$$\mathfrak{u}_Q(U) = \begin{cases} 1 & \text{if } Q \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

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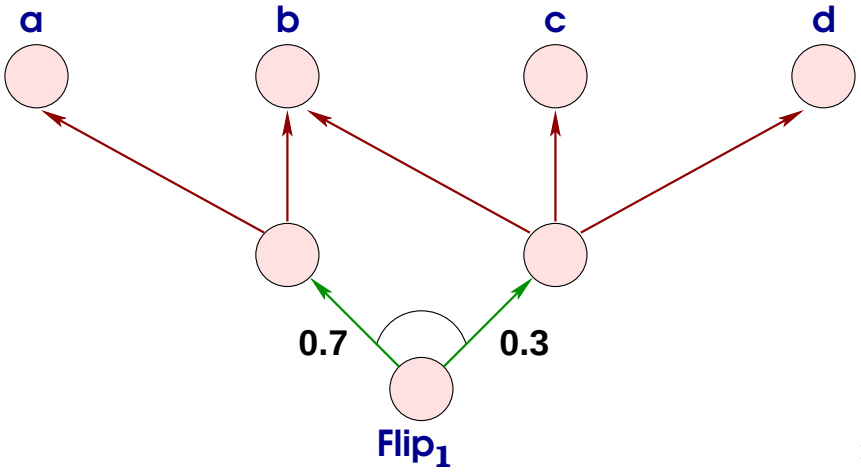
Non-deterministic (demonic) choice between Flip_1 and Flip_2:

$$\mathfrak{u}_{\{\text{Flip}_1,\text{Flip}_2\}}$$

(This notion is a special case of a “cooperative game with transferable utility function” in economics.)

Simple Belief Functions

Mix (demonic) non-deterministic and probabilistic choice:



$$= 0.7u_{\{a,b\}} + 0.3u_{\{b,c,d\}}$$

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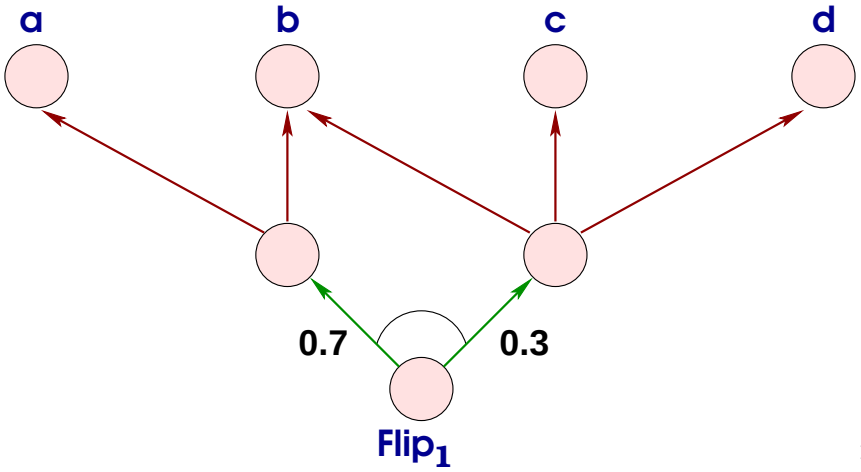

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A simple belief function is any $\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$, $a_i \in \mathbb{R}^+$, $Q_i \neq \emptyset$ compact saturated.

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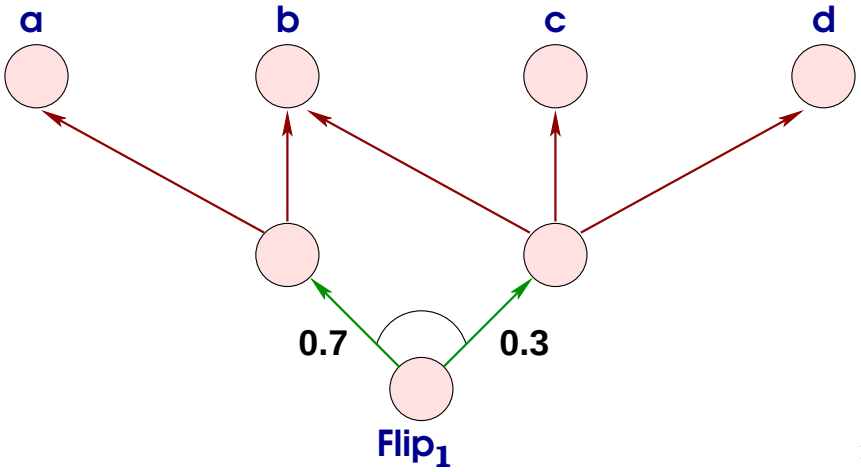

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(Looks like strictly alternating probabilistic automata [SegalaLynch95], or as in [MisloveOuaknineWorrell03], except we flip first *then* choose non-deterministically.)

Axiomatization: Capacities, (Cooperative) Games

Definition

A capacity ν is a map $\mathcal{O}(X) \to \mathbb{R}^+$, with $\nu(\emptyset) = 0$.

- A game is a *monotonic* capacity: $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$;

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(= for *valuations* [$\sim$ measures])

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$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \geq \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right)$$

(would be $=$ for valuations: the *inclusion-exclusion principle*.)

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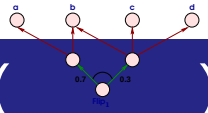
Lemma ()

Every simple belief function is a continuous belief function.

Conversely (1/2)

Theorem (JGL, "Completion of a misspecified model")

Let X be mild enough (e.g., stably compact).
For every continuous belief function ν on X, there is a unique continuous valuation ν^* on $\mathcal{Q}(X)$ such that $\nu(U) = \nu^*(\Box U)$ for all opens U.

Proof.

- Uniqueness is easy.
- Existence takes quite some computations, and rests on several combinatorial identities [JGL07].

$\Box$

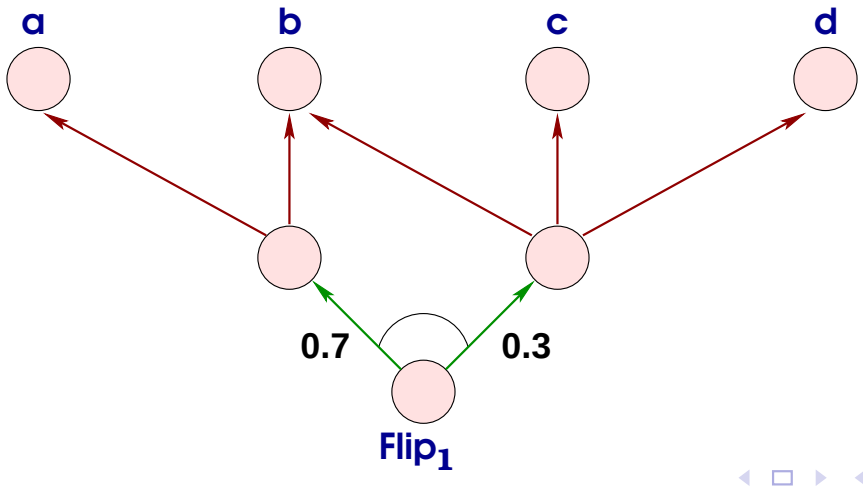
Conversely (2/2)

Let $\mathbf{Cd}_1(X)$ be the space of continuous belief functions of total mass 1.

Corollary

$\mathbf{Cd}_1(X) \cong \mathbf{V}_1(\mathcal{Q}(X))$.

I.e., continuous belief functions $\cong$ probabilistic choice (possibly non-discrete) then demonic (possibly infinitely branching) non-deterministic choice.



The Choquet Integral [1953-54]

You can always integrate any continuous function $h : X \to \mathbb{R}^+$ along any game ν:

$$\oint_{x \in X} h(x) d\nu = \int_0^{+\infty} \nu(h^{-1}]t, +\infty[) dt$$

(An ordinary Riemann integral)

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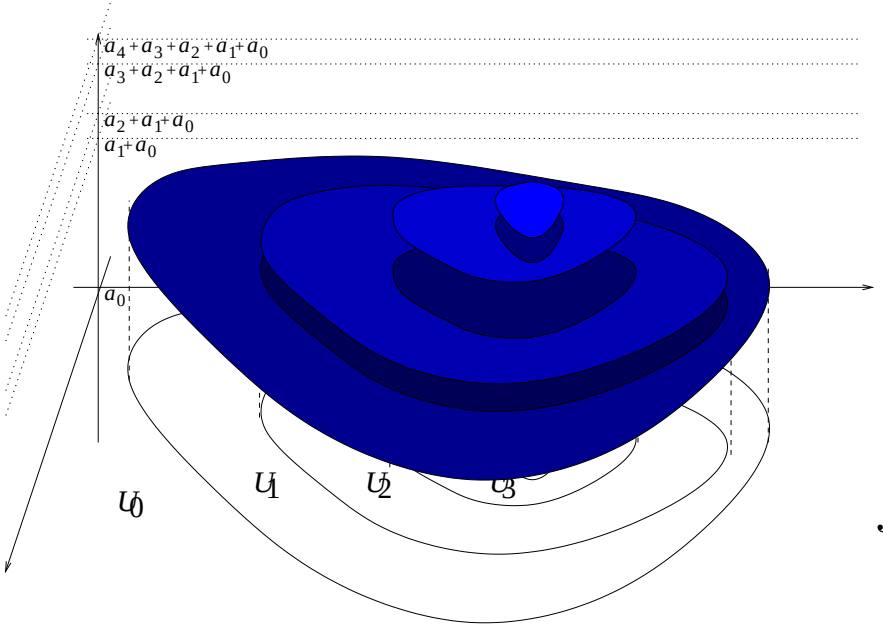


When $h = \sum_{i=1}^{n} a_i \chi_{U_i}$

(a *step* function)

$$\oint_{x \in X} h(x) d\nu = \sum_{i=1}^{n} a_i \nu(U_i)$$

Properties of the Choquet Integral

Linear in ν

$$\oint_{x \in X} f(x) d \sum_{i=1}^{n} a_i \nu_i$$

$$= \sum_{i=1}^{n} a_i \oint_{x \in X} f(x) d\nu_i$$

$$\oint_{x \in X} f(x) d\nu$$

Properties of the Choquet Integral

Linear in ν

$$\oint_{x \in X} f(x) d\nu$$

Scott−continuous in ν

$$\oint_{x \in X} f(x) d \sup^{\uparrow}_{i \in I} \nu_i = \sup^{\uparrow}_{i \in I} \oint_{x \in X} f(x) d\nu_i$$

Properties of the Choquet Integral

Linear in ν

$$\oint_{x \in X} f(x) d\nu$$

Scott−continuous in f

Scott−continuous in ν

$$\oint_{x \in X} \sup\nolimits^{\uparrow}_{i \in I} f_i(x) d\nu$$
$$= \sup\nolimits^{\uparrow}_{i \in I} \oint_{x \in X} f_i(x) d\nu$$

(Provided ν is continuous)

Properties of the Choquet Integral

Linear in f

$$\cancel{\oint_{x\in X} \sum_{i=1}^{n} a_i f_i(x) d\nu = \sum_{i=1}^{n} a_i \oint_{x\in X} f_i(x) d\nu}$$

(Unless ν is a valuation)

$$\oint_{x\in X} f(x) d\nu$$

Linear in ν

Scott−continuous in f

(Provided ν is continuous)

Scott−continuous in ν

Properties of the Choquet Integral

Positively homogeneous

$$\oint_{x\in X} a.f(x)d\nu = a.\oint_{x\in X} f(x)d\nu \quad (a \geq 0)$$

Linear in ν

$$\oint_{x\in X} f(x)d\nu$$

Scott−continuous in f

(Provided ν is continuous)

Scott−continuous in ν

Properties of the Choquet Integral

Positively homogeneous
Additive on comonotonic f, g

$$\oint_{x \in X} f(x) + g(x) d\nu \qquad (f \frown g)$$

$$= \oint_{x \in X} f(x) d\nu + \oint_{x \in X} g(x) d\nu$$

$$\oint_{x \in X} f(x) d\nu$$

Linear in ν

Scott−continuous in f

(Provided ν is continuous)

Scott−continuous in ν

Properties of the Choquet Integral

Positively homogeneous
Additive on comonotonic f, g
Superadditive

(Provided ν is convex)

$$\oint_{x\in X} f(x)+g(x)d\nu \geq \oint_{x\in X} f(x)d\nu + \oint_{x\in X} g(x)d\nu$$

$$\oint_{x\in X} f(x)d\nu$$

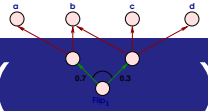
Linear in ν

Scott−continuous in f

(Provided ν is continuous)

Scott−continuous in ν

Integrating Along a Belief Function

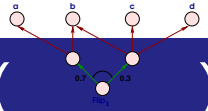
Lemma ()

Let $\nu = \sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$ *a simple belief function. Then:*

$$\oint_{x \in X} h(x) d\nu \quad = \quad \sum_{i=1}^{n} a_i \min_{x \in Q_i} h(x)$$

In other words:

Integrating Along a Belief Function

Lemma ()

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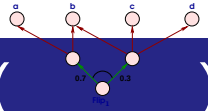
Lemma ()

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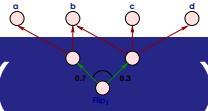
- C then picks x from Q_i so as to minimize payoff $h(x)$.

Integrating Along a Belief Function



Theorem ( )

Let X be stably compact, $\nu \in \mathbf{Cd}(X)$.

$$\oint_{x \in X} h(x) d\nu \quad = \quad \oint_{Q \in \mathcal{Q}(X)} \min_{x \in Q} h(x) d\nu^*$$

In other words:

- P draws $Q \in \mathcal{Q}(X)$ at random, with probability ν^*;
- C then picks x from Q so as to minimize payoff $h(x)$.

Demonic Non-Determinism and Unanimity Games

Lemma

$Q \mapsto \mathfrak{u}_Q$ is an order-isomorphism between $\mathfrak{Q}(X)$ and the space of unanimity games on X.

Note that $h_*(Q)$ (risk, least payoff; Smyth case) is just $\oint_{x \in X} h(x) d\mathfrak{u}_Q^*$ (Choquet "average").

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Also an isomorphism of topological spaces, provided we take weak topologies (next).

The Weak Topology

... is defined just as for valuations:

Definition (Weak Topology, $\mathbf{J}_{wk}(X)$)

For any space of continuous games $\mathbf{J}(X)$ on X, the *weak topology* on $\mathbf{J}(X)$ is generated by:

$$[U > r] \quad = \quad \{\nu \in \mathbf{J}(X) \mid \nu(U) > r\}$$

with U open in X, $r \in \mathbb{R}$. Write the resulting space $\mathbf{J}_{wk}(X)$.

E.g., when $\mathbf{J}(X) = \mathbf{Cd}_1(X)$, $\mathbf{J}_{wk}(X) = \mathbf{Cd}_{1\,wk}(X)$.

Weak Topology and Stable Compactness

Whenever $\mathbf{J}(X)$ is one of the standard spaces of continuous games with bounded total mass (valuations, belief functions, convex games, or plausibilities and concave games to come):

Theorem

If X is stably compact, then so is $\mathbf{J}_{wk}(X)$.

Proof.

À la [Jung04]; idea similar to proof of Banach-Alaoglu's Theorem:

- $\mathbf{J}_{wk}(X)$ appears as a retract of the corresponding space $J_{wk}(X)$ of non-continuous games;
- $J_{wk}(X)$ is the subspace of those (non-continuous) $\nu : \mathcal{O}(X) \to [0,1]_\sigma$ satisfying (infinitely many) inequations between patch-continuous maps (e.g., convexity, strictness), hence is patch-closed in $(\mathcal{O}(X) \to [0,1]_\sigma)_{wk}$;
- The weak topology on $\mathcal{O}(X) \to [0,1]_\sigma$ is the product topology on $[0,1]^{\mathcal{O}(X)}$;
- $[0,1]_\sigma^{\mathcal{O}(X)}$ is stably compact.

$\square$

Example Games

Model angelic non-determinism as games:

Definition

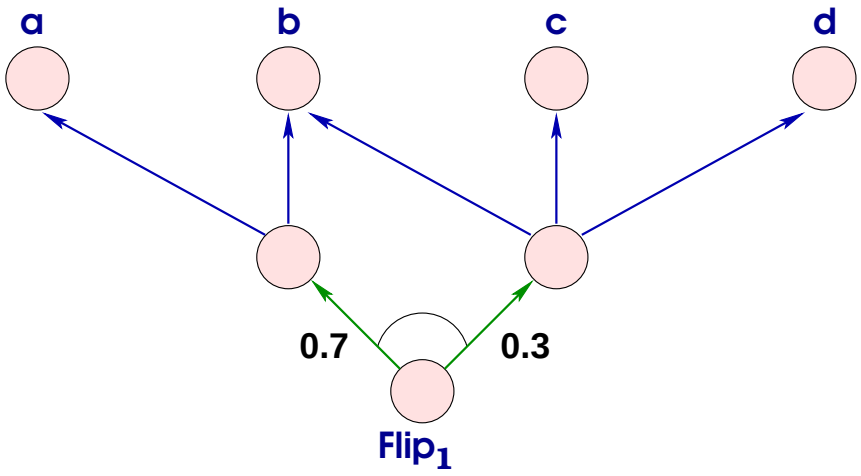
The example game $\mathfrak{e}_F$ is the set function such that:

$$\mathfrak{e}_F(U) = \begin{cases} 1 & \text{if } F \cap U \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

(No equivalent in economics.)

Simple Plausibilities

Mix (angelic) non-deterministic and probabilistic choice:



$$= 0.7\mathfrak{e}_{\{a,b\}} + 0.3\mathfrak{e}_{\{b,c,d\}}$$

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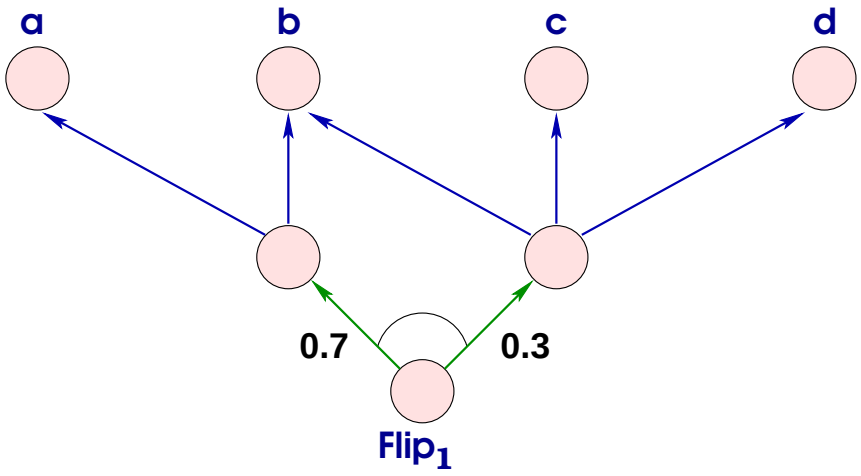

$$= 0.7\mathfrak{e}_{\{a,b\}} + 0.3\mathfrak{e}_{\{b,c,d\}}$$

Definition

A simple plausibility is any $\sum_{i=1}^{n} a_i \mathfrak{e}_{F_i}$, $a_i \in \mathbb{R}^+$, $F_i \neq \emptyset$ closed.

Axiomatization: Capacities, (Cooperative) Games

Definition

A capacity ν is a map $\mathcal{O}(X) \to \mathbb{R}^+$, with $\nu(\emptyset) = 0$.

- A game is a *monotonic* capacity: $U \subseteq V \Rightarrow \nu(U) \leq \nu(V)$;

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- A game is concave iff $\nu(U \cup V) \leq \nu(U) + \nu(V) - \nu(U \cap V)$;
  ($\geq$ for convex games)

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- A game is totally concave (i.e., a plausibility) iff:

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \leq \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right)$$

(would be $\geq$ for belief functions)

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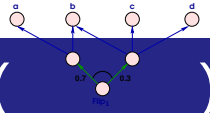
Lemma ()

Every simple plausibility is a continuous plausibility.

Conversely

Theorem (JGL, (Dual) "Completion of a misspecified model")

For every continuous plausibility ν on X, there is a unique continuous valuation ν_* on $\mathcal{H}_u(X)$ such that $\nu(U) = \nu_*(\Diamond U)$ for all opens U.

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(Yes).

By the way, let $\mathbf{Pb}_1(X)$ be the space of continuous plausibilities of total mass 1.

Dualizing Games: The Convexe-Concave Duality

As for valuations, one may measure saturated compacts Q ("à la [Tix99]"):

$$\nu^{\dagger}(Q) \quad = \quad \inf_{U \supseteq Q} \nu(U)$$

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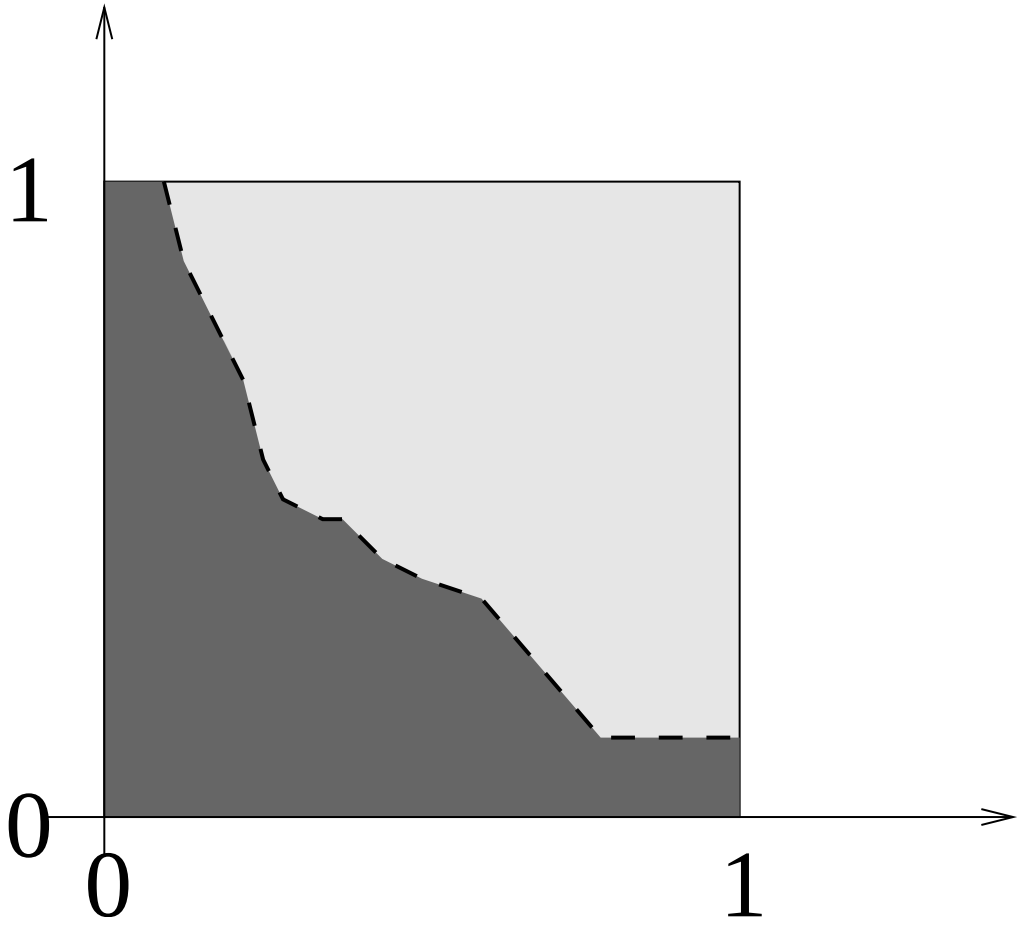
… or their complements $\overline{Q}$:

Definition ($\nu^{\perp}$)

Let X be stably compact.
For any continuous games ν of total mass 1, let:

$$\nu^{\perp}(\overline{Q}) \quad = \quad 1 - \nu^{\dagger}(Q)$$

Measuring Cocompacts



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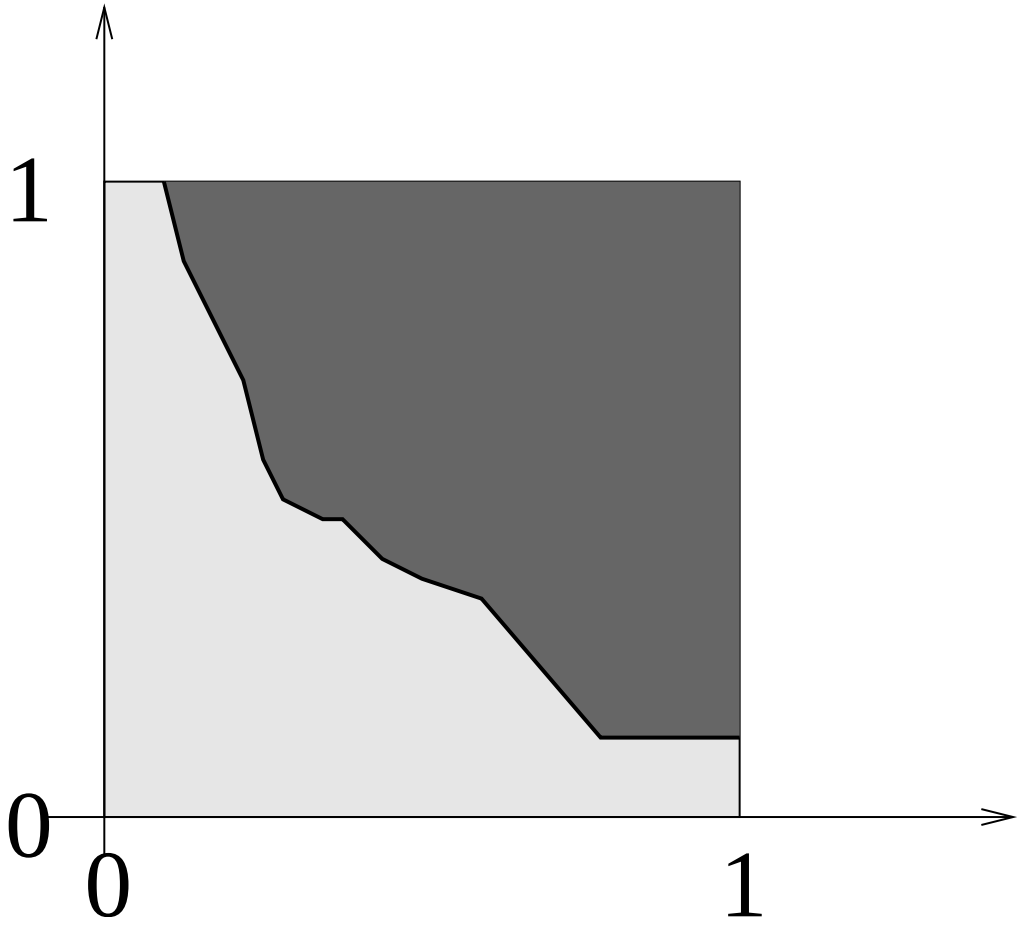

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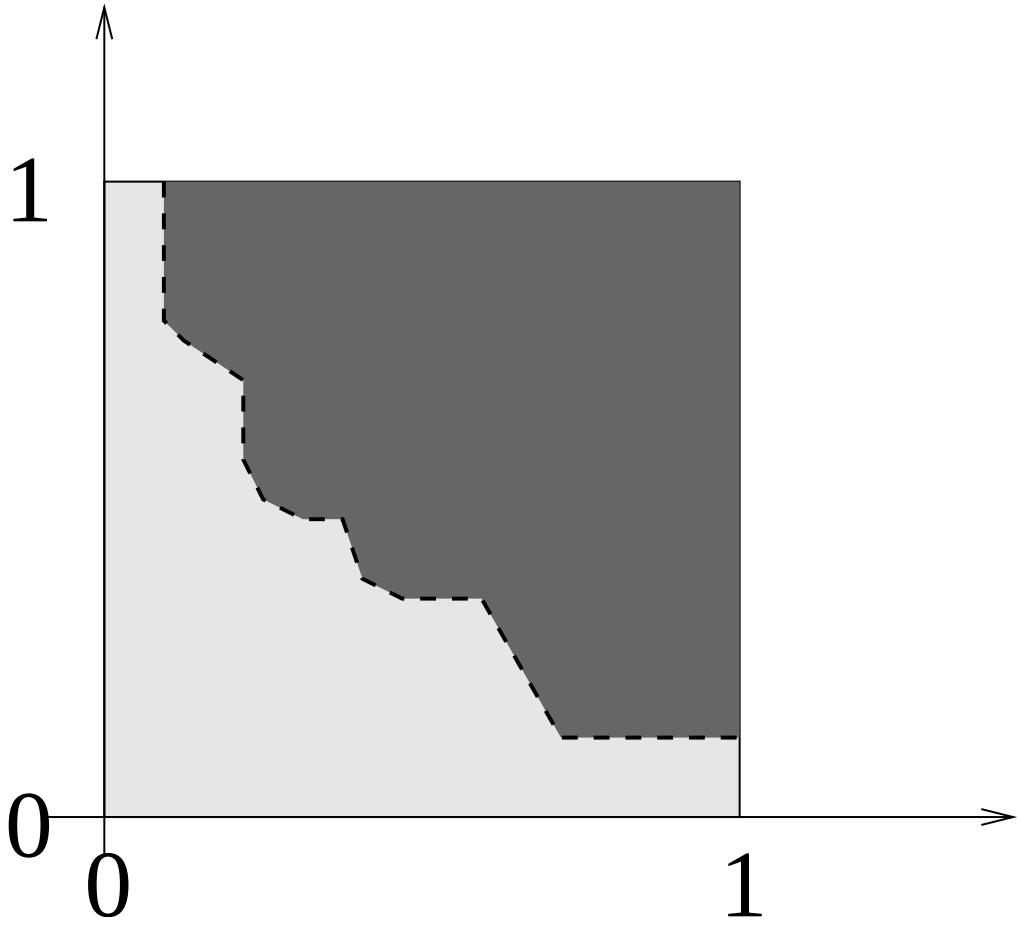

The Dual of a Continuous Game

Proposition

For any continous game ν with total mass 1 on X,

- $\nu^{\perp}$ *is a continuous game with total mass* 1 *on* X^{d}*;*
- $\nu^{\perp\perp} = \nu$.
-

$\nu \in \mathbf{J}(X)$	$\nu^{\perp} \in \mathbf{J}(X^{\mathrm{d}})$
convex	*concave*
concave	*convex*
belief function	*plausibility*
plausibility	*belief function*
valuation	*valuation [Tix99]*

Proof.

Similar arguments as in Tix's Diplomarbeit (1999), Satz 3.4. □

De Groot on Games: Convex-Concave Duality

Duality also work on spaces of games, not just valuations:

Theorem (Convex-Concave Duality)

Let X be stably compact; $_^{\perp} : \nu \mapsto \nu^{\perp}$ is an isomorphism:

$$\begin{array}{ccc} \mathbf{J}^{convex}_{1\ wk}(X)^{\mathrm{d}} & \cong & \mathbf{J}^{concave}_{1\ wk}(X^{\mathrm{d}}) \\ \mathbf{Cd}_{1\ wk}(X)^{\mathrm{d}} & \cong & \mathbf{Pb}_{1\ wk}(X^{\mathrm{d}}) \\ \mathbf{V}_{1\ wk}(X)^{\mathrm{d}} & \cong & \mathbf{V}_{1\ wk}(X^{\mathrm{d}}) \\ \{\mathfrak{u}_Q \mid Q \in \mathcal{Q}(X)\}^{\mathrm{d}} & \cong & \{\mathfrak{e}_F \mid F \in \mathcal{H}_u(X^{\mathrm{d}})\} \end{array}$$

(We already knew this.)

(i.e., $\mathcal{Q}(X)^{\mathrm{d}} = \mathcal{H}_u(X^{\mathrm{d}})$)

- I.e., Nature does not take sides, but duality exchanges demons (convex) with angels (concave).

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- I.e., Nature does not take sides, but duality exchanges demons (convex) with angels (concave).

Proof.

Show that the compact saturated subsets of $\mathbf{J}_{1\ wk}(X)$ are generated by $\langle Q > r\rangle = \{\nu \in \mathbf{J}_1(X) \mid \nu^{\dagger}(Q) > r\}$ (as before). □

Convex-Concave Duality Again

Those were topological dualities. Posets Y have another dual Y^{op}, obtained by reversing $\leq$.

Theorem (Convex-Concave Duality, Order-Theoretic)

Let X be stably compact; $_^{\perp} : \nu \mapsto \nu^{\perp}$ is a poset isomorphism:

$$\boxed{\begin{array}{ccc} \mathbf{J}_1^{convex}(X)^{op} & \cong & \mathbf{J}_1^{concave}(X^{\mathrm{d}}) \\ \mathbf{Cd}_1(X)^{op} & \cong & \mathbf{Pb}_1(X^{\mathrm{d}}) \\ \mathbf{V}_1(X)^{\mathrm{d}} & \cong & \mathbf{V}_1(X^{\mathrm{d}}) \\ \{\mathfrak{u}_Q \mid Q \in \mathcal{Q}(X)\}^{op} & \cong & \{\mathfrak{e}_F \mid F \in \mathcal{H}(X^{\mathrm{d}})\} \end{array}} \quad (\textit{i.e.}, \mathcal{Q}(X)^{op} = \mathcal{H}(X^{\mathrm{d}}))$$

Conversely

Returning to a Theorem we were too tired to prove:

Theorem (JGL, (Dual) "Completion of a misspecified model")

Let X be stably compact. For every continuous plausibility ν on X, there is a unique continuous valuation ν_* on $\mathcal{H}_u(X)$ such that $\nu(U) = \nu_*(\Diamond U)$ for all opens U.

Proof.

Existence is the only tricky part, as in the demonic case.

- $\nu^\perp$ is a belief function on X^{d};
- So there is a unique continuous valuation $(\nu^\perp)^*$ on $\mathcal{Q}(X^{\mathsf{d}})$ such that $\nu^\perp(\overline{Q}) = (\nu^\perp)^*(\Box\overline{Q})$ for all cocompacts Q (using demonic case).
- Now let $\nu_* = ((\nu^\perp)^*)^\perp$, and check that it fits (easy).

$\Box$

Some Free Theorems on Plausibilities (1)

Remember that $\mathbf{Cd}_1(X) \cong \mathbf{V}_1(\mathcal{Q}(X))$. Similarly:

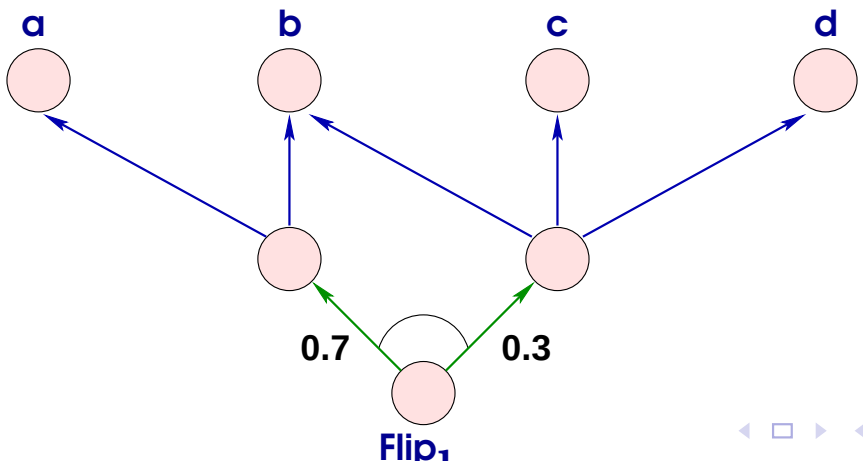
Corollary

$\mathbf{Pb}_1(X) \cong \mathbf{V}_1(\mathcal{H}_u(X))$ *whenever X is stably compact.*

Proof.

$\mathbf{Pb}_1(X) \cong \mathbf{Cd}_1(X^{\mathrm{d}})^{op} \cong \mathbf{V}_1(\mathcal{Q}(X^{\mathrm{d}}))^{op} = \mathbf{V}_1(\mathcal{H}_u(X)^{\mathrm{d}})^{op} \cong \mathbf{V}_1(\mathcal{H}_u(X))$. □

I.e., continuous plausibilities $\cong$ probabilistic choice then angelic non-deterministic choice.



Some Free Theorems on Plausibilities (2)

Theorem

If X is a coherent continuous dcpo with $\perp$, then not only $\mathcal{Q}(X)$, $\mathcal{H}(X)$ and $\mathbf{V}_1(X)$ are coherent continuous dcpos with $\perp$, but also $\mathbf{Cd}_1(X)$ and $\mathbf{Pb}_1(X)$.

Proof.

In this case, $\mathbf{Cd}_1(X) \cong \mathbf{V}_1(\mathcal{Q}(X))$, and $\mathbf{Pb}_1(X) \cong \mathbf{V}_1(\mathcal{H}(X))$ (because $\mathcal{H}(X) = \mathcal{H}_u(X)$, since the Scott topology coincides with the lower Vietoris topology in this case). $\square$

Some Free Theorems on Plausibilities (3)

Proposition

If X be stably compact, then $\mathbf{Pb}_1(X)$ is a coherent bicontinuous dcpo; every $\nu \in \mathbf{Pb}_1(X)$ is:

- *not only the directed limit $\sup^{\uparrow}_{i \in I} \nu_i$ of simple plausibilities $\nu_i \in \mathbf{Pb}_1(X)$ way-below ν;*
- *but also the filtered limit $\inf^{\downarrow}_{i \in I} \nu_i$ of simple belief functions $\nu_i \in \mathbf{Pb}_1(X)$ way-above ν.*

Proof.

- First part: $\mathbf{Pb}_1(X) \cong \mathbf{V}_1(\mathcal{H}_u(X))$; $\mathcal{H}_u(X) = \mathcal{H}(X)$ is continuous since X is; hence $\mathbf{V}_1(\mathcal{H}_u(X))$ is, too [Jones, Edalat].
- Second part: $\mathbf{Pb}_1(X)^{op} \cong \mathbf{Cd}_1(X^{\mathrm{d}}) \cong \mathbf{V}_1(\mathcal{Q}(X^{\mathrm{d}}))$; $\mathcal{Q}(X^{\mathrm{d}})$ is always continuous, hence $\mathbf{V}_1(\mathcal{Q}(X^{\mathrm{d}}))$, too.

$\square$

Some Free Theorems on Plausibilities (3)

Dually:

Proposition

If X is a coherent continuous dcpo with $\perp$, then $\mathbf{Cd}_1(X)$ is a coherent bicontinuous dcpo: every $\nu \in \mathbf{Cd}_1(X)$ is:

- *not only the directed limit $\sup^{\uparrow}_{i \in I} \nu_i$ of simple belief functions $\nu_i \in \mathbf{Cd}_1(X)$ way-below ν;*
- *but also the filtered limit $\inf^{\downarrow}_{i \in I} \nu_i$ of simple belief functions $\nu_i \in \mathbf{Cd}_1(X)$ way-above ν.*

Proof.

Duality reverses order, and changes sups into infs.
Then apply previous theorem. □

Outline

1 Introduction

2 De Groot Duality

3 Angelic and Demonic Nondeterminism

4 Probabilistic Choice

5 Belief Functions, Plausibilities

6 Concluding Remarks

I Would in fact Have Much More to Say...

... but I won't have time, and I guess you've got enough material to see the idea. Here are three directions:

- Better models of probabilistic+non-deterministic choice:
  - Continuous previsions [JGL07];
  - Convex powercones [MOW, TKP].

(Btw, those two are isomorphic, up to small details [JGL08a, KeimelPlotkin].)

Previsions are certain functionals $F : \langle X \to \mathbb{R}^+ \rangle \to \mathbb{R}^+$.
In this case again, convex-concave duality works wonders.
Idea: (except that, formally, the following is utterly nonsensical.)

$$\oint_{x \in X^{\mathrm{d}}} h(x) d\nu^{\perp} \quad = \quad - \oint_{x \in X} -h(x) d\nu$$

By analogy, define the convex-concave dual of F by $F^{\perp}(h) = -F(-h)$.

Again allows one to transfer results from demonic to angelic case [JGL08a].

I Would in fact Have Much More to Say...

... but I won't have time, and I guess you've got enough material to see the idea. Here are three directions:

- Better models of probabilistic+non-deterministic choice:
  - Continuous previsions [JGL07];
  - Convex powercones [MOW, TKP].

(Btw, those two are isomorphic, up to small details [JGL08a, KeimelPlotkin].)

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  - Hausdorff-like hemi-metrics as instances of Hutchinson hemi-metric (on games and previsions);
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  - all topologizing the usual spaces of choices (Smyth, Hoare, games, previsions);
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- De Groot duality is the key to the fundamental Theorem of Noetherian Spaces [JGL07] (The sober Noetherian spaces are the well-founded posets having properties T and W — with the upper topology.)

  ... Itself the key to defining the right completions needed in forward algorithms for coverability in WSTS [FinkelJGL09].

Conclusion

Two dualities:

- De Groot duality between spaces X, X^{d}...
- extends to the convex-concave duality between models of:
  - demonic non-determinism (+ probabilities);
  - angelic non-determinism (+ probabilities).
- Remarkably consistent results across all views of choice.
- Allows one to lift results from one world (e.g., demonic) to the other (angelic).

"Nature (probability) does not take sides, while duality exchanges the sides of demons and angels."