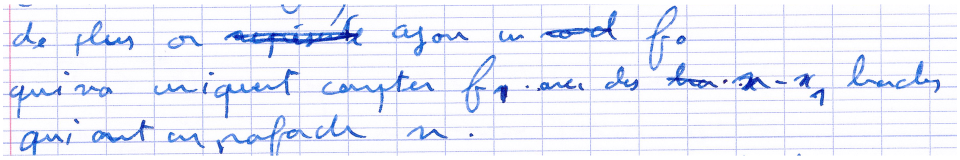
Homework assignment, λ-calculus, 2018

Please turn in on April 10th, 2018 at the latest

Note. The exam may contain a continuation of this homework assignment. I have no idea at the moment whether this will be the case, but in case that happens, you'll be happy you have devoted some thought to this homework assignment.

Important. Please be clear. I'll show no mercy for answers that I would have trouble reading or understanding. You don't have any excuse ! This is homework, and you have got time to complete it. Those from Cachan already know what I mean, but explicitly, here is something that you should avoid at all costs :



It is unreadable. Scratch marks and spelling errors make it worse. Also, the answer is completely off the mark, but that is harder to see.

Here is a sample that could be used as a model :

Q 5

> Si F une fonction inflationnaire d'un treillis complet L dans lui-même, et si $\eta \leq \eta'$, alors
>
> $$lfp_\eta(F) \leq lfp_{\eta'}(F)$$

En effet :

Il suffit de constater que l'ensemble $FP_{\eta'}$ des points fixes supérieurs à η' est, par transitivité, inclus dans l'ensemble FP_η des points fixes supérieurs à η.

Comme $lfp_\eta(F)$ minore $FP_\eta \supset FP_{\eta'}$, $lfp_\eta(F)$ est un minorant de $FP_{\eta'}$, et

$$lfp_\eta(F) \leq lfp_{\eta'}(F)$$

puisque $lfp_{\eta'}(F)$ est le plus grand des minorants de $FP_{\eta'}$.

It is clear, easy to read. The argument is correct, written in a concise way, without any irrelevant case analysis or detour.

$\star \star \star$

1 A bit of call-by-value λ-calculus

A *value* in the λ-calculus defined as a variable or a λ-abstraction. The *call-by-value λ-calculus* is defined as the λ-calculus, except that β-reduction is replaced by β_V*-reduction* :

$$(\beta_V) \qquad (\lambda x.u)V \to u[x := V]$$

where V is restricted to be a value. We write $\to_V$ for reduction in the call-by-value λ-calculus. The usual notion of reduction, β-reduction, defines *call-by-name λ-calculus*.

1. Define $\langle u\rangle^\bullet\kappa$, for all λ-terms u and κ, by :

$$\begin{aligned} \langle x\rangle^\bullet\kappa &= x\kappa \\ \langle \lambda x.u\rangle^\bullet\kappa &= \kappa(\lambda x, k.\langle u\rangle^\bullet k) && (k \text{ fresh}) \\ \langle uv\rangle^\bullet\kappa &= \langle u\rangle^\bullet(\lambda f.f(\lambda k.\langle v\rangle^\bullet k)\kappa) && (f, k \text{ fresh}) \end{aligned}$$

   Show that $u \to^* v$ implies $\langle u\rangle^\bullet\kappa \to_V^* \langle v\rangle^\bullet\kappa$, for all terms u, v, and for every value κ. State clearly any auxiliary lemmas you need, and number them (A), (B), ..., in that order.

2. We use the simple type discipline. We fix a type (formula) A, and define $\neg_A F$ as $F \to A$. Define $\langle F\rangle^*$ and $(F)^\bullet$, for every type F, by :

$$\begin{aligned} \langle F\rangle^* &= \neg_A\neg_A\langle F\rangle^\bullet \\ \langle b\rangle^\bullet &= b && (b \text{ base type}) \\ \langle F \to G\rangle^\bullet &= \langle F\rangle^* \to \langle G\rangle^* \end{aligned}$$

   For every typing context Γ, $\langle\Gamma\rangle^*$ denotes the typing context obtained by replacing every binding $x : F$ in Γ by $x : \langle F\rangle^*$. Show that if $\Gamma \vdash u\colon F$ and $\langle\Gamma\rangle^* \vdash \kappa\colon \neg_A\langle F\rangle^\bullet$ are derivable, then $\langle\Gamma\rangle^* \vdash \langle u\rangle^\bullet\kappa\colon A$ is derivable.

3. Give an example of a λ-term that has a normal form in the (call-by-name) λ-calculus but does not have any in the call-by-value λ-calculus.

2 A maximal strategy

For any λ-term u, define $\nu(u)$ as the length of the longest reduction starting from u if $u \in SN$ (the set of strongly normalizing terms), as ∞ otherwise. Note that $\nu(u) = \infty$ if and only if there is an infinite reduction starting from u.

4. For all terms $u_0, u_1, \ldots, u_n$ ($n \geq 1$), show that $\nu((\lambda x.u_0)u_1 \cdots u_n)$ is equal to $\nu(u_0[x := u_1]u_2 \cdots u_n) + 1$ if x is free in u_0, and to $\nu(u_0 u_2 \cdots u_n) + \nu(u_1) + 1$ otherwise.

The *maximal strategy* selects the leftmost redex $(\lambda x \cdot u)v$ such that x is free in u or v is normal, if any (otherwise, reduction stops). Let $\rightarrow_{\max}$ denote β-reduction according to the perpetual strategy. Explicitly, $\rightarrow_{\max}$ is defined inductively by :

— $(\lambda x \cdot u)v \rightarrow_{\max} u[x := v]$ if x is free in u or if v is normal ;

— $(\lambda x \cdot u)v \rightarrow_{\max} (\lambda x \cdot u)v'$ if x is not free in u and $v \rightarrow_{\max} v'$;

— $uv \rightarrow_{\max} u'v$ if u is neutral and $u \rightarrow_{\max} u'$;

— $uv \rightarrow_{\max} uv'$ if u is normal, neutral, and $v \rightarrow_{\max} v'$;

— $\lambda x \cdot u \rightarrow_{\max} \lambda x \cdot u'$ if $u \rightarrow_{\max} u'$.

Beware that $\rightarrow_{\max}$ is *not* meant to be closed under context application.

Note also that the maximal strategy is deterministic : either u is normal or it has a unique $\rightarrow_{\max}$-redex.

For every term u, let $\nu_{\max}(u)$ denote the length of the unique $\rightarrow_{\max}$-reduction starting from u (possibly ∞).

5. Justify that $\nu(v) \leq \nu(u)$ for every subterm v of any term u.
6. Show that $\nu_{\max}(u) = \nu(u)$, for every term u. Hint 1 : examine the head form of u. Hint 2 : if the claim failed, there would be a counterexample with $\nu(u)$ minimal (in $\mathbb{N} \cup \{\infty\}$), and among those with that value of $\nu(u)$, there would be one of minimal size.
7. Show the *anti-standardization theorem* : if u is not strongly normalizing, then the $\rightarrow_{\max}$-reduction starting from u is infinite. (We say that the maximal strategy is *perpetual*.)

3 The λ_I-calculus

A λ_I*-redex* is a redex $(\lambda x.u)v$ such that x is free in u. A λ_I*-term* is a term in which all the subterms of the form $\lambda x.u$ are such that x is free in u. Hence, for example, $\lambda x.y$ is not a λ_I-term, but $\lambda x, y.y(\lambda z.z)x$ is a λ_I-term.

8. Show that for every λ_I-redex $(\lambda x.u)v$, its contractum $u[x := v]$ has exactly the same free variables as the original redex.

9. Show that for every λ_I-term u, if $u \to v$ then v is a λ_I-term.
10. Show that for every λ_I-term u, u is (weakly) normalizing if and only if it is strongly normalizing.
11. We define a new encoding of natural numbers by $\lceil n \rceil' = \lceil n+1 \rceil$, for every $n \in \mathbb{N}$. Recall that $\lceil n \rceil$ is the Church numeral for n (see the lecture notes on the untyped λ-calculus). This is a λ_I-term provided $n \geq 1$.

    Give a λ_I-term $\lceil intcase \rceil'$ such that :

    $$\lceil intcase \rceil' \lceil 0 \rceil' u f \to^* u$$
    $$\lceil intcase \rceil' \lceil n+1 \rceil' u f \to^* f \lceil n \rceil'$$

    for all $n \in \mathbb{N}$. You are allowed, and even encouraged, to reuse the constructions given in Section 2.3 of the lecture notes on the untyped λ-calculus.

One can do similar re-codings of all the constructions of Section 2.3 of the lecture notes on the untyped λ-calculus, and that shows that, despite its limitations, the λ_I-calculus also encode all recursive functions, using the modified Church numerals $\lceil n \rceil'$.