Weighted Strategy Logic with Boolean goals over one-counter games.

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Games are used for the synthesis of reactive system. We look at objectives that can be express through temporal logics.
One logic, two semantics and many sub-logics

Inclusion graph
One logic, two semantics and many sub-logics

- $SL_{float}$
- $NG_{float}$

Undecidable

- $AG$
- $BG$

2Exptime complete

- $CG$  $DG$

k-Exptime hard for any $k$

$ATL^*_SC$

$ATL^*$

$LTL$

Inclusion graph
Example of a formula in BG

Let $\phi : \exists x \forall y \exists z \ [\circ/x][\Box/y] Fq \rightarrow [\circ/z][\Box/y] Fp$
Definition (BG)

Formulas are build upon the rules:

\[
\phi := \exists x \phi \mid \forall x \phi \mid \phi
\]

\[
\varphi := [agent/strategy] \phi \mid \phi \lor \psi \mid \neg \phi \mid \land \phi \mid \psi
\]

\[
\psi \in LTL
\]

Normal form

\[
\varphi \land \bigvee_{i \in I} \bigvee_{j \in J} B_{i,j} \psi_{i,j}
\]

with \( \varphi \) a bunch of quantifications, \( B \) a set of bindings of form \([\circ/x] \) and the \( \psi_{i,j} \) a bunch of LTL formulas.
Example: Nash equilibrium [MMPV10]

One can express qualitative Nash equilibrium for LTL objectives:

Let $a_1, \ldots a_l$ be $l$ agents, each $a_i$ with their own objective $\psi_i$. Then the following express existence of a Nash equilibrium:

$$
\exists x_1 \ldots x_l \ \forall y_1 \ldots y_l \ \bigwedge_{i \leq l} \left[ a_1/x_1 \right] \ldots \left[ a_{i-1}/x_{i-1} \right] \left[ a_i/y_i \right] \left[ a_{i+1}/x_{i+1} \right] \ldots \left[ a_l/x_l \right] \ \psi_i

\Rightarrow \left[ a_1/x_1 \right] \ldots \left[ a_{i-1}/x_{i-1} \right] \left[ a_i/x_i \right] \left[ a_{i+1}/x_{i+1} \right] \ldots \left[ a_l/x_l \right] \ \psi_i
$$
Multi-objectives strategies

Multi-purposes strategies are the key. In concurrent-games, a strategy can be shared by multiple players:

$$\exists x \ y \ \forall z \ \left( [a_1/x][a_2/y][a_3/z]G(p) \leftrightarrow [a_1/z][a_2/y][a_3/x]G(p) \right)$$

For a given objective, checking a player privileges against another player becomes possible - in contrast to usual $\omega$-objectives.
Non behavioural logic

\[ \psi_0 := \forall w, x \exists y, z \]

\[ \left( [\circ / w][\Box / x] Fp \right) \iff \left( [\circ / y][\Box / z] Fq \right) \]
Non behavioural logic

\[ \psi_0 := \forall w, x \exists y, z \]

\[
(\lfloor \circ / w \rfloor [\square / x] Fp) \iff (\lfloor \circ / y \rfloor [\square / z] Fq)
\]
Introduction to BG problematics

Non behavioural logic

\( \psi_0 := \ \forall w, x \ \exists y, z \)

\[ ([\circ / w][\Box / x] \ Fp) \iff ([\circ / y][\Box / z] \ Fq) \]
Math. representation of the dependencies

Non-behavioural:

\[(path \mapsto action)^\forall strats \implies (path \mapsto action)^\exists strats\]

with \(x_i\) only depending on universal choices of strategies quantified before \(x_i\).

Behavioural:

\[path \mapsto \left((action)^\forall strats \implies (action)^\exists strat\right)\]

with \(x_i\) only depending on universal choices of strategies quantified before \(x_i\).
Complexity

Theorem

The following formulas admit a $2 - \text{EXPTIME}$ model checking:

- **LTL objectives over turn based game.**
- **Conjunctive/disjunctive goals** [MMPV12] :
  \[
  \varnothing \bigwedge_{i \in I} B_i \psi_i \quad \text{and} \quad \varnothing \bigvee_{j \in J} B_j \psi_j
  \]
- **"Alternating" formulas** [MMS14]
  \[
  \varnothing B_1 \psi_1 \lor \left( B_2 \psi_2 \land (B_3 \psi_3 \lor (B_4 \psi_4 \land \ldots )) \right)
  \]

For the full fragment we knew that:

1. Non-Elementary upper bound
2. 2-Exptime lower bound
Conjunctive and disjunctive formulas

Theorem (MMPV12)

*Both the conjunctive and disjunction formulas are behavioural*

\[ \phi := Q_1 x_1, \ldots, \exists x_a \ldots Q_l x_l \ \land_{i \leq n} \beta_i \psi_i \text{ with } Q_j \in \{\exists, \forall\}. \]

We look for the objectives ”realizable” on \( \pi \).

We play for all ”realizable” objectives on \( \pi \).

Existential strategies will separate different objectives into different paths.
Alternating formulas

Theorem (MMS14)

The alternating formulas are behavioural.

\[ \phi := \varnothing \mathcal{B}_1 \psi_1 \lor \left( \mathcal{B}_2 \psi_2 \land (\mathcal{B}_3 \psi_3 \lor (\mathcal{B}_4 \psi_4 \land \ldots)) \right). \]

There is an implicit order on objective \( \psi_1 > \psi_2 \cdots > \psi_n \). Such ordering doesn’t exist in the full fragments (ex: \( \beta_1 \psi_1 \leftrightarrow \beta_2 \psi_2 \)).

A strategy make its choices according to the order. You do not have to explore the full possibilities of the other objectives.
Hardness Results

Theorem

*BG is Tower – hard over concurrent games*
QLTL

\[ QLT\ell : \exists p_1 \ \forall p_2 \ldots \exists p_l \ \phi \text{ with } \phi \in LTL \text{ over the atomic set } \{p_1, \ldots p_l\}. \]

Semantics :

- \( \exists p_i \mapsto \) we can extend previous colouring of \( \mathbb{N} \) with a new color \( p_i \).
- \( \forall p_i \mapsto \) for all extensions of the previous colouring with a new color \( p_i \).
- \( \phi \in LTL \mapsto \) Coloured \( \mathbb{N} \) satisfy \( \phi \)

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & \ldots \\
p_1 & p_1 & p_1 & p_1 & p_1 & \ldots \\
p_2 & p_2 & p_2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_l & p_l & p_l & p_l & p_l & \ldots \\
\end{array}
\]
Theorem

QLTL satisfiability is $k$-hard for formulas with $k$ quantifiers alternations.

We will use the Buchi automaton associated with $\phi$.

Remainder: States of the Buchi automaton are labelled by sub-formulas of $\phi$. 
States of Buchi automaton for $\phi$(LTL part)

- $p_1, p_2, \ldots \in q_1$
- $p_1, \neg p_2, \ldots \in q_2$
- $\neg p_1, p_2, \ldots \in q_3$
- $\neg p_1, \neg p_2, \ldots \in q_4$

Define the path that must satisfy $\phi$
Sketch of proof

States of Buchi automaton for $\phi$(LTL part)

**How it works**

1. Check that the corresponding atomic propositions or negation are in $q_i$.
2. Check that $q_{i+1}$ is accessible from $q_i$.
3. Check the sequence $(q_i)_{i \in \mathbb{N}}$ see infinitely often an accepting state.

Define the path that must satisfy $\phi$
Sketch of proof: $q_i$ follows the choices among the $p'_j$s.

$\alpha$ force change

$(s \times \alpha \times \gamma)^\mathbb{N}$
Sketch of proof: $q_i$ follows the choices among the $p'_j$'s

\[
\begin{align*}
\alpha \text{ force change} & \quad \rightarrow \quad (s \times \alpha \times \gamma)^\mathbb{N} \\
\end{align*}
\]
Sketch of proof: $q_i$ follows the choices among the $p'_j$s

- $\alpha$ force change
- $a_2 \rightarrow p_2 \quad 2nd\ objective$
- $a_1 \rightarrow p_1$
- $b \rightarrow q$

$\alpha \times (s \times \beta \times \gamma)^\mathbb{N}$

Buchi states

- $q$
- $q'$
- $b$
- $s$
- $a_1$
- $\neg p_1$
- $p_1$
- $a_l$
- $\neg p_l$
- $p_l$

$\alpha$
- $\beta_1$
- $\beta_l$

$\delta$

$\gamma$
Sketch of proof: \( q_i \) follows the choices among the \( p_j \)s

If both 1st and 2nd objectives are ok then 3rd must be ok

\[ \alpha \text{ force change} \rightarrow \ (s \times \alpha \times \gamma)^\mathbb{N} \]
Checking for an infinite number of accepting states

\[(s \times \alpha \times \gamma)^\mathbb{N}\]
Checking for an infinite number of accepting states

If $s_\alpha$ stops

$(s \times \alpha \times \gamma)^N$
Checking for an infinite number of accepting states

If $s_\alpha$ stops

There is $s_\gamma$

$(s \times \alpha \times \gamma)^\mathbb{N}$
Checking for an infinite number of accepting states

If $s_\alpha$ stops, there is $s_\gamma$ such that

$s_\alpha + s_\gamma \text{ get to } \alpha \times \delta$

\[(s \times \alpha \times \gamma)^\mathbb{N}\]
Checking a state is accessible from its predecessor

\[(s \times \alpha \times \gamma)^\mathbb{N}\]
Checking a state is accessible from its predecessor

If $s_\alpha$ stops

$$\left( s \times \alpha \times \gamma \right)^\mathbb{N}$$

Buchi states
Checking a state is accessible from its predecessor

If $s_\alpha$ stops

There is $s_\gamma$

$$(s \times \alpha \times \gamma)^\mathbb{N}$$
Checking a state is accessible from its predecessor

If \( s_\alpha \) stops

There is \( s_\gamma \)

can’t find \( s_{\text{blue}} \) with \( s_\alpha + s_{\text{blue}} \) get to \( s \times \alpha \) and \( s_\gamma + s_{\text{blue}} \) to \( b \times \gamma \)
Is concurrency essential to the hardness result?

One cannot use the same technique on turn-based games:

- Commutable, hence we have a set and not a sequence
Aftermath

• "Alternatively built" formulas model checking is 2-EXPTIME complete [MMS14]

\[ \varnothing \ B_1 \psi_1 \lor \left( B_2 \psi_2 \land (B_3 \psi_3 \lor (B_4 \psi_4 \land \ldots)) \right) \]

• Full fragment is \( k - \text{Exptime} \) complete for all \( k \).

\[ \varnothing \land \bigvee_{i \in I} \bigvee_{j \in J} B_{i,j} \psi_{i,j} \]

Open questions

What happens on turn base games?

What link exists between non-behavioral formulas, order on the objectives and model checking?
Open questions

- Can we exhibit a periodic property for BG as we can for CTL, LTL, ATL and ATL*?
- If so, can we use such property to find an algorithm to check BG formulas.

There exists a periodic property, but we can’t generate an algorithm for the model checking.
Periodicity

**Theorem**

In a 1 counter game $\mathcal{G}$ and given a BG formula $\phi$, there exists two integers $h, \Lambda$ such that for any configuration $(q, c)$ with $c > h$:

$$(q, c) \models \phi \text{ if and only if } (q, c + \mathbb{N} \times \Lambda) \models \phi$$
Introduction to BG problematics

Hardness proof of BG

BG over one counter
Thank you!

Open questions

What happens on turn base games?

What link exists between non-behavioral formulas, order on the objectives and the model checking problem?