Automates d’arbre
TD n°5 : Alternating Automata

Exercise 1:
Prove that (Membership):
Instance: an AWA \( A \) and a word \( w \)
Question: \( w \in L(A) \) ?
is in PTIME.
Same question for top-down ATA.

Solution:
Let \( n \) be the size of \( w \). Define \( S_i \) with \( 0 \leq i \leq n \) by decreasing induction :
- \( S_n = Q_f \)
- \( S_i = \{ q \in Q \mid \exists q_i, w_i \rightarrow \phi \in A, S_{i+1} \models \phi \} \)
Then \( w \in L(A) \) iff \( I \cap S_0 \neq \emptyset \). All this can be done in PTIME. Idem for ATA.

Exercise 2:
Let \( \Phi \) be a propositional formula in CNF with \( n \) variables \( x_1, \ldots, x_n \). Construct an AWA \( A_\Phi \) of size polynomial in the size of \( \Phi \) such that \( L(A_\Phi) = \{ w \in \{0,1\}^n \mid w \models \Phi \} \).

Solution:
Let \( \Phi = \bigwedge_{j=1}^m C_j \) with \( C_j \) clauses. The expected AWA is the following :
- \( Q = \{ q_0 \} \cup \{ q_{\alpha,k}^j \mid 1 \leq j \leq m, 1 \leq k \leq n, \alpha \in \{0,1\} \} \)
- \( I = \{ q_0 \} \)
- \( F = \{ q_j^1 \mid 1 \leq j \leq m \} \)
- \( \Delta = \)
  * \( q_{0,1} \mapsto \bigwedge_{j|x_j \in C_j} q_{j,1}^{1,1} \land \bigwedge_{j|x_j \notin C_j} q_{j,1}^{0,1} \)
  * \( q_{0,0} \mapsto \bigwedge_{j|\neg x_j \in C_j} q_{j,1}^{1,1} \land \bigwedge_{j|\neg x_j \notin C_j} q_{j,1}^{0,1} \)
  * \( q_{1,k}^{1,k}, 1 \mapsto q_{1,k+1}^{1,k+1} \) for all \( k < n \) and all \( j \)
  * \( q_{1,k}^{0,k}, 1 \mapsto q_{0,k+1}^{0,k+1} \) for all \( j \) and all \( k < n \) such that \( x_k \in C_j \)
  * \( q_{1,k}^{1,k}, 0 \mapsto q_{1,k+1}^{1,k+1} \) for all \( j \) and all \( k < n \) such that \( \neg x_k \in C_j \)
  * \( q_{1,k}^{0,k}, 0 \mapsto q_{j,k+1}^{0,k+1} \) for all \( j \) and all \( k < n \) such that \( \neg x_k \notin C_j \)

Exercise 3:
Prove that (Membership):
Instance: an NFHA \( A \) where the horizontal languages are given by AWA (and not finite automata) and a word \( w \)
Question: \( w \in L(A) \) ?
is NP-complete.

Solution:
- in NP : first, guess a run i.e. a coloring \( \rho \) by states of your tree. Second, check this is an accepting run i.e. for all position \( p \) of \( t \), check that there exists a transition of the form
that \( \rho(p.1) \ldots \rho(p.k) \in L \) (which can be done in P by exercise 1) and \( \rho(\epsilon) \) is a final state.

- NP-hard : we reduce SAT. Let \( \Phi \) in CNF. We construct \( \tilde{A}_\Phi \) this way : \( Q = \{ 0, 1, q_f \} \), \( F = \{ q_f \} \), \( \Sigma = \{ \@, \# \} \) and \( \Delta = \{ \#(\epsilon) \rightarrow 1, \#(\epsilon) \rightarrow 0, a(A_\Phi) \rightarrow q_f \} \) where \( A_\Phi \) is from exercise 2. Then \( \Phi \) is satisfiable iff \( a(\#, \ldots, \#) \in L(\tilde{A}_\Phi) \).

**Exercise 4 :**

1) Given a NFTA \( A = \langle Q, F, Q_f, \Delta \rangle \), let \( A' \) be the AWA on the singleton alphabet \( \{ 1 \} \) with states \( Q \times F \) and transitions : 
\[
\delta((q, f), 1) = \bigvee_{f(q_1, \ldots, q_m) \rightarrow q \in \Delta} \bigwedge_{i=1}^{n} \bigvee_{g \in F} (q_i, g)
\]

Give the set of initial/final states of \( A' \) such that \( L(A) \neq \emptyset \) iff \( L(A') \neq \emptyset \). What can we deduce?

2) Given an AWA \( A \) on a singleton alphabet and whose formulae are positive and in DNF, construct a NFTA \( A' \) in polynomial time such that \( L(A) \neq \emptyset \) iff \( L(A') \neq \emptyset \). What can we deduce?

**Solution:**

1) 
- \( F = \{ (q, f) \mid f \rightarrow q \in \Delta \} \)
- \( I = \{ (q, f) \mid q \in Q_f \} \)
- We deduce that emptiness for AWA on singleton alphabet is P-hard.

2) 
- Let \( A = \langle Q, \{ * \}, F, I, \Delta \rangle \) such a AWA. We construct the NFTA \( A' = \langle Q, \{ f_k(k) \mid 0 \leq k \leq n \}, F', \Delta' \rangle \) with :
  * \( F' = I \)
  * \( f_{m_j}(q_1, \ldots, q_{m_j}) \rightarrow q \in \Delta' \) if \( (q, 1) \rightarrow \bigvee C_j \in \Delta \) where \( \{ q_1, \ldots, q_{m_j} \} \) is the set of all positive literals of \( C_j \)
- We deduce that the previous problem where formulae are in DNF is in P.