Exercise 1:

Give a recognizable tree language \( L \) and a tree morphism \( h \) such that \( h(L) \) is not recognizable.

Solution:

Let \( \mathcal{F} = \{g(1), a(0)\} \) and \( \mathcal{F}' = \{f(2), a(0)\} \). If \( h \) maps \( g(x) \) to \( f(x, x) \) and \( a \) to \( a \) then \( h(T(\mathcal{F})) \) is not recognizable by the pumping lemma.

Exercise 2:

1) We can see the set of runs of an NFTA \( \mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta) \) as a tree language on \( \mathcal{F} \times Q = \{(f, q)(n) \mid f(n) \in \mathcal{F}, q \in Q\} \) as the smallest set \( \text{Run}(\mathcal{A}) \) included in \( T(\mathcal{F} \times Q) \) such that :

- if \( f q_1, ..., q_n \rightarrow q \in \Delta \), then \( (a, q) \in \text{Run}(\mathcal{A}) \)
- if \( f q_1, ..., q_n \rightarrow q \in \Delta \) and \( t_1, ..., t_n \in \text{Run}(\mathcal{A}) \) with \( t_i(\epsilon) = (, q_i) \) then \( (f, q)(t_1, ..., t_n) \in \text{Run}(\mathcal{A}) \).

Then the set of accepting runs can be seen as \( \text{Acc}(\mathcal{A}) = \{t \in \text{Run}(\mathcal{A}) \mid t(\epsilon) = (, q), q \in Q_f\} \).

Prove that \( \text{Acc}(\mathcal{A}) \) is in the smallest class \( \text{Stab} \) of sets which contains all the \( T(\mathcal{F}) \) for any finite ranked set \( \mathcal{F} \) and which is stable by image of linear morphisms and inverse image of morphisms. For example, you should be able to prove that \( \text{Acc}(\mathcal{A}) = \beta^{-1}(\gamma(\delta^{-1}(T(\mathcal{F}')))) \) where \( \gamma \) is linear.

Deduce that \( \text{Stab} = \text{Rec} \).

Solution:

1) We define \( \beta, \gamma \) and \( \delta \) this way :

- \( \delta : T(\mathcal{F}_A) \rightarrow T(\mathcal{F}_\bot) \) where \( \mathcal{F}_\bot = \mathcal{F} \cup \{(1) \mid q \in F\} \cup \{\bot(0)\} \) and \( \mathcal{F}_A = \{(f, q_1, ..., q_n, q)(n) \mid f(n) \in \mathcal{F}, q, q_i \in q\} \cup \{(1) \mid q \in Q\} \) such that :

- \( * q(x) \rightarrow q(x) \) if \( q \in F \)
- \( * q(x) \rightarrow \bot \) if \( q \notin \mathcal{F} \)
- \( * (f, q_1, ..., q_n, q)(x_1, ..., x_n) \rightarrow f(x_1, ..., x_n) \) if \( f(q_1, ..., q_n) \rightarrow q \in \Delta \)
- \( * (f, q_1, ..., q_n, q)(x_1, ..., x_n) \rightarrow \bot \) else

- \( \gamma : T(\mathcal{F}_A) \rightarrow T(\mathcal{F}_Q) \) linear where \( T(\mathcal{F}_Q) = \mathcal{F} \cup \{(1) \mid q \in Q\} \) such that :

- \( ** q(x) \rightarrow q(x) \)
- \( ** (f, q_1, ..., q_n, q)(x_1, ..., x_n) \rightarrow q(f(q_1(x_1), ..., q_n(x_n))) \)

- \( \beta : T(\mathcal{F} \times Q) \rightarrow T(\mathcal{F}_Q) \) such that :

- \( *** (f, q)(x_1, ..., x_n) \rightarrow q(q(f(x_1, ..., x_n))) \)

Then \( \text{Acc}(\mathcal{A}) = \beta^{-1}(\gamma(\delta^{-1}(T(\mathcal{F}_\bot \setminus \bot)))) \).

2) \( \text{Stab} \subseteq \text{Rec} : \text{Rec} \) is stable under inverse image, linear image and contains all the \( T(\mathcal{F}) \).

\( \text{Stab} \subseteq \text{Rec} : \) Let \( L \in \text{Rec} \) and \( \mathcal{A} \) a NFTA recognizing \( L \). Define \( \alpha : T(\mathcal{F} \times Q) \rightarrow T(\mathcal{F}) \) linear such that :

- \( *** (f, q)(x_1, ..., x_n) \rightarrow f(x_1, ..., x_n) \)

Then \( L = \alpha(\text{Acc}(\mathcal{A})) \) and by 1), \( L \in \text{Stab} \).

Exercise 3:

A bottom-up tree transducer (NUTT) is a tuple \( U = (Q, \mathcal{F}, \mathcal{F}', Q_f, \Delta) \) where \( Q \) is a finite set (of states), \( \mathcal{F} \) and \( \mathcal{F}' \) are finite ranked sets (of input and output), \( Q_f \subseteq Q \) (final states) and \( \Delta \) is a finite set of rules of the form :
Exercise 4:

3) Prove that the image of a recognizable tree language is recognizable.

1) Prove that tree morphisms are a special case of NUTT that is if \( \mu : T(F) \rightarrow T(F') \) is a morphism, then there exists a NUTT \( U_\mu \) such that \( R(U_\mu) = \{(t, \mu(t)) \mid t \in T(F)\} \). Be sure that if \( \mu \) is linear then \( U_\mu \) is too.

2) Prove that the domain of a NUTT \( U \), that is \( \{t \in T(F) \mid \exists t' \in T(F'), (t, t') \in U\} \), is recognizable.

3) Prove that the image of a recognizable tree language \( L \) by a linear NUTT \( U \), that is \( \{t' \in T(F') \mid \exists t \in L, (t, t') \in U\} \), is recognizable.

Solution:

1) \( Q = \{q\}, Q_f = \{Q\} \) and \( \Delta = f(q(x_1),...,q(x_n)) \rightarrow q(\mu(f)(x_1,...,x_n)) \) linear when \( \mu \) is

2) \( Q = Q_U, F = F_U \) and \( \Delta = f(q_1,...,q_n) \rightarrow q \) if there exists \( u \) such that \( f(q_1,...,q_n) \rightarrow q(u) \in \Delta_U \)

3) Let \( Q = Q_U \) and \( A \) a NFTA on \( F \). For every pair of rules \( r = f(q_1(x_1),...,q_n(x_n)) \rightarrow q(u) \in \Delta_U \) and \( r' = f(q_1'(x_1),...,q_n'(x_n)) \rightarrow q'(q) \in \Delta_A \), we define :

\[ Q^{r,r'} = \{q^{r,r'}_p \mid p \in Pos(u)\} \]

\[ \Delta^{r,r'} = \]

\* \( g(q^{r,r'}_p) \rightarrow g^{r,r'}_p \) for \( p \in Pos(u) \) such that \( u(p) = g \in F' \)

\* \( q^{r,r'}_p \rightarrow q^{r,r'}_p \) if \( u(p) = x_i \) (linearity assure that we only have one of this kind for every \( i \))

\* \( q^{r,r'}_p \rightarrow (q, q') \)

For every rule \( r = g(x) \rightarrow q'(u) \in \Delta_U \), we define :

\[ Q^r = \{q^r_p \mid p \in Pos(u)\} \times Q_A \]

\[ \Delta^r = \]

\* \( g((q^r_p, q''),...,q''_p) \rightarrow (q''_p, q'') \) for \( p \in Pos(u) \) such that \( u(p) = g \in F' \) and \( q''_p \in Q_A \)

\* \( (q, q'') \rightarrow (q''_p, q'') \) if \( u(p) = x \) and \( q''_p \in Q_A \) (linearity assure that we only have one of this kind)

\* \( (q''_p, q'') \rightarrow (q, q'') \)

Then this NFTA works :

\[ \tilde{Q} = Q_U \times Q_A \cup \bigcup_{(r,r')} Q^{r,r'} \cup \bigcup_{r} Q^r \]

\[ \tilde{F} = F_U \times F_A \]

\[ \tilde{\Delta} = \bigcup_{(r,r')} \Delta^{r,r'} \cup \bigcup_{r} \Delta^r \]

Exercise 4:

1) Let \( F = \{f(2), g(1), a(0)\} \). Is the relation \( R = \{(g(t), t) \mid t \in T(F)\} \) recognizable? And if \( F = \{g(1), a(0)\}\)?

2) Here assume that \( F = \{g(1), a(0)\} \). Is \( R^* \) recognizable?

From now, let \( F = \{a_i(1) \mid 1 \leq i \leq n\} \cup \{0(0)\} \).

3) Prove that any rewrite system \( \rightarrow \) (i.e. the one step rewriting relation) on \( F \) is recognizable.

4) Prove that if \( S = \{(a_1^k(a_1(a_2^k(a_1(0))))), a_1^k(a_2^k(0))) \mid k, p \in \mathbb{N}\} \), then \( S^* \) is not recognizable.

Solution:

1) No by the pumping lemma. Yes.
Exercise 8:

Exercise 6:

Exercise 5:

A GTT is a pair \((A_1, A_2)\) of NFTA on the same alphabet and which can share some states (synchronization states).

A pair \((t, t')\) is recognized by the GTT \((A_1, A_2)\) if there exists a context \(C \in \mathcal{C}^n(\mathcal{F})\), trees \(t_1, ..., t_n, t'_1, ..., t'_n\) and states \(q_1, ..., q_n\) of both automata such that \(t = C[t_1, ..., t_n]\), \(t' = C[t'_1, ..., t'_n]\) and for all \(i, t_i \rightarrow^*_A q_i\) and \(t'_i \rightarrow^*_A q_i\).

We write \(L(A_1, A_2)\) the relation recognized by the GTT \((A_1, A_2)\), i.e. the set of pairs of trees recognized by this GTT.

Exercise 5:

Let \(\mathcal{F} = \{\times(2), +\times(2), 0(0), 1(0)\}\) and \(R = \{0 \times x \rightarrow 0\}\). Prove that \(\rightarrow^*_R = \{(t, t') \mid t \rightarrow^*_R t'\}\) is recognizable by a GTT.

Solution:

See example 3.2.5 of TATA.

Exercise 6:

Prove that GTT is closed by inverse i.e. if \(R\) is recognized by a GTT then we can effectively construct a GTT recognizing \(R^{-1} = \{(t', t) \mid (t, t') \in R\}\).

Solution:

If \(R\) is recognized by \((A_1, A_2)\) then \(R^{-1}\) is recognized by \((A_2, A_1)\).

Exercise 7:

Prove that GTT \(\subset\) Rec.

Solution:

See proposition 3.2.7 of TATA.

Exercise 8:

Prove that GTT is closed by composition i.e. if \(R_1\) and \(R_2\) are recognized by GTT, then we can effectively construct a GTT recognizing \(R_2 \circ R_1 = \{(t, t'') \mid \exists t', (t, t') \in R_1 \land (t', t'') \in R_2\}\).
Solution:
See proposition 3.2.16 of TATA.

Remark. Unlike Rec, GTT is also closed by iteration i.e. if $R$ is recognized by a GTT, then we can effectively construct a GTT recognizing $R^* = \{(t, t') \mid \exists t_1, \ldots, t_n, t_1 = t \land t_n = t' \land \forall i, (t_i, t_{i+1}) \in R\} = \bigcup_{n \in \mathbb{N}} R^n$ (see Theorem 3.2.14 of TATA).

Exercise 9:
Let $R$ be a finite left-linear and right-ground rewrite system.
1) Prove that $\rightarrow_R^*$ is recognizable by a GTT.
2) Deduce that (First Order Reachability):
   Instance: $R$ a finite left-linear and right-ground rewrite system, $u, v$ two ground terms
   Question: $u \rightarrow_R^* v$?
   is decidable.
3) Deduce from 1 that (Confluence):
   Instance: $R$ a finite left-linear and right-ground rewrite system
   Question: $R$ is confluent, i.e. for all ground terms $u, v, w$ such that $u \rightarrow_R^* w \rightarrow_R^* v$, there exists a ground term $z$ such that $u \rightarrow_R^* z \rightarrow_R^* v$?
   is decidable.
   hint: use (and prove) the decidability of (Inclusion):
   Instance: $L, L'$ two relations recognized by GTT
   Question: $L \subseteq L'$?

Solution:
1) Let us do it for $R = \{s \rightarrow t\}$, the general case can be easily deduced.
   Let $A_1$ be the NFTA with $Q = \{q_T\} \cup Pos(s)$ and $\Delta =$
   $\ast f(q_T, \ldots, q_T) \rightarrow q_T$ for all $f$
   $\ast f(p_1, \ldots, p_n) \rightarrow p$ for all $p \in Pos(s)$ such that $s(p)$ is a variable and all $f$
   Let $A_2$ be the NFTA with $Q' = Pos(t)$ and $\Delta' =$
   $\ast f(p_1, \ldots, p_n) \rightarrow p$ for all $p \in Pos(t)$ such that $s(p) = f$ of arity $n$
   Then the GTT $(A_1, A_2)$ recognizes a relation $R'$ such that $R'^* \rightarrow_R^*$. Thus, by stability by iteration, $\rightarrow_R^* \in GTT$.
2) As $\rightarrow_R^* \in GTT \subset Rec$ and use the membership problem in Rec.
3) First, $L \subseteq L' \Leftrightarrow L \cap (T(F) \setminus L') = \emptyset$, so (Inclusion) is just a problem of emptiness which is decidable in $GTT \subset Rec$. Then, $R$ confluent is equivalent to $\rightarrow_R^* \subseteq \rightarrow_R^* \circ \rightarrow_R^*$ which are both GTT by 1 and exercise 2 and 4.