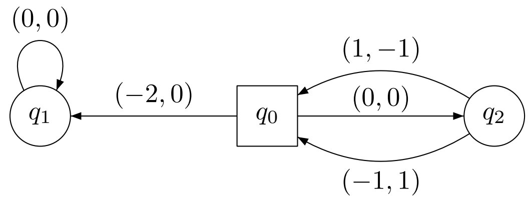
2.9 Verification of dynamic and parameterized systems 2011

Lecture 6: Multi-dimensional energy games Laurent Doyen

6.1. Multi-dimensional energy games. A *d-dimensional energy game* consists of a game graph $\langle Q, E \rangle$ (with $E \subseteq Q \times Q$), and a weight function $w : E \to \mathbb{Z}^d$ (with $d \geq 1$). The *energy level vector* of a play $\rho = q_0\, q_1\, q_2 \dots$ at position $k \geq 0$ is $\mathsf{EL}(\rho, k) = \Sigma_{i=0}^{k-1} w(q_i, q_{i+1})$.

A strategy σ_1 for player 1 is winning from state q with initial credit $c_0 \in \mathbb{N}^d$ for the *multi-energy* objective if for all strategies σ_2 the outcome ρ of σ_1, σ_2 from q is such that $c_0 + \mathsf{EL}(\rho, k) \in \mathbb{N}^d$ for all $k \geq 0$.



The *unknown initial credit problem* is to decide, given a multi-dimensional energy game and an initial state q whether there exists an initial credit and a winning strategy for player 1 for the multi-energy objective.

6.2. Strategy complexity. To win multi-energy games, it is sufficient for player 1 to play *finite-memory strategies*, while for player 2 it is sufficient to play *memoryless strategies*.

Theorem 6A For all multi-energy games, the answer to the unknown initial credit problem is YES if and only if there exists an initial credit $c_0 \in \mathbb{N}^d$ and a *finite-memory* winning strategy σ_1 for player 1 for the multi-energy objective; the answer to the unknown initial credit problem is NO if and only if there exists a *memoryless* strategy σ_2 for player 2, such that for all initial credit vectors $c_0 \in \mathbb{N}^k$ and all strategies σ_1 for player 1 we have $c_0 + \mathsf{EL}(\rho, k) \notin \mathbb{N}^d$ for some $k \geq 0$.

We prove the claim for player 1. One direction is trivial. For the other direction, assume that σ_1 is a (not necessary finite-memory) winning strategy for player 1 in G with initial credit $c_0 \in \mathbb{N}^d$. We construct from σ_1 a finite-memory strategy σ_1^{FM} that is winning against all strategies of player 2 for initial credit c_0. Consider the unfolding of the game graph G in which player 1 plays according to σ_1. This infinite tree, noted $T_{G_{\sigma_1}}$, has as set of nodes all the prefixes of plays in G compatible with σ_1. We associate to each node $\pi = q_0 q_1 \dots q_n$ in this tree the energy vector $c_0 + \mathsf{EL}(\pi, n)$. Since σ_1 is winning, we have that $c_0 + \mathsf{EL}(\pi, n) \in \mathbb{N}^k$. Now, consider the set $Q \times \mathbb{N}^k$, and the relation $\sqsubseteq$ on this set defined as follows: $(q_1, v_1) \sqsubseteq (q_2, v_2)$ if $q_1 = q_2$ and $v_1 \leq v_2$ (i.e., $v_1(i) \leq v_2(i)$ for all $i = 1, \dots, k$). The relation $\sqsubseteq$ is a well quasi order. As a consequence, on every infinite branch $\rho = q_0 q_1 \dots q_n \dots$ of $T_{G_{\sigma_1}}$, there exist two positions $i < j$ such that $q_i = q_j$ and $c_0 + \mathsf{EL}(\rho, i) \leq c_0 + \mathsf{EL}(\rho, j)$. We say that node j *subsumes* node i. Now, let $T^{\mathsf{FM}}_{G_{\sigma_1}}$ be the tree $T_{G_{\sigma_1}}$ where we delete the subtree below each node n_2 that subsumes one of its ancestor node n_1. Clearly, $T^{\mathsf{FM}}_{G_{\sigma_1}}$ is finite. Also, it is easy to see that player 1 can play in the node n_2 as she plays from the n_1 because its energy level in n_2 is greater than in n_1. From $T^{\mathsf{FM}}_{G_{\sigma_1}}$, we can construct a finite-state machine that encodes a winning strategy σ_1^{FM} in the generalized energy game G with initial energy level c_0. ■

The argument to show that memoryless strategies are sufficient for player 2 is similar to the proof in Section 3.3. Note that the initial credit vector may be different when player 2 uses an arbitrary or a memoryless strategy. However, if a finite initial credit vector is sufficient for player 1 against memoryless strategies of player 2, then there exists a (possibly larger) finite initial credit vector for player 1 to win against arbitrary strategies of player 2. In the example of Section 6.1, the initial credit $(2,0)$ in q_0 is sufficient when player 2 is memoryless, while if player 2 can use arbitrary strategies, then player 1 needs initial credit at least $(2,1)$ or at least $(3,0)$.

6.3. Computational complexity. We show that the unknown initial credit problem for multi-energy games is coNP-complete.

coNP upper bound. First, we need the following result about finding zero circuits in multi-weighted directed graphs (a graph is a one-player game). A zero circuit is a finite sequence $q_0 q_1 \dots q_n$ $(n > 0)$ such that $q_0 = q_n$, $(q_i, q_{i+1}) \in E$ for all $0 \leq i < n$, and $\sum_{i=0}^{n-1} w(q_i, q_{i+1}) = (0, 0, \dots, 0)$. The circuit need not be simple. There is a polynomial-time algorithm to decide if a d-dimensional directed graph contains a zero circuit. See [Kosaraju,Sullivan 1988].

We show that the unknown initial credit problem for multi-energy games is in coNP. Since player 2 can be restricted to play memoryless strategies, a coNP algorithm is to guess a memoryless strategy σ_2 for player 2 and check in polynomial time that player 1 is not winning using the following argument.

Consider the graph $G_{\sigma_2} = \langle Q, E' \rangle$ (where $E' = \{(q, q') \in E \mid q \in Q_1 \vee (q \in Q_2 \wedge q' = \sigma_2(q))\}$) as a one-player game (in which all states belong to player 1). We show that if there exists an initial energy level $c_0 \in \mathbb{N}^d$ and an infinite play $\rho = q_0 q_1 \dots q_n \dots$ in G_{σ_2} such that ρ satisfies the multi-energy objective with initial credit vector c_0 (i.e., player 1 wins), then there exist a reachable circuit in G_{σ_2} that has nonnegative effect in all dimensions (and the converse holds trivially). To show this, we extend ρ with the energy information as follows: $\rho' = (q_0, v_0)(q_1, v_1) \dots (q_n, v_n) \dots$ where $v_0 = c_0$ and for all $i \geq 1$, $v_i = v_{i-1} + w(q_{i-1}, q_i)$. As ρ satisfies the multi-energy objective, we know that $v_i \in \mathbb{N}^d$ for all $i \geq 0$. Define the following well-quasi order on the set $Q \times \mathbb{N}^d$: $(q, v) \sqsubseteq (q', v')$ if $q = q'$ and $v(j) \leq v'(j)$ for all $1 \leq j \leq d$. By definition of wqo, there exist two positions $i_1 < i_2$ in ρ' such that $(q_{i_1}, v_{i_1}) \sqsubseteq (q_{i_2}, v_{i_2})$. Hence the path from q_{i_1} to q_{i_2} in ρ is a circuit with nonnegative effect in all dimensions.

Based on this, it is sufficient to decide in polynomial time if the graph G_{σ_2} has a reachable circuit with nonnegative effect in all dimensions. This problem can be reduced to decide the existence of a reachable zero circuit in a graph G'_{σ_2} obtained from G_{σ_2} as follows. In every state of G_{σ_2}, we add d self-loops with respective multi-weight $(-1, 0, \dots, 0)$, $(0, -1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, -1)$, i.e. each self-loop removes one unit of energy in one dimension. It is easy to see that G_{σ_2} has a circuit with nonnegative effect in all dimensions if and only if G'_{σ_2} has a zero circuit, which can be solved in polynomial time using he algorithm of Kosaraju & Sullivan. The result follows.

coNP lower bound. We show that the unknown initial credit problem for multi-weighted two-player game structures is coNP-hard. We present a reduction from the complement of the 3SAT problem which is NP-complete.

We show that the problem of deciding whether player 1 has a winning strategy for the unknown initial credit problem for multi-energy games is at least as hard as deciding whether a 3SAT formula is unsatisfiable. Consider a 3SAT formula ψ in CNF with clauses $C_1, C_2, \dots, C_k$ over variables $\{x_1, x_2, \dots, x_n\}$, where each clause consists of disjunctions of exactly three literals (a literal is a variable x_i or its negation $\overline{x}_i$). Given the formula ψ, we construct a game graph as shown in Figure 1. The game graph is as follows: from the initial state, player 1 chooses a clause, then from a clause player 2 chooses a literal that appears in the clause (i.e., a literal that makes the clause true). From every literal the next state is the initial state. We now describe the weight vectors assigned to each edge. The dimension of the weights is $d = 2n$, i.e. there is a dimension for each variable and its negation. For edges from the initial state to the clauses, and from the clauses to the literals, the weight in every dimension is 0. For the edges from a literal y to the initial state, the weight in the dimension of y is 1, the weight in the dimension of the complement of y is -1, and for all other dimensions the weight is 0. For example, the weight is $(1, -1, 0, \dots, 0)$ from x_1, and $(-1, 1, 0, \dots, 0)$ from $\overline{x}_1$.

The correctness of this reduction is established as follows. We consider *assignments* that map truth values to each literal. An assignment is *valid* if for every literal the truth value assigned to the literal and its

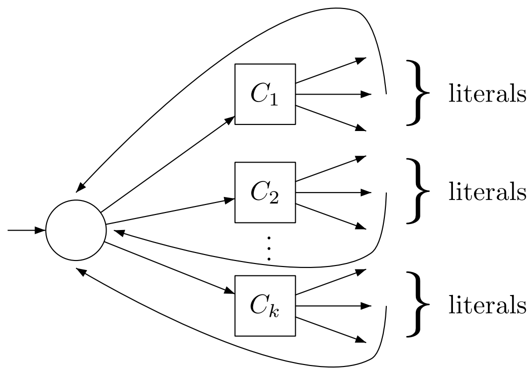


Figure 1: Game graph construction for a 3SAT formula.

complement are complementary: for all $1 \leq i \leq n$, if x_i is assigned true (resp., false), then the complement of x_i is assigned false (resp., true). A non-valid assignment is called *conflicting* (i.e., for some $1 \leq i \leq n$, both x_i and $\overline{x}_i$ are assigned the same truth value). If the formula ψ is satisfiable, then there is a valid assignment that satisfies all the clauses. If the formula ψ is not satisfiable, then every assignment that satisfies all the clauses must be conflicting. We now present the two directions of the hardness proof.

ψ satisfiable implies player 2 winning. We show that if ψ is satisfiable, then player 2 has a winning strategy. Since ψ is satisfiable, there is a valid assignment that satisfies every clause. The memoryless strategy is constructed from the assignment as follows: for a clause C_i, the strategy chooses a literal as successor that appears in C_i and is set to true by the assignment. Consider the outcome of an arbitrary strategy for player 1: the literals visited in the play are all true according to the assignment, and the infinite play must visit some literal infinitely often. Consider a literal x that occurs infinitely often in the play, then the complement literal $\neg x$ is never visited, and for each visit to the literal x, the dimension corresponding to $\neg x$ decreases by 1. Therefore, the energy in this dimension is never incremented, and it is decremented infinitely many times. It follows that the play is winning for player 2 for every finite initial credit. It follows that the strategy for player 2 is winning, and the answer to the unknown initial credit problem is "No".

ψ not satisfiable implies player 1 is winning. We now show that if ψ is not satisfiable, then player 1 is winning. By determinacy, it suffices to show that player 2 is not winning, and by existence of memoryless winning strategy for player 2 (**Theorem 6A**), it suffices to show that there is no memoryless winning strategy for player 2. Fix an arbitrary memoryless strategy for player 2, (i.e., in every clause player 2 chooses a literal that appears in the clause). If we consider the assignment naturally derived from the memoryless strategy, then since ψ is not satisfiable it follows that the assignment is conflicting. Hence there must exist clauses C_i and C_j, and variable x such that the strategy of player 2 chooses the literal x in C_i and the complement literal $\overline{x}$ in C_j. The strategy for player 1 that alternates between clause C_i and C_j from the initial state is winning with initial credit of 1 in the dimension of x and $\overline{x}$, and 0 in the other dimensions. Hence the answer to the unknown initial credit problem is "Yes".

Theorem 6B The unknown initial credit problem for multi-energy games is coNP-complete.