5.1. Energy parity games. An *energy parity game* consists of a game graph $\langle Q, E\rangle$ (where Q is a finite set and $E \subseteq Q \times Q$), a weight function $w : E \to \mathbb{Z}$, and a priority function $p : Q \to \{0, 1, \dots, d\}$.

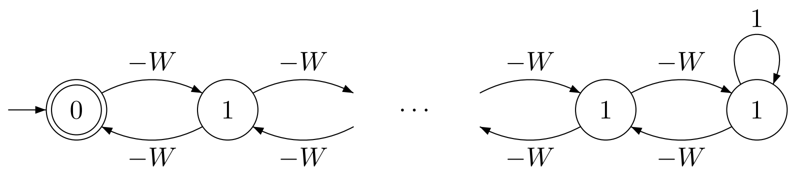
A strategy σ_1 for player 1 is winning from state q with initial credit c_0 for the *energy parity* objective if for all strategies σ_2 the play ρ produced from q by σ_1, σ_2 is such that

$$\min\{p(q) \mid q \in \mathsf{Inf}(\rho)\} \text{ is even, and } c_0 + \mathsf{EL}(\rho, k) \geq 0 \text{ for all } k \geq 0.$$

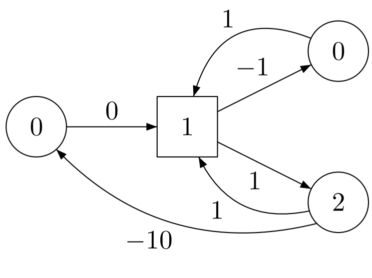
where $\mathsf{Inf}(q_0 q_1 \dots) = \{q \in Q \mid \forall i \geq 0 \cdot \exists j \geq i : q_i = q\}$ is the set of states visited infinitely often in ρ.

The special case of a priority function $p : Q \to \{0, 1\}$ defines energy Büchi games, and the special case where $p : Q \to \{1, 2\}$ defines energy coBüchi games.

5.2. Example. In the energy Büchi game below, player-1 needs memory of size $2 \cdot (n-1) \cdot W + 1$ and initial credit of $(n-1) \cdot W$ to win. To satisfy the parity (here reduced to Büchi) condition, the play has to visit the initial state infinitely often, and to maintain the energy positive, the play has to visit the state with the positive-weighted self-loop. Since the paths between these two state have weight $-(n-1) \cdot W$, the initial credit $(n-1) \cdot W$ is necessary, and the self-loop has to be taken $M = 2 \cdot (n-1) \cdot W$ times, which requires memory of size $M + 1$. Note the M is exponential if the weights are encoded in binary.



5.3. Strategy complexity. The previous example suggests that winning strategies may need to alternate between recharging energy, and visiting low even priorities. In the recharging phase, if the energy remains at the same level (through a loop of weight 0) then the least priority visited must be even.



A strategy σ for player 1 is *good-for-energy* in state q if for all outcomes $\rho = q_0 q_1 \dots$ of σ such that $q_0 = q$, for all cycles $\gamma = q_i \dots q_{i+k}$ in ρ (where $k > 0$ and $q_i = q_{i+k}$), either $\mathsf{EL}(\gamma, k) > 0$, or $\mathsf{EL}(\gamma, k) = 0$ and γ is even (i.e., $\min\{p(q) \mid q \in \gamma\}$ is even).

Lemma 5A Let Win be the set of winning states for player 1 in an energy parity game. Then, there exists a memoryless strategy for player 1 which is good-for-energy in every state $q \in \mathsf{Win}$.

To see this, note that a good-for-energy strategy in q can be viewed as a winning strategy in a finite cycle-forming game from q where the game stops when a cycle C is formed, and the winner is determined by the sequence of states in C (and is independent of cyclic permutations). In such finite cycle-forming game both players have memoryless optimal strategies.

Now, assume that player 1 wins an energy parity game from a state q. Towards contradiction, assume that player 1 has no good-for-energy strategy from q. Then, player 2 would have a memoryless winning strategy in the finite cycle-forming game. Fix this strategy in the original energy parity game and then all cycles have either negative weight, or weight is zero and the least priority is odd. It follows that player 1 looses the energy parity game from q (no matter the value of the initial credit), a contradiction. Hence, player 1 has a memoryless good-for-energy strategy σ_q from q. ■

A strategy uses *finite-memory* if it can be encoded by a deterministic transducer $\langle M, m_0, \alpha_u, \alpha_n \rangle$ where M is a finite set (the memory of the strategy), $m_0 \in M$ is the initial memory value, $\alpha_u : M \times Q \to M$ is an update function, and $\alpha_n : M \times Q_1 \to Q$ is a next-move function. The *size* of the strategy is the number $|M|$ of memory values. If the game is in a player-1 state q and m is the current memory value, then the strategy chooses $q' = \alpha_n(m, q)$ as the next state and the memory is updated to $\alpha_u(m, q)$. Formally, $\langle M, m_0, \alpha_u, \alpha_n \rangle$ defines the strategy σ such that $\alpha(\rho \cdot q) = \alpha_n(\hat{\alpha}_u(m_0, \rho), q)$ for all $\rho \in Q^*$ and $q \in Q_1$, where $\hat{\alpha}_u$ extends α_u to sequences of states as expected. A strategy is *memoryless* if $|M| = 1$.

Theorem 5B For all energy parity games G with n states and d priorities, if player 1 wins from a state q_0, then player 1 has a winning strategy from q_0 with memory of size $n \cdot d \cdot W$ and initial credit $(n-1) \cdot W$.

We prove by induction a slightly stronger statement, namely that player 1 has a winning strategy with memory of size $n \cdot d \cdot W$, where $n = |Q|$, and such that all its outcomes with initial credit $x \geq (n-1) \cdot W$ have energy level always at least $x - (n-1) \cdot W$ (and this strategy is winning from every state where player 1 wins in G, thus including q_0).

For the case of games with $d = 1$ priority, either the priority is odd and all states are loosing for player 1 (hence, the result holds trivially), or the priority is even and the energy parity game reduces to an energy game which can be won by player 1 with a memoryless strategy and initial credit $(n-1) \cdot W$ from every winning state. If the initial credit is $x \geq (n-1) \cdot W$, then the same strategy ensures that the energy level is always at least $x - (n-1) \cdot W$.

By induction, assume that the statement holds for all energy parity games G with $d-1$ priorities. Consider a winning state q_0 in an energy parity game G with d priorities. By Lemma 5A, player 1 has a memoryless strategy σ_{gfe} which is good-for-energy from every winning state of G. We consider two cases.

A. If the least priority in G is even (say it is 0). Let Win be the set of winning states for player 1 in G (thus $q_0 \in \mathsf{Win}$), and let Ω_0 be the player-1 attractor of priority-0 states in the subgraph of G induced by Win. We construct a winning strategy as follows (for clarity, we call the initial credit x though the strategy definition is independent of the value of x):

(1) play σ_{gfe} until the energy level has increased by $\Delta = (n-1) \cdot W$ (i.e., the energy level has reached $x + \Delta$) and proceed to (2) with energy level $x' \geq x + \Delta$, or play σ_{gfe} forever if the energy level never reaches $x + \Delta$;

(2) (a) if the current state of the game is not in Ω_0, then play a winning strategy in the subgame induced by $\mathsf{Win} \setminus \Omega_0$ (which has at most $d-1$ priorities) and such that the energy level never drops below $x' - (n-k-1) \cdot W$ where $k = |\Omega_0|$ (such a strategy exists by the induction hypothesis); (b) whenever the game reaches Ω_0, then play a memoryless strategy to reach a priority-0 state (this may decrease the energy level by $k \cdot W$), and proceed to (1) with energy level at least $x' - (n-k-1) \cdot W - k \cdot W = x' - (n-1) \cdot W \geq x$;

We show that this strategy is winning in G from every state in Win. First, we show that the energy level never drops below $x - (n-1) \cdot W \geq 0$ if the initial credit is $x \geq (n-1) \cdot W$ (and thus in particular never drops below 0). In phase (1), the energy level is always at least $x - (n-1) \cdot W \geq 0$ since σ_{gfe} is memoryless and good-for-energy. If the strategy switches to phase (2), then we have already proved that the energy never drops below $x' - (n-1) \cdot W \geq x$. Therefore, whenever the strategy switches back to phase (1), the energy level has not decreased (i.e., it is at least x), and the argument can be repeated. Second, we show that the parity condition is satisfied. We consider three possible cases: (i) if phases (1) and (2) are played infinitely often, then priority 0 is visited infinitely often and the parity condition is satisfied; (ii) if phase (1) is played finitely often, then eventually phase (2) is played forever, which means that we play a winning strategy in the subgame induced by $\mathsf{Win} \setminus \Omega_0$. Therefore, by induction hypothesis the parity condition is satisfied in the game (since the parity objective is independent of finite prefixes); (iii) if phase (2) is played finitely often, then eventually phase (1) is played forever, which implies that eventually all visited cycles have weight 0, which entails that their least priority is even (by definition of good-for-energy strategies), hence so is the least priority visited infinitely often.

Now, we analyze the amount of memory needed by this strategy. In this analysis, we denote by $M(d,n)$ the size of the memory needed by our winning strategy in game G with n states and d priorities. In phase (1), we need to remember the energy level variation, which is between $-(n-1) \cdot W$ and $(n-1) \cdot W$, thus can be done with memory of size at most $(2n-1) \cdot W$. In phase (2), the subgame strategy has memory size bounded by $M(d-1,n)$, and the attractor strategy is memoryless. Hence, the size of the memory needed is at most $M(d,n) \leq (2n-1) \cdot W + 1 + M(d-1,n)$.

B. If the least priority in G is odd (say it is 1). Let Win be the set of winning states for player 1 in G (thus $q_0 \in \mathsf{Win}$), and let Ω_1 be the player-2 attractor of priority-1 states in the subgraph of G induced by Win.

We claim that the set Win' of states in the subgame G' induced by $\mathsf{Win} \setminus \Omega_1$ that are winning (for energy parity objective) is nonempty, and player 1 is winning in the subgame induced by $\mathsf{Win} \setminus \mathsf{Attr}_1^*(\mathsf{Win}')$. Towards contradiction, assume that player 2 has a (memoryless) spoiling strategy π on $\mathsf{Win} \setminus \Omega_1$ in the energy parity game $\langle G', w, p \rangle$, and consider the (memoryless) extension of π to Ω_1 that enforces to reach priority-1 states. Let ρ be an outcome of this strategy. Either ρ visits Ω_1 (and also priority-1 states) infinitely often and thus violates the parity condition, or ρ eventually stays in $\mathsf{Win} \setminus \Omega_1$ and thus violates the energy parity condition since it is an outcome of the spoiling strategy π. This contradicts that player 1 wins in G from Win.

We construct a winning strategy on $\mathsf{Attr}_1^*(\mathsf{Win}')$ as follows (for clarity, we call the initial credit x though the strategy definition is independent of the value of x):

(1) play a memoryless strategy to reach Win' (let Δ_1 be the maximal drop of energy cost), and proceed to (2) with energy level $x' = x - \Delta_1$;

(2) in Win', play a winning strategy in the subgame induced by Win' (which has at most $d-1$ priorities) and such that the energy level never drops below $x' - \Delta_2$, where $\Delta_2 = |\mathsf{Win}'| \cdot W$ (such a strategy exists by induction hypothesis); note that $\Delta_1 + \Delta_2 \leq |\mathsf{Attr}_1^*(\mathsf{Win}')| \cdot W$.

We apply the same construction recursively to the subgame induced by $\mathsf{Win} \setminus \mathsf{Attr}_1^*(\mathsf{Win}')$, and call the corresponding energy drops Δ_3, Δ_4, etc.

We show that this strategy is winning in G from every state in Win. First, the sum $\Delta_1 + \Delta_2 + \ldots$ of energy drops is bounded by $|\mathsf{Win}| \cdot W$ and thus initial credit of $(n-1) \cdot W$ is enough. Second, the parity condition is satisfied since we eventually play a winning strategy in a subgame where priorities are greater than 1, and without visiting priority 1. The amount of memory needed is the sum of the memory size of the strategies in the sets Win' (of size k_1, k_2, etc.), and of the memoryless attractor strategy (of size 1), hence at most $M(d,n) \leq M(d-1,k_1) + \cdots + M(d-1,k_m) + 1$ where $k_1 + \cdots + k_m < n$.

Combining the recurrence relations obtained in **A** and **B** for $M(d,n)$, we verify that $M^{\mathsf{even}}(d,n) \leq 2 \cdot n \cdot W + n \cdot (d-2) \cdot W \leq n \cdot d \cdot W$ when the smallest priority is even, and $M^{\mathsf{odd}}(d,n) \leq n \cdot (d-1) \cdot W$ when d the smallest priority is odd. Hence $M(d,n) \leq n \cdot d \cdot W$. ■

It follows from the proof of **Theorem 5B** that memoryless strategies are sufficient for player 1 to win energy coBüchi games (why?), and that alternation between attractor strategy and good-for-energy strategy is sufficient for energy Büchi games.

5.4. Computational complexity.

By arguments similar to energy games (Section 3.3), memoryless strategies are sufficient for player 2. Hence, a coNP algorithm guesses a memoryless player-2 strategy and checks in polynomial time that in the graph induced by this strategy, there exists a strongly connected component with smallest priority i such that i is even, and with either a strictly positive cycle, or a zero cycle through a state with priority i (such cycle detection problem can be encoded in linear programming).

Theorem 5C The problem of deciding the existence of a finite initial credit for energy parity games is in NP $\cap$ coNP.

We present an NP algorithm for the problem that guesses the set of winning states in G. The result holds for energy parity games with one priority (i.e., simple energy games) by guessing a memoryless winning strategy. Assume by induction that the result holds for games with less than d priorities, and let G be an energy parity game with d priorities and n states.

First, if the least priority in G is even (assume w.l.o.g. that the least priority is 0), an NP algorithm guesses (i) the set Win of winning states in G, and (ii) a memoryless good-for-energy strategy σ_{gfe} on Win. Let Ω_0 be the player-1 attractor of priority-0 states in the subgraph of G induced by Win. By induction, we can check in NP that player 1 is winning in the subgraph of G induced by $\mathsf{Win} \setminus \Omega_0$ (because this game has less than d priorities). This is sufficient to establish that player 1 wins in G with initial credit $n \cdot W$, using the following strategy: (1) play strategy σ_{gfe} as long as the energy level is below $2 \cdot n \cdot W$; (2) while the game is in $\mathsf{Win} \setminus \Omega_0$, we know that player 1 can play a winning strategy that needs initial credit at most $(n-k) \cdot W$ where $k = |\Omega_0|$ and such that the energy level drops by at most $(n-k-1) \cdot W$, and therefore (3) if the game leaves $\mathsf{Win} \setminus \Omega_0$, then the energy level is at least $2n - (n-k-1) \cdot W = (n+k+1) \cdot W$ which is enough for player 1 to survive while enforcing a visit to a priority-0 state (within at most $|\Omega_0| = k$ steps) and to proceed to step (1) with energy level at least $n \cdot W$. This strategy is winning with initial credit $n \cdot W$. The time complexity of this algorithm is $T(n) = p(n) + T(n-1)$ where $p(\cdot)$ is a polynomial (linear) function for the time complexity of guessing Win and σ_{gfe}, checking that σ_{gfe} is good-for-energy, and computing the the player-1 attractor of priority-0 states Ω_0. Therefore $T(n) = O(n^2)$.

Second, if the least priority in G is odd (assume w.l.o.g. that the least priority is 1), consider the set Win of winning states in G, and Ω_1 the player 2 attractor of priority-1 states in the subgame of G induced by Win. The set Win' of states in the subgame induced by $\mathsf{Win} \setminus \Omega_1$ that are winning (for energy parity objective) is nonempty, and player 1 is winning in the subgame induced by $\mathsf{Win} \setminus \mathsf{Attr}_1^*(\mathsf{Win}')$. An NP algorithm guesses the sets Win and Win', and checks that player 1 is winning in Win' (which can be done in NP, since $\mathsf{Win} \setminus \Omega_1$ has less than d priorities), and that player 1 is winning in $\mathsf{Win} \setminus \mathsf{Attr}_1^*(\mathsf{Win}')$ which can be done in NP, as shown by an induction proof on the number of states in the game since the case of games with one state is clearly solvable in NP.

5.5. Algorithm.

We assume without loss of generality that the least priority in the input game graph is either 0 or 1; if not, then we can reduce the priority in every state by 2. The algorithm considers two cases: (a) when the minimum priority is 0, and (b) when the minimum priority is 1. The details of the two cases are given below. The algorithm is based on a procedure to construct memoryless good-for-energy strategies. To obtain a good-for-energy strategy, we modify the weights in the game so that every simple cycle with (original) sum of weight 0 gets a strictly positive weight if it is even, and a strictly negative weight if it is odd. Formally, the new weight function w' is defined by $w'(q, q') = w(q, q') + \Delta(q)$ where $\Delta(q) = (-1)^k \cdot \frac{1}{(n+1)^{k+1}}$ for all $(q, q') \in E$ where $k = p(q)$ is the priority of q, and $n = |Q|$. Winning strategies in the energy game with modified weights w' correspond to good-for-energy strategies in the original game. By scaling the weights with a factor $|Q|^{d+1}$, the problem of deciding the existence of a memoryless good-for-energy strategy in energy parity games can be solved in time $O(|E| \cdot |Q|^{d+2} \cdot W)$.

(a) If the least priority in the game is 0, then we compute the winning states of Player 1 as the limit of a decreasing sequence $A_0, A_1, \ldots$ of sets. Each iteration removes from A_i some states that are winning for Player 2. The set $A'_i \subseteq A_i$ contains the states having a good-for-energy strategy (line 8) which is a necessary condition to win, according to Lemma 5A. We decompose A'_i into X_i and $A'_i \setminus X_i$, where X_i is the set of states from which Player 1 can force a visit to priority-0 states, and $A'_i \setminus X_i$ has less priorities than A'_i. The winning states Z_i in $A'_i \setminus X_i$ for Player 2 are also winning in the original game (as in $A'_i \setminus X_i$ Player 1 has no edge going out of $A'_i \setminus X_i$). Therefore we remove Z_i and Player-2 attractor

Algorithm 1: SolveEnergyParityGame

Input : An energy parity game $\langle G, p, w \rangle$ with state space Q.

Output: The set of winning states in $\langle G, p, w \rangle$ for player 1.

begin
- **if** $Q = \emptyset$ **then return** $\emptyset$;
- Let k^* be the minimal priority in G. Assume w.l.o.g. that $k^* \in \{0, 1\}$;
- Let G_0 be the game G ;
- $i \leftarrow 0$;
- **if** $k^* = 0$ **then**
  - $A_0 \leftarrow Q$ `/* over-approximation of Player-1 winning states */` ;
  - **repeat**
    - $A'_i \leftarrow \mathsf{SolveEnergyGame}(G_i, w')$ (where w' is defined in Section 5.5) ;
    - $X_i \leftarrow \mathsf{Attr}_1^*(A'_i \cap p^{-1}(0))$;
    - Let G'_i be the subgraph of G_i induced by $A'_i \setminus X_i$;
    - $Z_i \leftarrow (A'_i \setminus X_i) \setminus \mathsf{SolveEnergyParityGame}(G'_i, p, w)$;
    - $A_{i+1} \leftarrow A'_i \setminus \mathsf{Attr}_2^*(Z_i)$;
    - Let G_{i+1} be the subgraph of G_i induced by A_{i+1} ;
    - $i \leftarrow i + 1$;
  - **until** $A_i = A_{i-1}$;
  - **return** A_i;
- **if** $k^* = 1$ **then**
  - $B_0 \leftarrow Q$ `/* over-approximation of Player-2 winning states */` ;
  - **repeat**
    - $Y_i \leftarrow \mathsf{Attr}_2^*(B_i \cap p^{-1}(1))$;
    - Let G_{i+1} be the subgraph of G_i induced by $B_i \setminus Y_i$;
    - $B_{i+1} \leftarrow B_i \setminus \mathsf{Attr}_1^*(\mathsf{SolveEnergyParityGame}(G_{i+1}, p, w))$;
    - $i \leftarrow i + 1$;
  - **until** $B_i = B_{i-1}$;
  - **return** $Q \setminus B_i$;

end

to Z_i in A_{i+1}. The correctness argument for this case is similar to the proof of Theorem 5C, namely that when $A_i = A'_i = A_{i-1}$, Player 1 wins by playing a winning strategy in $A'_i \setminus X_i$ (which exists by an inductive argument on the number of recursive calls of the algorithm), and whenever the game enters X_i, then Player 1 can survive while forcing a visit to a priority-0 state, and then uses a good-for-energy strategy to recover enough energy to proceed.

(b) The second part of the algorithm (when the least priority in the game is 1) computes a decreasing sequence $B_0, B_1, \ldots$ of sets containing the winning states of Player 2. The correctness is proven in a symmetric way using the same argument as in the second part of the proof of Theorem 5C.

We obtain the following result, where d is the number of priorities in the game, and W is the largest weight.

Theorem 5D The problem of deciding the existence of a finite initial credit for energy parity games can be solved in time $O(|E| \cdot d \cdot |Q|^{d+3} \cdot W)$.