Expressing theories in the $\lambda \Pi$-calculus modulo theory and in the Dedukti system

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Predicate logic

(Peano) arithmetic, (Euclidean) geometry, (Zermelo) set theory...

Theories in Predicate logic (Hilbert and Ackermann, 1928)

A logical framework where formalisms can be defined as theories

- $\land$, $\lor$, $\forall$... defined once for all
- proof, model... defined once for all
- soundness, completeness... proved once for all
- $Z \subseteq ZF \subseteq ZFC$
- if $\mathcal{T} \vdash A \Rightarrow B$ and $\mathcal{T}' \vdash A$, then $\mathcal{T} \cup \mathcal{T}' \vdash B$
But...

The Theory of classes (aka Second-order logic)
Simple type theory (aka Higher-order logic)
The Calculus of constructions
The Calculus of inductive constructions

... not theories expressed in Predicate logic
A Babel tower

Before: a proof of xyz (rarely: using the axiom of choice)
Now: a Coq proof of the four color theorem”, “an Isabelle/HOL proof of the correctness of seL4”

A proof of $A$ in $S$ cannot be used in $S'$

A proof of $A$ in $S$, a proof of $A \implies B$ in $S'$, a proof of $B$ in nothing
Five limitations of Predicate logic

1. No bound variables (except $\forall$, $\exists$), no function symbol $\mapsto$ →
2. No proofs-as-terms principle
3. No computation: a proof of $2 + 2 = 4$
4. No theory-independent cut-elimination theorem
5. No constructive proofs
Partial solutions: more logical frameworks

1. \(\lambda\)-Prolog, Isabelle
2. LF, aka \(\lambda\Pi\)-calculus, aka \(\lambda\)-calculus with dependent types
3. Deduction modulo theory

Combine \(\lambda\Pi\)-calculus and Deduction modulo theory: \(\lambda\Pi\)-calculus modulo theory (variant of the Martin-Löf logical framework)
Solves 1., 2., 3., 4., and 5.

Implemented in DEDUKTI http://dedukti.gforge.inria.fr/
Simple type theory in Dedukti: 8 variables and 3 rules

\[\begin{align*}
\text{type} &: \ Type \\
\ o &: \ type \\
\ i &: \ type \\
\text{arrow} &: \ type \rightarrow type \rightarrow type \\
\eta &: \ type \rightarrow Type \\
\eta(\text{arrow} \ a \ b) &: \eta(a) \rightarrow \eta(b) \\
\Rightarrow &: \eta(o) \rightarrow \eta(o) \rightarrow \eta(o) \\
\forall &: \Pi a : type \ ((\eta(a) \rightarrow \eta(o)) \rightarrow \eta(o)) \\
\varepsilon &: \eta(o) \rightarrow Type \\
\varepsilon(\Rightarrow \ p \ q) &: \varepsilon(p) \rightarrow \varepsilon(q) \\
\varepsilon(\forall \ a \ p) &: \Pi x : \eta(a) \ \varepsilon(p \ x)
\end{align*}\]
What does “expressing a logic in a framework” mean?

Adequacy theorem (in principle)
Large library of formal proofs translated and checked (in facts)
Dedukti libraries (650 MB)

- Constructive predicate logic (Resolution proofs): The iProverModulo TPTP library (38.1 MB)
- Classical logic (tableaux proofs): The Zenon modulo Set Theory Library (595 MB)
- FoCaLiZe: The Focalide library (1.89 MB)
- Simple type theory: The Holide library (21.5 MB)
- The Calculus of constructions with universes: The Matita arithmetic library (1.11 MB)
Minimal logic in the $\lambda\Pi$-calculus

$i : Type$

for each variable $x$, $x : i$

for each function symbol $f$, $f : i \to \ldots \to i \to i$

for each predicate symbol $P$, $P : i \to \ldots \to i \to Type$

- $|x| = x$
- $|f(t_1, \ldots, t_n)| = (f \mid t_1 \mid \ldots \mid t_n)|$
- $|P(t_1, \ldots, t_n)| = (P \mid t_1 \mid \ldots \mid t_n)|$
- $|A \Rightarrow B| = |A| \to |B|$, i.e. $\Pi z : |A| \mid |B|$
- $|\forall x \ A| = \Pi x : i \mid |A|$

$A$ provable if and only if there exists $\pi$ such that $\pi : |A|$
\( o \) aka \( Prop, \) \( bool \ldots \)

\[ \iota : Type, \ o : Type \]

for each predicate symbol \( P, \ P : \iota \rightarrow \ldots \rightarrow \iota \rightarrow o \)

\( \top, \bot \) of type \( o \)

\( \Rightarrow, \ \wedge, \ \vee \) of type \( o \rightarrow o \rightarrow o \)

\( \forall, \ \exists \) of type \( (\iota \rightarrow o) \rightarrow o \)

\( o \) embedded in \( Type \) with \( \varepsilon \) of type \( o \rightarrow Type \)

Meaning defined by rewrite rules e.g.

\[ \varepsilon(\wedge \ x \ y) \rightarrow \Pi z : o ((\varepsilon(x) \rightarrow \varepsilon(y) \rightarrow \varepsilon(z)) \rightarrow \varepsilon(z)) \]
The impredicative expression of connectives and quantifiers

\[ \varepsilon(\land x\; y) \rightarrow \Pi z : o \ ( (\varepsilon(x) \rightarrow \varepsilon(y) \rightarrow \varepsilon(z)) \rightarrow \varepsilon(z)) \]

\[ \Pi z : o : \text{a quantification over all propositions} \]

But... yields a type (\( : Type \)) and not a proposition (\( : o \))

Not even in the image of the embedding \( \varepsilon \)

Propositions-as-types: \( o \sqsubseteq Type (\varepsilon) \) not \( o = Type \)
Ongoing work

More proofs: PVS (predicate subtyping), Coq (universe polymorphism: rewriting modulo AC), SMT-solvers

Reverse engineering of proofs: Half of the HOL-Light standard library is constructive \textit{a posteriori}
Can we express (part of) the Matita arithmetic library in HA?