Quantitative informational aspects in discrete physics

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Joint work with Pablo Arrighi
I. From Gandy’s hypotheses to discrete physics
Gandy’s hypotheses

Homogeneity of *time* and *space*

Bounded *velocity* of information

Bounded *density* of information
If Gandy’s hypotheses are verified

Any system (can be simulated by | is) a cellular automaton

- Each cell has a finite state space (bounded density)
- State of a cell depends on state of a finite number of cells at previous time step (bounded velocity)
- Local evolution function the same everywhere and everywhen (homogeneity)
Bounded velocity of information

\[ v \leq c \]
Bounded density of information

The amount of information that can be stored in a region of space of radius $R$ is bounded

Gandy (1980): the three hypotheses imply the physical Church-Turing thesis: any physical system (can be simulated by $\mid$ is) a cellular automaton

Bekenstein (1981): $I_{\text{max}} = \frac{1}{4 \ln(2)} \frac{c^3}{\hbar G} 4\pi R^2$  

($R^2$, not $R^3$)
Discrete physics

Model phenomena with differential equations, cellular automata.

The problem needs to be addressed phenomenon by phenomenon.
This talk

The simplest motion: constant force applied to a particle: free fall

Compute the amount of information involved

A criterion to compare theories (Newtonian physics, Special Relativity, General Relativity)
Motion in a cellular automaton

States: \( \{ q, \ldots, \} \)

All cells are quiescent (\( q \)) except one: the particle
E.g. : 1D space

Similar for 2D and 3D space
Evolution rules preserve this invariant
II. A digression: what is the Planck constant the magnitude of?
Not just a constant in some equation: the speed of *something*.

$h$: the action of *what*?

No name: Rømer constant, Einstein constant: speed of light.
The Planck constant.

Just one $c$
but $h$, $\hbar$...
Units

Often unit system such that $c = 1$, $\mathcal{G} = 1$, $h = 1$
A unique unit system for distance, time, mass

Here, just $c = 1$ and $\mathcal{G} = 1$, keep distances in m
Time and mass are distances (everything in m)

$t$ in s $\rightarrow$ $ct$ in m
$m$ in kg $\rightarrow$ $(\mathcal{G}/c^2)m$ in m

$c : m \, s^{-1}$ and $\mathcal{G}/c^2 : m \, kg^{-1}$ unit conversion constants (like 0.0254 m in$^{-1}$)
Units

velocity: \( m \text{ s}^{-1} \quad \longrightarrow \quad m^0 \) (scalar)
momentum: \( \text{kg} \ m \text{ s}^{-1} \quad \longrightarrow \quad m \)
acceleration: \( \text{m} \text{ s}^{-2} \quad \longrightarrow \quad m^{-1} \)
force: \( \text{kg} \ m \text{ s}^{-2} \quad \longrightarrow \quad m^0 \)
energy: \( \text{kg} \ m^2 \text{ s}^{-2} \quad \longrightarrow \quad m \)
action: \( \text{kg} \ m^2 \text{ s}^{-1} \quad \longrightarrow \quad m^2 \)
The Planck constant

\[ \hbar = 1.054 \times 10^{-34} \text{ m}^2 \text{ kg s}^{-1} \text{ homogeneous to m}^2 \]

The Planck constant in m²: \( a_P = \hbar \left( \frac{G}{c^2} \right) (1/c) = 2.61 \times 10^{-70} \text{ m}^2 \)

The Planck area

\[ l_{max} = \frac{1}{4 \ln(2)} \frac{c^3}{\hbar G} 4\pi R^2 = \frac{1}{4 \ln(2) a_P} 4\pi R^2 \]

\( 4 \ln(2) a_P = 7.2 \times 10^{-70} \text{ m}^2 \) is the area of one bit

Not \( h, \hbar, \) or \( a_P: 4 \ln(2) a_P: \) the area of one bit
III. Constant force in Newtonian physics
What is free fall?

Gravitational attraction

Approximated (force variation neglected: constant force)
Newton’s law

\[ f = \frac{mm'}{(R - y)^2} \]

Approximation:

\[ f = \frac{mm'}{R^2} = m'g \]

where \( g = m/R^2 \)

Earth: \( g = 1.09 \times 10^{-16} \text{ m}^{-1} \) \( (gc^2 = 9.81 \text{ m s}^{-2}) \)
Trajectory

\[ f = m'a \]

\[ a = g \]

\[ v = gt \]

\[ y = \frac{1}{2}gt^2 \]
Parabolic motion

Velocity unbounded
No bounded velocity of information

Moreover: velocity part of the internal state: no bounded density of information (even if finite precision)

No cellular automaton
IV. Constant force in Special Relativity
Constant force in Special Relativity

Momentum increases linearly with time

\[ m'g = \frac{d}{dt} \left( m' \frac{1}{\sqrt{1 - v^2}} v \right) = m' \frac{1}{\sqrt{1 - v^2^3}} \frac{dv}{dt} \]

\[ \frac{dv}{dt} = g \sqrt{1 - v^2^3} \]

\[ v = \frac{gt}{\sqrt{1 + (gt)^2}} \]

\[ y = \frac{1}{g} (\sqrt{1 + (gt)^2} - 1) \]

Hyperbolic motion
Asymptote \( y' = t - (1/g) \)

When \( t \geq \theta = \frac{1-(g\Delta)^2}{2g^2\Delta} \) indistinguishable: \( |y - y'| \leq \Delta \)

After \( \theta \): momentum still increases linearly, but little impact on velocity
Discrete motion

Space and time coordinates in $\Delta \mathbb{Z}$

$\tilde{y}$ discretization of $y$

Number of time steps before curve indistinguishable from asymptote: $L = \lceil \theta / \Delta \rceil$
A cellular automaton

States: \( \{ q, 0, \ldots, L - 1, \infty \} \)

\( 0, \ldots, L - 1 \): clock, momentum

In state \( \infty \), motion at velocity \( c \):

In state \( k \)

or

depending on \( \tilde{y}((k + 1)\Delta) = \tilde{y}(k\Delta) \) or \( \tilde{y}((k + 1)\Delta) = \tilde{y}(k\Delta) + \Delta \)
Number of states / amount of information in the particle

If $\Delta$ fixed in advance

Number of states

$$2 + \frac{\theta}{\Delta} = \frac{1}{2g^2\Delta^2} + \frac{3}{2}$$

Amount of information

$$I = \log_2\left(\frac{1}{2g^2\Delta^2} + \frac{3}{2}\right)$$
Fix the size $\rho$ of one bit
Size of the particle at least

$$\rho l = \rho \log_2 \left( \frac{1}{2g^2\Delta^2} + \frac{3}{2} \right)$$

$\Delta$ smallest size such that

$$\rho \log_2 \left( \frac{1}{2g^2\Delta^2} + \frac{3}{2} \right) \leq \Delta$$

$\Delta - \rho \log_2 \left( \frac{1}{2g^2\Delta^2} + \frac{3}{2} \right)$ monotonic in $\Delta$: numerical solution

$\rho = 1.6 \times 10^{-35}$ m: 320 bits, $\Delta = 5.1 \times 10^{-33}$ m
($\rho = 2.7 \times 10^{-35}$ m: 319 bits, $\Delta = 8.6 \times 10^{-33}$ m)
Towards an uncertainty principle

Internal state: clock, but also momentum $p = m'gt = m'gk\Delta$
Uncertainty on the momentum $P = m'g\Delta$
Size of the particle (uncertainty on the position)

$$\Delta = \rho \log_2 \left( \frac{1}{2(P/m')^2} + \frac{3}{2} \right)$$

More accurate the momentum, less accurate the position
Qualitative uncertainty principle (rather than quantitative

$$2^{\Delta/\rho}(P/m')^2 = 1/2 \quad ?!$$
V. Constant force in General Relativity
Gravitation in General Relativity

Body at a distance $R - y$, metric tensor

\[
\begin{pmatrix}
g_{tt} & 0 \\
0 & -\frac{1}{g_{tt}}
\end{pmatrix}
\]

where $g_{tt} = 1 - 2m/(R - y)$

Motion of a particle: geodesic $t(\tau), y(\tau)$ for this metric
The equations of the motion

\[
\frac{d^2 t}{d\tau^2} = -\frac{1}{g_{tt}} \frac{dg_{tt}}{dy} \frac{dt}{d\tau} \frac{dy}{d\tau}
\]

\[
\frac{d^2 y}{d\tau^2} = -\frac{1}{2} \frac{dg_{tt}}{dy} \left( g_{tt}(\frac{dt}{d\tau})^2 - \frac{1}{g_{tt}} (\frac{dy}{d\tau})^2 \right)
\]

\[
g_{tt}(\frac{dt}{d\tau})^2 - \frac{1}{g_{tt}} (\frac{dy}{d\tau})^2 = 1
\]
Constant force in General Relativity

\[ g_{tt} = 1 - \frac{2m}{R - y} \]

First-order expansion

\[ g_{tt} = 1 - \frac{2m}{R} - \frac{2m}{R^2} y = 1 - \frac{2m}{R} - 2gy = 2g(y_1 - y) \]

where \( y_1 = (1 - 2m/R)/(2g) \)
Trajectory

\[ y = y_1 \tanh^2(gt) \]
**Limit point**

**Special Relativity:** velocity goes to infinity when $\tau$ does but mapping from $t$ to $\tau$ slows down thus **velocity bounded**

**General Relativity:** velocity goes to infinity when $\tau$ does but mapping from $t$ to $\tau$ slows down (infinite $t$: finite $\tau$) **particle has a limit position**
Even better than an asymptote: a limit point

Cellular automaton
Final state: the particle stays at the same place

Number of states / amount of information in the particle:
\[ \rho = 1.6 \times 10^{-35} \text{ m: 168 bits, } \Delta = 2.6 \times 10^{-33} \text{ m} \]
\[ (\rho = 2.7 \times 10^{-35} \text{ m: 167 bits, } \Delta = 4.5 \times 10^{-33} \text{ m}) \]
Conclusion

Constant force motion can be described by a cellular automaton in (Special and General) Relativity, but not in Newtonian physics.

Exact up to $\Delta$: no accumulation of rounding errors.

Small amount of information: 319 / 167 bits.

General Relativity simpler than Special Relativity (simpler than Newtonian physics).

Not all motions possible: parabolic motion (contradicts bounded velocity and bounded density: $\sqrt{\cdot}$ would not contradict bounded velocity but would still contradict bounded density).
But...

An existence proof, rather than a satisfactory automaton

Position: on the grid
momentum, velocity (in a fixed coordinate system): internal state

The particle “knows” the laws of physics

An accurate simulation rather than an accurate description of the physical phenomenon

Calls for more natural automata: e.g. messenger particles