Modal Logics with Presburger Constraints

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Overview

Presburger constraints

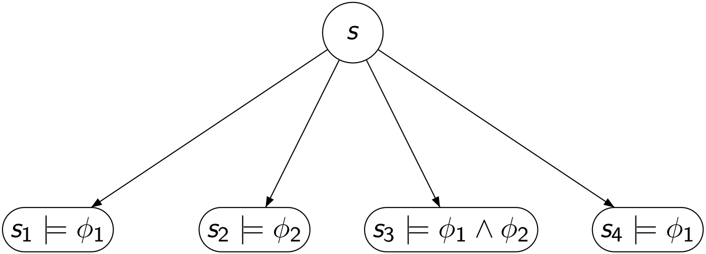
- Presburger arithmetic:
  - First-order theory of $\langle \mathbb{N}, +, = \rangle$.
  - Quantifier elimination.
  - Satisfiability in 3-EXPTIME.

- Presburger constraints on graphs/trees:
  - Constraints in counter automata.
  - Constraints on the number of event occurrences.
    [Bouajjani & Echahed & Habermehl, LICS 95]
  - Constraints on XML documents.
    [Dal Zilio & Lugiez, RTA 03]
    [Seidl et al, ICALP 04]
  - Graded modal logics ($\Diamond_{\geqslant 3}\ p$). [Fine, NDJFL 72]
  - Description logics (($\geqslant 3\ R \cdot C$)). [Hollunder & Baader, KR 91]
  - Hennessy-Milner Logic (HML).

Regularity constraints

- Regularity constraints in graphs: sequences of states/transitions belong to some regular language.
- Extended Temporal Logic (ETL). [Wolper, IC 83]
- Again, constraints for XML documents. [Dal Zilio & Lugiez, RTA 03, Seidl et al, ICALP 04]
- Propositional dynamic logic PDL. [Pratt 76]
- Description logics (ALCReg). [Baader, IJCAI 91]

Presburger and regularity constraints in graphs



$$s \models (\sharp\phi_1 = \sharp\phi_2 + 1) \wedge \phi_1 \phi_2^* \phi_1^+ .$$

Logics that count in PSPACE

- Minimal graded modal logic. [Tobies, CADE 99]
- Majority logic. [Pacuit & Salame, KR 04]
- Rank-1 modal logics. [Schröder & Pattinson, LICS 06]
- Constraints on sets with cardinalities.
  [Kuncak & Manette & Rinard, Dagstuhl 05]

Other logics that count

- Graded μ-calculus. [Kupferman & Sattler & Vardi, CADE 02]
- Logic with fixpoint operators. [Seidl et al, ICALP 04]
- Sheaves logic is decidable. [Dal Zilio & Lugiez, RTA 03]
- CTL* with counting. [Moller & Rabinovich, IC 03]
- + jungle of logics:
  - MSO logics. [Seidl & Schwentick & Muscholl, PODS 03]
  - Spatial logics, description logics (with number restrictions),
  - FO + counting quantifiers ...

Goal

- To study languages with counting and regularity constraints.
- To provide conditions for satisfiability in PSPACE.
- PSPACE upper bound implies
  - Presburger constraints in quantifier-free fragment,
  - no general fixpoint operators.

Models

Kripke-like structures $\mathcal{M} = \langle T, (R_{\mathtt{r}})_{\mathtt{r} \in \Sigma}, (<_s^{\mathtt{r}})_{s \in T}, l \rangle$

- T: set of nodes.
- l: labelling of nodes by propositional variables.
- $R_{\mathtt{r}}$: binary relation on T with finite-branching.
- $<_s^{\mathtt{r}}$: total ordering on the set of $R_{\mathtt{r}}$ successors of node s.

Syntax

$$\begin{array}{lll} \phi & ::= & p \mid \neg\phi \mid \phi \wedge \phi \mid t \sim b \mid t \equiv_k c \mid \mathcal{A}(\mathtt{r}, \phi_1, \ldots, \phi_n) \\ t & ::= & a \times \sharp^{\mathtt{r}}\phi \mid t + a \times \sharp^{\mathtt{r}}\phi \end{array}$$

$(a \in \mathbb{Z},\ b \in \mathbb{N},\ k, c \in \mathbb{N})$

- p: propositional variable, $\mathtt{r}$: relation symbol.
- $\mathcal{A}$: finite-state automaton.
- $\sim \in \{<, >, =\}$.
- $\sharp^{\mathtt{r}}\phi$: number of nodes accessible by $\mathtt{r}$ satisfying ϕ.

BINARY ENCODING OF INTEGERS

The satisfaction relation

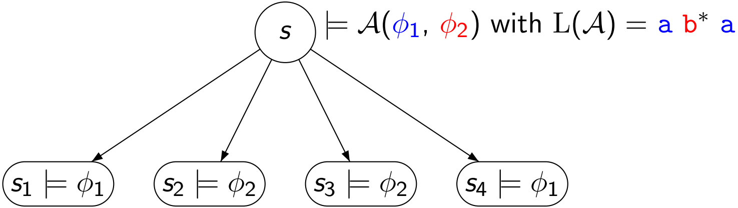
- $\mathcal{M}, s \models \sum_i a_i \sharp^{\mathtt{r}_i} \phi_i \sim b \stackrel{\text{def}}{\Leftrightarrow} \sum_i a_i R^{\sharp}_{\mathtt{r}_i,\phi_i}(s) \sim b$ with

$$R^{\sharp}_{\mathtt{r}_i,\phi_i}(s) = \mathrm{card}(\{s' \in T : \langle s, s' \rangle \in R_{\mathtt{r}_i},\ \mathcal{M}, s' \models \phi_i\}).$$

- $\mathcal{M}, s \models \mathcal{A}(\mathtt{r}, \phi_1, \ldots, \phi_n) \stackrel{\text{def}}{\Leftrightarrow}$ there is $\mathrm{a}_{i_1} \cdots \mathrm{a}_{i_\alpha} \in \mathrm{L}(\mathcal{A})$ s.t.
  - $R_{\mathtt{r}}(s) = s_1 < \ldots < s_\alpha$,
  - for every $j \in \{1, \ldots, \alpha\}$, $\mathcal{M}, s_j \models \phi_{i_j}$.

Examples

- $\Diamond\phi \approx \sharp^{\mathtt{r}}\phi \geqslant 1 \quad \Box\phi \approx \sharp^{\mathtt{r}}\neg\phi = 0 \quad \Diamond_{\geqslant n}\phi \approx \sharp^{\mathtt{r}}\phi \geqslant n.$
- There are as many words accessible by $\mathtt{r}_1$ satisfying ϕ_1 as worlds accessible by $\mathtt{r}_2$ satisfying ϕ_2: $\sharp^{\mathtt{r}_1}\phi_1 = \sharp^{\mathtt{r}_2}\phi_2$.



Tree model property

Lemma: ϕ has a model iff ϕ has a (unranked, ordered) tree model.

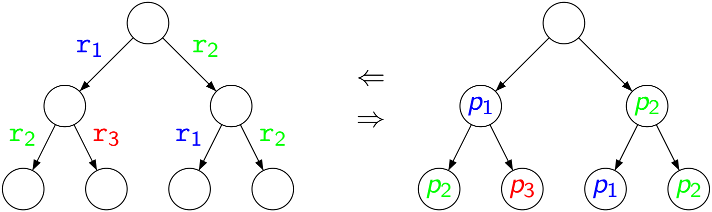
Unfold $\langle T, (R_{\mathtt{r}})_{\mathtt{r} \in \Sigma}, (<_s^{\mathtt{r}})_{s \in T}, l \rangle \rightarrow \langle T', (S_{\mathtt{r}})_{\mathtt{r} \in \Sigma}, (<_s^{\prime \mathtt{r}})_{s \in T'}, l' \rangle$

- T': set of finite sequences $s \ \mathtt{r}_1 \ s_1 \ldots \mathtt{r}_k \ s_k$.
- $(s \ \mathtt{r}_1 \ s_1 \ldots \mathtt{r}_n \ s_n) \ S_{\mathtt{r}} \ (s \ \mathtt{r}_1 \ s_1 \ldots \mathtt{r}_n \ s_n \ \mathtt{r}_{n+1} \ s_{n+1})$ iff $\langle s_n, s_{n+1} \rangle \in R_{\mathtt{r}}$ and $\mathtt{r} = \mathtt{r}_{n+1}$.
- $l'(s \ \mathtt{r}_1 \ s_1 \ldots \mathtt{r}_n \ s_n) = l(s_n)$.
- ordering $<_{s'}^{\prime \mathtt{r}}$ on sequences induced by orderings on last elements.

Reduction to one relation

Lemme

There is logspace reduction from EML *satisfiability into* EML *satisfiability restricted to a single relation symbol.*



ϕ is transformed into $\psi_{uni} \wedge \psi_{subst}$:

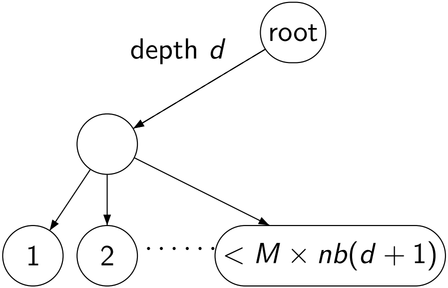
- ψ_{uni} states that a unique p_i is true at each (non root) node,
- ψ_{subst} is obtained from ϕ by replacing
  - $\#^{\mathtt{r}_i}\varphi$ by $\#(\varphi \wedge p_i)$
  - $\mathcal{A}(\mathtt{r}_i, \varphi_1, \ldots)$ by $\mathcal{A}'(\mathtt{r}, \neg p_i, \varphi_1 \wedge p_i, \ldots)$ ($\mathcal{A}'$ variant of $\mathcal{A}$).

n-maximally consistent sets

- Fischer/Ladner closure $cl(\phi)$:
  - closure under subformulae and negation,
  - $t \sim b \in \mathrm{cl}(\phi)$ implies $t \sim' b \in \mathrm{cl}(\phi)$ for $\sim' \in \{<, >, =\}$,
  - $t \equiv_k c \in \mathrm{cl}(\phi)$ implies $t \equiv_K c' \in \mathrm{cl}(\phi)$ for $c' \in \{0, \ldots, K-1\}$ (K lcm).
- Relative closure of level n: $cl(n, \phi)$
  - $cl(0, \phi) = cl(\phi)$,
  - if $\#\psi$ occurs in $cl(n, \phi)$ then $\psi \in cl(n+1, \phi)$,
  - similar condition for automata-based subformulae.
- Introduction of *n*-maximal consistency (extension from Hintikka sets).

M-bounded models

- $nb(d)$: number of d-maximal consistent sets (wrt ϕ).
- $nb(d)$ is exponential in $|\phi|$.
- $\langle \mathcal{M}, s \rangle$ is M-bounded for ϕ:



Ladner-like algorithms

- Modal logic K in PSPACE. [Ladner, SIAM 77]
- Nondeterministic algorithm that do not rely on automata, tableaux, sequents etc. See also [Spaan, 93]
- Correctness is partly based on the tree model property.

Procedure $SAT(X, \phi, d)$

- Stop: X not d-maximal consistent or contains only propositions.

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- Guess number nb of children in $\{0, \ldots, nb(d+1) \times M\}$
  - for each $i = 1, .., nb$,
    - Guess i^{th} child as Y_x with $x \in \{1, \ldots, nb(n+1)\}$
    - if not $SAT(Y_x, \phi, d+1)$ then `abort`.
    - for $\psi \in Y_x$, increment C_ψ.
    - update states of automata (guess transitions).

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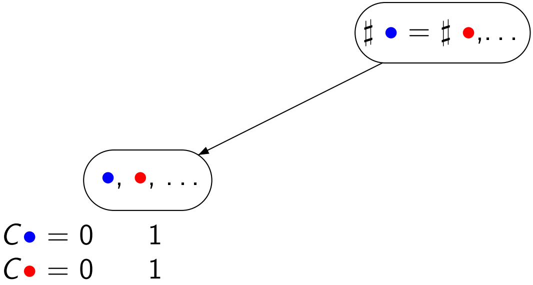
    - for $\psi \in Y_x$, increment C_ψ.
    - update states of automata (guess transitions).
- Check the constraints in X ($\#\psi$ replaced by C_ψ) and the automata constraints.

Guess and check

$\sharp \bullet = \sharp \bullet, \ldots$

$C_{\bullet} = 0$
$C_{\bullet} = 0$

Guess and check



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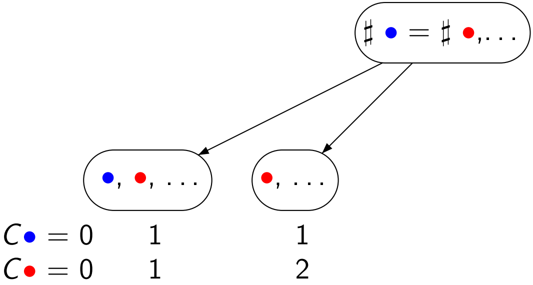

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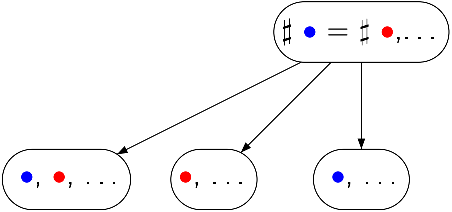

$C_{\bullet} = 0$	1	1	2
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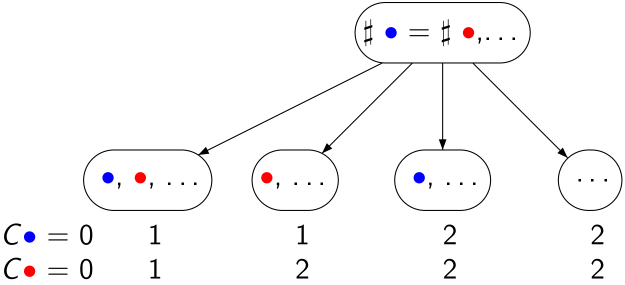
Guess and check



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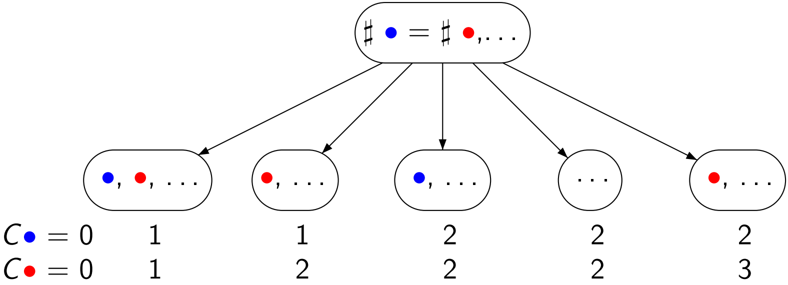

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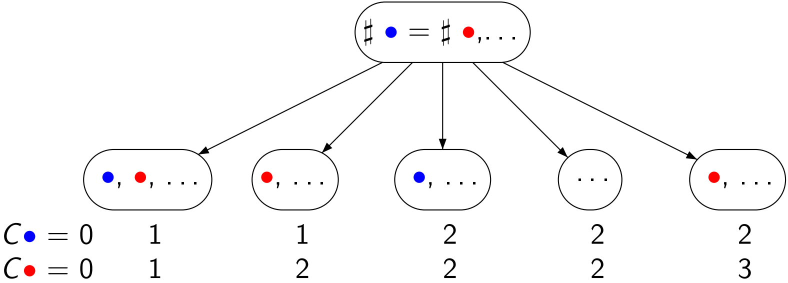


Final check: $C_\bullet = C_\bullet$?

Space bound and correctness

Proposition

For all 0-maximally consistent sets X, and computations of $SAT(X, \phi, 0)$,

- *the recursive depth is linear in $|\phi|$,*
- *each call requires space polynomial in the sum of*
  - *space for encoding 0-maximally consistent sets*
  - *$+ \ log(M)$.*

Proposition

There is a set $X \subseteq cl(\phi)$ such that $\phi \in X$ and $SAT(X, \phi, 0)$ has an accepting computation iff ϕ is EML satisfiable in some M-bounded model.

Complexity results

- Boundedness Lemma: There is a polynomial $p(\cdot)$ such that for every formula ϕ, ϕ is EML satisfiable iff ϕ is satisfiable in some $2^{p(|\phi|)}$-bounded model.
- Proof can be divided as follows
  - Small solutions for constraint systems [Borosh & Treybis, AMS 76; Papadimitriou, JACM 81]
  - Product automata over an enriched alphabet.
  - Reduction of the number of equations based on Claim 7.3 in [Seidl et al.,TLG 07]
  - Constraint systems from maximally consistent sets.
- Corollary: EML satisfiability is PSPACE-complete.

From sets to constraint systems

d-maximally consistent set X

$\Downarrow$

Boolean combination of arithmetical constraints $\mathcal{S}_X$

- $(d+1)$-maximally consistent sets $Y_1, \ldots, Y_{nb(n+1)}$.
- $nb(d+1)$ exponential in $|\phi|$.
- Term $t_\psi = \sum_{\{i \mid \psi \in Y_i\}} x_i$ built over the variables $x_1, \ldots, x_{nb(n+1)}$.
- x_i: number of occurrences of a successor satisfying Y_i.

Small solution property

- Atomic constraint: $\sum_j a_j \times x_{i_j} = b$ with $a_j \in \mathbb{Z}$, $b \in \mathbb{N}$.
- Constraint system $\mathcal{S}$: conjunction of atomic constraints.
- $\mathcal{S}$ has a positive solution iff there is a positive solution such that all the coefficients are bounded by

$$n \times (ma)^{2m+1}$$

  - n: number of variables,
  - a: maximal absolute value, among constants in $\mathcal{S}$,
  - m: number of atomic constraints in $\mathcal{S}$.

[Papadimitriou, JACM 81]

Building the constraint system

- $\sum_i a_i \# \psi_i = b \in X$ becomes $\sum_i a_i t_{\psi_i} = b \in \mathcal{S}_X$,
- $\sum_{Y_i \text{ is not satisfiable}} x_i = 0$,
- $+$ constraints from automata-based formulae
- X is satisfiable iff $\mathcal{S}_X$ has a positive solution.
- If $\mathcal{S}_X$ has a positive solution, it has a solution with values bounded by M (exponential in $|\phi|$).
- X is EML satisfiable iff $\mathcal{S}_X$ has a positive solution.

Linear sets for the Parikh images

- FSA $\mathcal{A} = \langle \Sigma, Q, \delta, I, F \rangle$.
- Parikh image of a b a a b $= \langle 3, 2 \rangle$.
- Parikh image $\pi(\mathrm{L}(\mathcal{A})) \subseteq \mathbb{N}^{|\Sigma|}$.
- $\pi(\mathrm{L}(\mathcal{A}))$ is a finite union of linear sets

$$\{\sigma_0 + \sum_{i=1}^{m} y_i \sigma_i : y_i \geqslant 0\}$$

  - each σ_j is in $\{0, \ldots, |Q|\}^{|\Sigma|}$.
  - m bounded by $(|Q| + 1)^{|\Sigma|}$.

[Seidl et al, ICALP 04]

Product automaton over an enriched alphabet

$$\mathcal{A}_1(\ldots), \ldots, \mathcal{A}_l(\ldots), \neg\mathcal{A}'_1(\ldots), \ldots, \neg\mathcal{A}'_{l'}(\ldots) \in X$$

- with argument subformulae in $\{\psi_1, \ldots, \psi_P\} \subseteq \mathrm{cl}(d+1, \phi)$.
- Enriched alphabet $\Sigma = \{Y_1, \ldots, Y_N\}$ made of $(d+1)$-maximally consistent sets.
- $\mathcal{B} = \langle \Sigma, Q', \ldots \rangle$ with $\mathrm{card}(Q')$ exponential in $|\phi|$ such that $w = Y_{j_1} \cdots Y_{j_\alpha} \in \Sigma^*$ iff
  - $\forall\ i$, $\exists\ \psi_1 \in Y_{j_1}$, ..., $\psi_\alpha \in Y_{j_\alpha}$ s.t. $\psi_1 \cdots \psi_\alpha \in \mathrm{L}(\mathcal{A}_i)$.
  - $\forall i$, $\nexists\ \psi_1 \in Y_{j_1}$, ..., $\psi_\alpha \in Y_{j_\alpha}$ s.t. $\psi_1 \cdots \psi_\alpha \in \mathrm{L}(\mathcal{A}'_i)$.

Parikh image of $\mathrm{L}(\mathcal{B})$

- $\pi(\mathrm{L}(\mathcal{B})) = \mathrm{L}_1 \cup \cdots \cup \mathrm{L}_m$ with $\mathrm{L}_i = \{\sigma_0 + \sum_{j=1}^{h} y_j \sigma_j : y_j \geqslant 0\}$.
- Each σ_j is in $\{0, \ldots, |Q'|\}^N$ [Seidl et al., ICALP 04].
- h is bounded by $(|Q'| + 1)^{|\Sigma|} \leqslant 2^{p(|\phi|) \times 2^{|\phi|}}$.
- The system (with $N + h$ variables)

$$\begin{pmatrix} z_1 \\ z_2 \\ \ldots \\ z_N \end{pmatrix} = \sigma_0 + \sum_{j=1}^{h} y_j \sigma_j$$

admits a (small) solution whose values are at most doubly exponential in $|\phi|$.

Reducing the number of variables (I)

- $H : \mathbb{N}^N \rightarrow \mathbb{N}^P$ with

$$H(\begin{pmatrix} n_1 \\ n_2 \\ \ldots \\ n_N \end{pmatrix})(i) \stackrel{\text{def}}{=} \sum_{\psi_i \in Y_j} n_j.$$

- $v \in H(\pi(\mathrm{L}(\mathcal{B}))$ iff there is $w \in \mathrm{L}(\mathcal{B})$ s.t. for $j \in \{1, \ldots, P\}$,

$$\mathrm{card}(\{w(k) : k < |w|, \psi_j \in w(k)\}) = v(j)$$

- $v \in H(\pi(\mathrm{L}(\mathcal{B})))$ iff $v \in H(\mathrm{L}_i)$ for some $i \in \{1, \ldots, m\}$.
- $H(\mathrm{L}_i) = \{H(\sigma_0) + \Sigma_{j=1}^{h} y_j H(\sigma_j) : y_j \geqslant 0\}$.

Reducing the number of variables (II)

- $H(\sigma_j)$ has dimension $P \leqslant |\phi|$ and each coefficient is $\leqslant N \times 2^{p(|\phi|) \times |\phi|}$.
- $\mathrm{card}(\{H(\sigma_j) : 1 \leqslant j \leqslant h\}) \leqslant 2^{p_1(|\phi|)}$.
- We pose $\{h_1, \ldots, h_\alpha\} = \{H(\sigma_j) : 1 \leqslant j \leqslant h\}$.
- Same projections over $z_1, \ldots, z_P$:

$$(\star) \begin{pmatrix} z_1 \\ z_2 \\ \ldots \\ z_P \end{pmatrix} = H(\sigma_0) + \sum_{j=1}^{\alpha} y_j h_j \quad (\star\star) \begin{pmatrix} z_1 \\ z_2 \\ \ldots \\ z_P \end{pmatrix} = H(\sigma_0) + \sum_{j=1}^{h} y'_j \, H(\sigma_j)$$

Reducing the number of variables (III)

- $\mathcal{S}^\star$ [resp. $\mathcal{S}^{\star\star}$]: disjunction of all the systems of the form $(\star)$ [resp. $(\star\star)$].
- Each disjunct of $\mathcal{S}^\star$ has a polynomial amount of equations, an exponential amount of variables $(P + \alpha)$ and coefficients are at most exponential in $|\phi|$.
- Consequently, $\mathcal{S}_X$ has small solutions of exponential values in $|\phi|$ (if any).

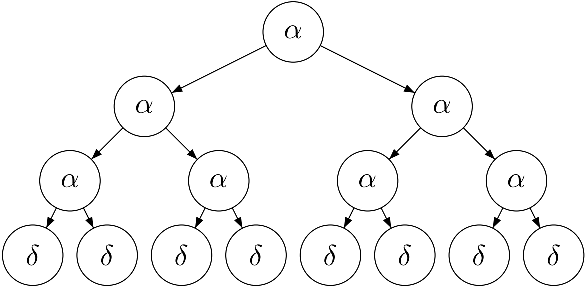
Comparison with other logics

- EML is a fragment of fixpoint-free fragment of the logic from [Seidl et al, ICALP 04]
- Sheaves Logic [Dal Zilio & Lugiez, 06]: a node is counted exactly once vs a node may contribute several times.
- Spatial Logic without quantification and fixpoint $SL_{|\exists,\mu}$ [Dal Zilio & Lugiez & Meyssonnier, POPL 04] [Boneva & Talbot, RTA 05] (Presburger is implicit)

Sheaves logic

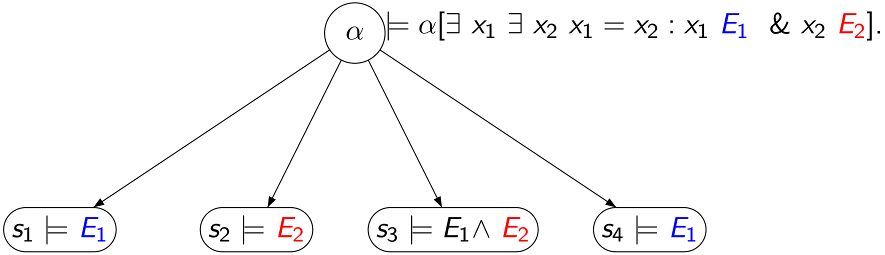
Semantics:

- interpreted on unranked ordered trees.
- satisfaction relation similar to EML except that a node counts exactly once.



Satisfaction in SL

$$E \;:=\; \alpha[D] \mid \delta \mid \neg E \mid E \wedge E \mid \texttt{true}$$
$$D \;:=\; \exists x_1, \ldots, x_p : \phi(x_1, \ldots, x_p) : x_1 E_1 \& \cdots \& x_p E_p \mid \mathcal{A}(E_1, \ldots, E_p) \mid \texttt{true} \mid \neg D \mid D \wedge D'$$



From SL to EML

Lemme

There is logspace reduction from SL satisfiability into EML *satisfiability.*

E is transformed into $E_{uni} \wedge E_{subst}$:

- E_{uni} states conditions related to
  - propositions for tags (only at internal nodes) and datatypes (only at leaves),
  - documents $D = \exists x_1, \ldots, x_p : \varphi(x_1, \ldots, x_p) : x_1 E_1 \& \cdots \& x_p E_p$ $(q_D^1, \ldots, q_D^n)$,
  - q_D^i implies $(E_i)_{subst}$.
- E_{subst} is obtained from E by replacing $\exists x_1, \ldots, x_p : \varphi(x_1, \ldots, x_p) : x_1 E_1 \& \cdots \& x_p E_p$ by $\varphi(\# q_{E_1}, \ldots, \# q_{E_n}) \wedge \neg(\sharp(\neg q_{E_1} \wedge \ldots \wedge \neg q_{E_n}) > 0)$.

Complexity

- Reduction from modal logic K with no propositional variables into SL satisfiability.
  PSPACE-hardness follows from [Hemaspaandra, JLC 01].
- Definition:
  - $t(\texttt{true}) = \texttt{true}$, $t(\texttt{false}) = \neg\texttt{true}$,
  - $t(\phi \wedge \phi') = t(\phi) \wedge t(\phi')$,
  - $t(\phi \vee \phi') = \neg(\neg t(\phi) \wedge \neg t(\phi'))$,
  - $t(\Diamond\phi) = \alpha[\exists\ x : x \geqslant 1 : x\ \ t(\phi)]$,
  - $t(\Box\phi) = \alpha[\exists\ x : x = 0 : x\ \neg t(\phi)] \vee \delta$.
- Negation-free ϕ is satisfiable iff $t(\phi)$ is satisfiable.
- Corollary SL satisfiability is PSPACE-complete.

How to enrich EML safely?

- $\mathtt{PDL}_{\mathtt{tree}}$ is interpreted over finite unranked ordered trees.
- Relation/direction expressions:

$$\mathbf{r} ::= \uparrow \mid \rightarrow \mid \mathbf{r}^* \mid \mathbf{r} \circ \mathbf{r} \mid \mathbf{r} \cup \mathbf{r} \mid \mathbf{r}^{-1}.$$

- Satisfiability for $\mathtt{PDL}_{\mathtt{tree}}$ is EXPTIME-complete [Afanasiev et al., JANCL 05].

$\mathcal{L}$: a $\mathtt{PDL}_{\mathtt{tree}}$-like logic

- Mix of EML and $\mathtt{PDL}_{\mathtt{tree}}$. [Afanasiev et al., JANCL 05]
- Models: finite labeled unranked ordered trees.
- Relation symbols: $\Sigma = \{\downarrow, \downarrow^*, \rightarrow, \rightarrow^*, \leftarrow, \leftarrow^*, \uparrow, \uparrow^*\}$
- Formulas:
  - $\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid t \sim b$
  - $t ::= a \times \sharp^{\mathtt{r}}\phi \mid t + a \times \sharp^{\mathtt{r}}\phi$ ($a \in \mathbb{Z}$, $b \in \mathbb{N}$, $\mathtt{r} \in \Sigma$),

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Proposition

Satisfiability for $\mathcal{L}$ is undecidable.

Proof by reduction from halting problem for Minsky machines.
Only $\{\downarrow, \downarrow^*, \rightarrow^*, \leftarrow\}$ is used.

Conclusion

- PSPACE upper bound extending what is known for graded modal logics.
- Sheaves logic SL is PSPACE-complete too.
- Decidability status of
  - the restriction of $\mathcal{L}$ to formulae with no subformula of the form $\Sigma_i a_i \sharp^{\mathtt{r}_i} \phi_i$ where for some $j \neq j'$, $\mathtt{r}_j \neq \mathtt{r}_{j'}$,
  - EML extended with $\leftarrow$.
  - EML extended with VPA.

?