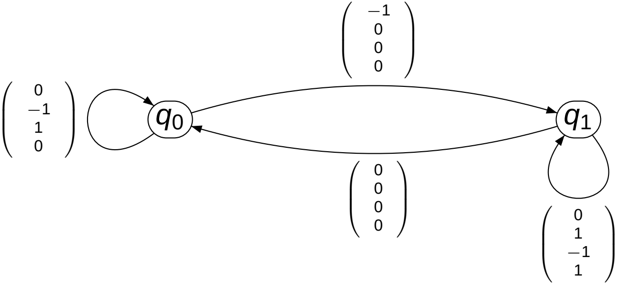
On Selective Unboundedness of VASS

Stéphane Demri

Laboratoire Spécification et Vérification (LSV)
ENS de Cachan & CNRS & INRIA

INFINITY'10, September 2010, Singapore

Vector addition systems with states



- $\approx$ Petri nets (greater practical appeal).

- Boundedness problem:

  Input: VASS $\mathcal{V}$ and configuration $(q, \vec{x}) \in Q \times \mathbb{N}^n$.

  Question: is $\{(q', \vec{x'}) \in Q \times \mathbb{N}^n : (q, \vec{x}) \xrightarrow{*} (q', \vec{x'})\}$ finite?

- Boundedness problem is ExpSpace-complete.

  [Lipton, TR 76; Rackoff, TCS 78]

  (decidability shown in [Karp & Miller, JCSS 69])

VASS and witness runs

- Properties on VASS may be characterized by witness runs.

- $(\mathcal{V}, (q, \vec{x}))$ is unbounded iff there is a finite run

$$(q, \vec{x}) \xrightarrow{*} (q', \vec{y}) \xrightarrow{+} (q', \vec{y'})$$

  for some $q' \in Q$ with $\vec{y} \prec \vec{y'}$.

- $(\mathcal{V}, (q, \vec{x}))$ has an infinite run iff there is a finite run

$$(q, \vec{x}) \xrightarrow{*} (q', \vec{y}) \xrightarrow{+} (q', \vec{y'})$$

  for some $q' \in Q$ with $\vec{y} \preceq \vec{y'}$ ($\preceq$ defined componentwise).

- Need for languages to express witness runs.
  [Jančar, TCS 90; Yen, IC 92; Atig & Habermehl, RP'09]
  [M. Praveen & Lodaya, FST&TCS'09]

Ingredients in Rackoff's proof

- Existence of witness runs can be restricted to doubly exponential runs.

- EXPSPACE upper bound is then obtained by using a nondeterministic algorithm and apply Savitch's theorem.

- Two main parts of the proof:
  - A technical lemma about small pseudo-runs obtained by using small solutions of systems of inequalities.
    [Borosh & Treybig, TCS 76]
    "Pseudo" refers to the presence of negative values.

  - Induction on the dimension verified by composing adequately subruns.

Generalization
(VAS version)

- Introducing a language for witness runs [Yen, IC 92]:

$$\exists\, \vec{x_1}, \ldots, \vec{x_n},\ \pi_1, \ldots, \pi_n\ \vec{x_0} \xrightarrow{\pi_1} \vec{x_1} \xrightarrow{\pi_2} \vec{x_2} \cdots \xrightarrow{\pi_n} \vec{x_n}$$

  satisfying the constraint $\mathcal{C}(\vec{x_1}, \ldots, \vec{x_n}, \pi_1, \ldots, \pi_n)$.

- Arithmetical constraints $\mathcal{C}$ about numbers of transitions and differences of counter values.

- Language can express witness run characterizations of most known decidable problems.

- Technical lemma in Rackoff's proof extends easily to this generalization.

- But satisfiability happens to be equivalent to the reachability problem for VASS [Atig & Habermehl, RP'09].

Increasing path formulae

- A restriction that is still powerful:

$$\exists\, \vec{x_1},\ldots,\vec{x_n},\ \pi_1,\ldots,\pi_n\ \vec{x_0} \xrightarrow{\pi_1} \vec{x_1} \xrightarrow{\pi_2} \vec{x_2} \cdots \xrightarrow{\pi_n} \vec{x_n} \wedge \mathcal{C}(\vec{x_1},\ldots,\vec{x_n}) \wedge \vec{x_1} \preceq \vec{x_n}$$

- Satisfiability problem is in ExpSpace.

  [Atig & Habermehl, RP'09]

- Many decision problems are captured except a few of them, including the regularity detection problem.

- $\vec{x_1} \preceq \vec{x_n}$ useful for the induction on dimension.

- Run from $\vec{x_0}$ obtained by firing $\pi_1(\pi_2\cdots\pi_n)\cdot(\pi_2\cdots\pi_n)$ still satisfies the property.

Generalized unboundedness property

- In this work, we propose a new class of properties
  1. for which the EXPSPACE upper bound can be guaranteed,
  2. characterizing decision problems for which complexity was still open (regularity, reversal-boundedness, strong promptness, etc.).
- Intervals $]-\infty,+\infty[$, $[a,+\infty[$, $]-\infty,b]$ or $[a,b]$ within $\mathbb{Z}$.
- Generalized unboundedness property: $\mathcal{P} = (\mathcal{I}_1,\ldots,\mathcal{I}_K)$ is a nonempty sequence of n-tuples of intervals.
- Run below satisfies $\mathcal{P} \overset{\text{def}}{\Leftrightarrow}$ (mainly constraints on the π_i's)

$$(q_0,\vec{x_0}) \xrightarrow{\pi'_0} (q_1,\vec{x_1}) \xrightarrow{\pi_1} (q_2,\vec{x_2}) \xrightarrow{\pi'_1} (q_3,\vec{x_3}) \cdots \xrightarrow{\pi'_{K-1}} (q_{2K-1},\vec{x}_{2K-1}) \xrightarrow{\pi_K} (q_{2K},\vec{x}_{2K})$$

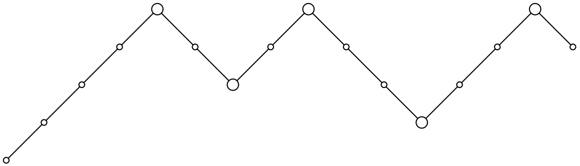
(P0) For $l \in [1,K]$, $q_{2l-1} = q_{2l}$.
(P1) For $l \in [1,K]$, $j \in [1,n]$, $\vec{x}_{2l}(j) - \vec{x}_{2l-1}(j) \in \mathcal{I}_l(j)$.
(P2) For $l \in [1,K]$, $j \in [1,n]$, if $\vec{x_{2l}}(j) - \vec{x_{2l-1}}(j) < 0$, then there is $l' < l$ such that $\vec{x}_{2l'}(j) - \vec{x}_{2l'-1}(j) > 0$.

Generalized unboundedness problem

- If the run ρ satisfies $\mathcal{P}$, then $\vec{x_2} - \vec{x_1} \geq \vec{0}$ but not necessarily $\vec{x_n} - \vec{x_1} \geq \vec{0}$.
- $(\mathcal{V}, (q_0, \vec{x_0}))$ satisfies $\mathcal{P}$ $\stackrel{\text{def}}{\Leftrightarrow}$ there is a finite run ρ from $(q_0, \vec{x_0})$ admitting a decomposition satisfying $\mathcal{P}$.
- Generalized unboundedness problem:

  Input: $(\mathcal{V}, (q_0, \vec{x_0}))$, $\mathcal{P}$.
  Question: Does $(\mathcal{V}, (q_0, \vec{x_0}))$ satisfy $\mathcal{P}$?
- Properties strictly weaker than [Yen, IC 92], incomparable with [Atig & Habermehl, RP'09].
- Disjunctions of properties can be easily handled.

Reversal-boundedness



- Reversal-bounded counter automata: each run has a bounded number of reversals.

- Reachability sets are effectively Presburger-definable.
  [Ibarra, JACM 78]

- Reversal-boundedness detection problem:
  Input: $(\mathcal{V},(q,\vec{x}))$.
  Question: Is $(\mathcal{V},(q,\vec{x}))$ reversal-bounded?

- Decidability for VASS (and for variants too).
  [Finkel & Sangnier, MFCS'08]

- Reversal-boundedness detection problem is undecidable for counter automata. [Ibarra, JACM 78]

Place boundedness problem

- Place boundedness problem:

  Input: $(\mathcal{V},(q,\vec{x}))$ and $i \in [1,n]$.

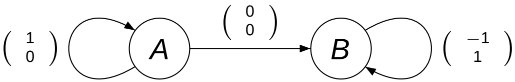
  Question: Is the set $\{\vec{x'}(i) : (q,\vec{x}) \xrightarrow{*} (q',\vec{x'})\}$ finite?

- Decidability by [Karp & Miller, JCSS 69].

- i-unboundedness not characterized by a witness run:

$$(q,\vec{x}) \xrightarrow{*} (q_1,\vec{x}_1) \xrightarrow{\pi} (q_2,\vec{x}_2)$$

  with $\vec{x}_1 \prec \vec{x}_2$, $q_1 = q_2$ and $\vec{x}_1(i) < \vec{x}_2(i)$.



- Characterization for (standard) unboundedness based on a first ∞ on a branch of the coverability graphs.

Strong promptness detection problem

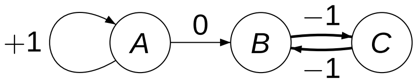
- Strong promptness detection problem:

  Input: $((Q, n, \delta), (q, \vec{x}))$ and a partition (δ_I, δ_E) of δ.

  Question: Is there $k \in \mathbb{N}$ such that for every $(q, \vec{x}) \xrightarrow{*} (q', \vec{x}')$, there is no $(q', \vec{x}') \xrightarrow{\pi} (q'', \vec{x}'')$ with $\pi \in \delta_I^{>k}$?

  Check whether the length of sequences of internal transitions is bounded. [Valk & Jantzen, Acta Inf. 85]

- $(\mathcal{V}, (A, 0))$ below is not strongly prompt and there is no $(A, 0) \xrightarrow{*} (q, \vec{x}) \xrightarrow{\pi} (q, \vec{y})$ with $\vec{x} \preceq \vec{y}$ and $\pi \in \delta_I^{+}$.



- There is a logspace reduction from strong promptness detection problem to the place boundedness problem.

Regularity detection problem

- Nonregularity of $(\mathcal{V}, (q_0, \vec{x_0}))$ is equivalent to the existence of a run

$$(q_0, \vec{x_0}) \xrightarrow{\pi_0'} (q_1, \vec{x_1}) \xrightarrow{\pi_1} (q_2, \vec{x_2}) \xrightarrow{\pi_1'} (q_3, \vec{x_3}) \xrightarrow{\pi_2} (q_4, \vec{x_4})$$

  such that

  1. $q_1 = q_2$, $q_3 = q_4$,
  2. $\vec{x_1} \prec \vec{x_2}$,
  3. there is $i \in [1, n]$ such that $\vec{x_4}(i) < \vec{x_3}(i)$,
  4. for all $j \in [1, n]$ such that $\vec{x_4}(j) < \vec{x_3}(j)$, we have $\vec{x_1}(j) < \vec{x_2}(j)$.

  [Valk & Vidal-Nacquet, JCSS 81]

- Nonregularity condition = disjunction of generalized unboundedness properties of the form $(\mathcal{I}_1^i, \mathcal{I}_2^i)$:

  1. $\mathcal{I}_1^i(i) = [1, +\infty[$,
  2. $\mathcal{I}_2^i(i) =]-\infty, -1]$,
  3. for $j \neq i$, we have $\mathcal{I}_1^i(j) = [0, +\infty[$ and $\mathcal{I}_2^i(j) =]-\infty, +\infty[$.

Pseudo-runs

- Pseudo-run: finite sequence in $Q \times \mathbb{Z}^n$ (instead of $Q \times \mathbb{N}^n$) respecting the transitions.

- The pseudo-run

$$(q_0, \vec{x_0}) \xrightarrow{\pi_0'} (q_1, \vec{x}_1) \xrightarrow{\pi_1} (q_2, \vec{x}_2) \xrightarrow{\pi_1'} (q_3, \vec{x}_3) \cdots \xrightarrow{\pi_{K-1}'} (q_{2K-1}, \vec{x}_{2K-1}) \xrightarrow{\pi_K} (q_{2K}, \vec{x}_{2K})$$

weakly satisfies $\mathcal{P} \overset{\text{def}}{\Leftrightarrow}$ it satisfies (P0), (P1), (P2) and

(P3) for $j \in [1, n]$, every pseudo-configuration $\vec{x}$ such that $\vec{x}(j) < 0$ occurs after some $\vec{x}_{2l}$ for which $\vec{x}_{2l}(j) - \vec{x}_{2l-1}(j) > 0$.

- Existence of a pseudo-run weakly satisfying $\mathcal{P} \Leftrightarrow$ existence of a run satisfying $\mathcal{P}$.

- Simple argument: repeat the paths π_i's hierarchically in order to eliminate negative values.

Small pseudo-run property

If ρ is a pseudo-run weakly satisfying $\mathcal{P}$, then there is ρ' starting from the same pseudo-configuration, weakly satisfying $\mathcal{P}$ and of length at most

$$poly(K, \mathrm{scale}(\mathcal{P}), \mathrm{card}(Q), \mathrm{scale}(\mathcal{V}))^{exp(n)}$$

- n: dimension of $\mathcal{V}$.
- $\mathrm{card}(Q)$: number of control states in the VASS.
- $\mathrm{scale}(\mathcal{V})$: maximal absolute value in transitions of $\mathcal{V}$.

- K: number of tuples of intervals in $\mathcal{P}$.
- $\mathrm{scale}(\mathcal{P})$: maximal absolute value in intervals of $\mathcal{P}$.

Ingredients of the proof

- Small solutions for systems of inequalities.
  [Borosh & Treybig, TCS 76]

- Induction on the dimension.

- Progress in the satisfaction of the property $\mathcal{P}$.

- Local repetition of pseudo-runs similar to the properties of global repetition with increasing path formulae.
  See e.g. [Rackoff, TCS 78; Atig & Habermehl, RP'09]

Corollaries

- Problems below in EXPSPACE:
  - The generalized unboundedness problem.
  - The regularity detection problem.
  - The strong promptness detection problem.
- For each fixed $n \geq 1$, their restrictions to VASS of dimension at most n are in PSPACE.

About reversal-boundedness

- Reversal-boundedness detection problem for VASS is EXPSPACE-complete.
  Reduction to the place boundedness problem and use of [Hopcroft & Pansiot TCS 79].

- Similar results for the variant of reversal-boundedness introduced in [Finkel & Sangnier, MFCS'08].

- EXPSPACE-hardness by an adaptation of [Lipton, TR 76].

- For each fixed $n \geq 1$, its restriction to VASS of dimension at most n is in PSPACE.

Conclusion

- EXPSPACE-easiness of the generalized unboundedness problem. The property $\mathcal{P}$ is also part of the input.

- Use of witness pseudo-run characterizations.

- Locality of the increasing path formulae.

- Complexity upper bound for
  - reversal-boundedness detection problems,
  - place boundedness problem,
  - strong promptness detection problem,
  - regularity detection problem.

- Possible continuations.

  1. Determine the robustness of the proof technique.
  2. Which subclasses of VASS decrease the complexity to PSPACE? See e.g. [M. Praveen & Lodaya, FST&TCS'09]