Decidable Problems for Counter Systems

Day 4
Reversal-Bounded Counter Automata

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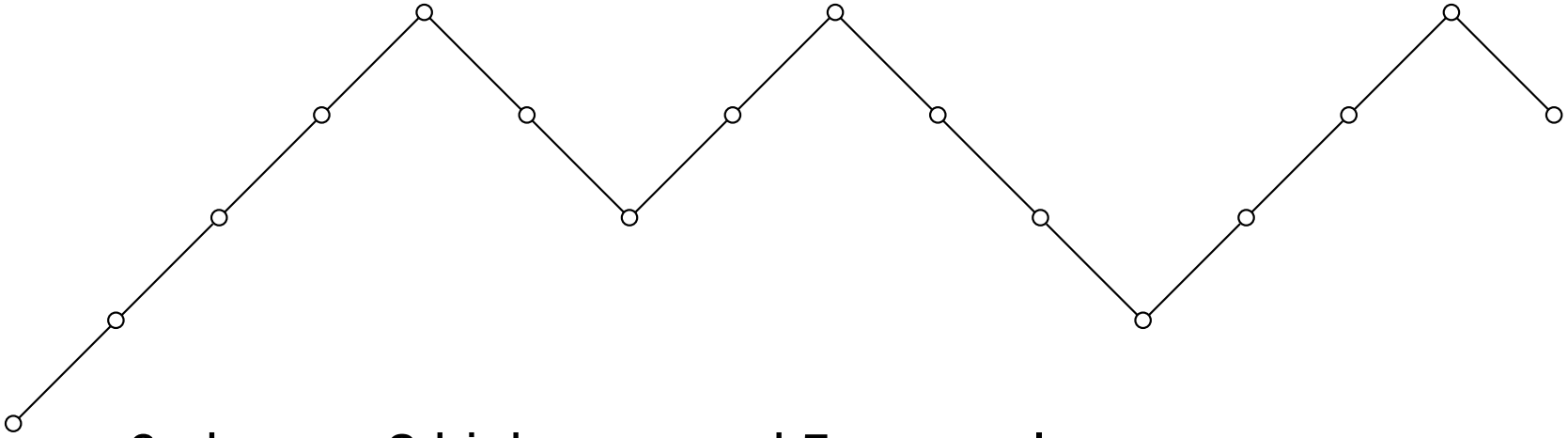
Plan of the talk

- Previous lectures:
  - Classes of counter systems, Presburger arithmetic.
  - LTL-like dialects for counter systems.
  - Petri nets, VASS, covering problem and control state reachability problem in EXPSPACE.
- Definitions about reversal-bounded counter automata.
- Proof for effective semilinearity of reachability sets.
- Variants.
- Repeated reachability and related logical problems.

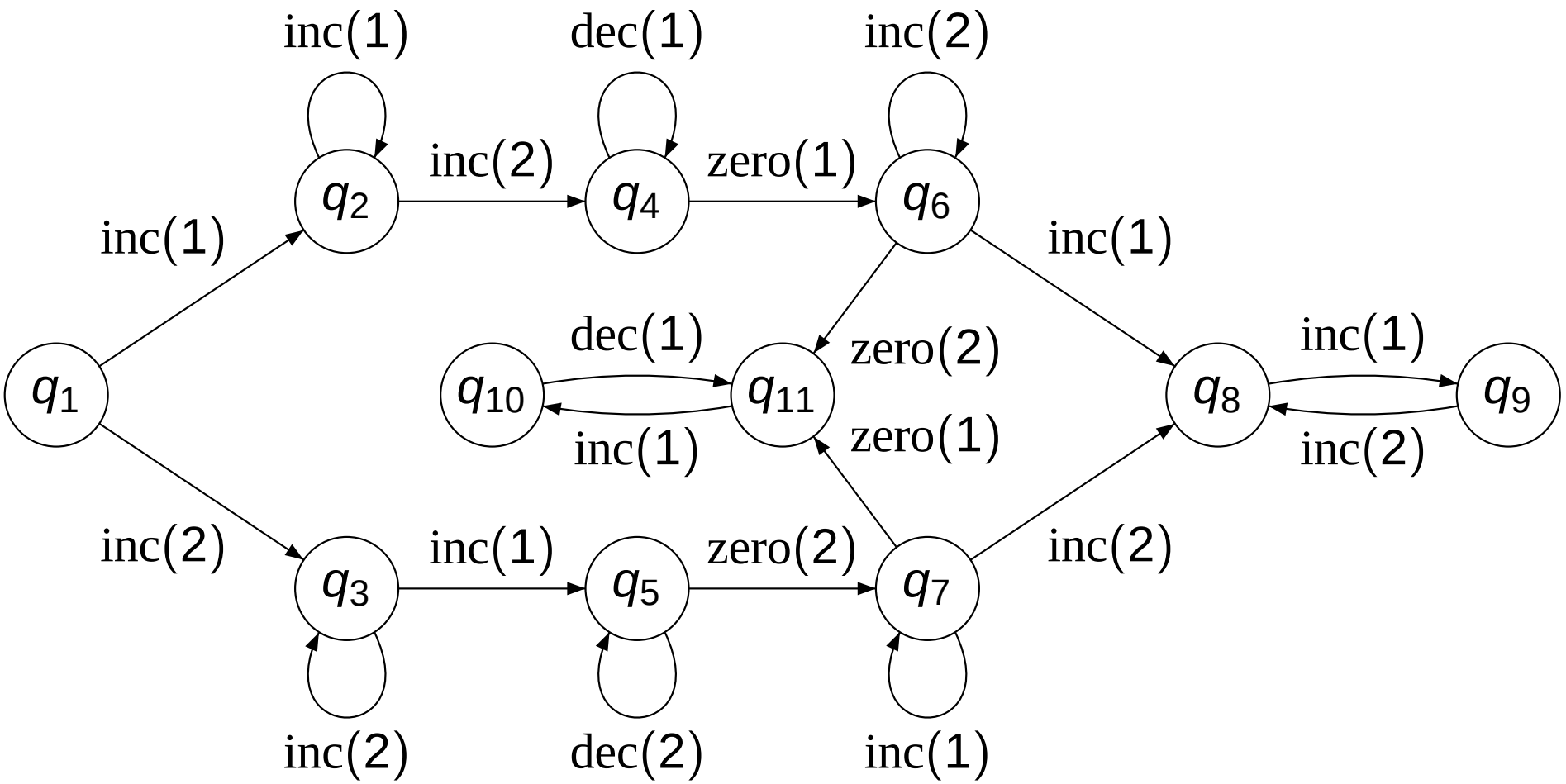
Reversal-Bounded Counter Automata

Reversals



- 6 phases, 3 biphases, and 5 reversals.
- Initialized CA $(\mathcal{S}, (q, \vec{x}))$ is r-reversal-bounded $\overset{\text{def}}{\Leftrightarrow}$ every run from $(q, \vec{x})$ has strictly less than $r + 1$ reversals.

  [Ibarra, JACM 78]
- $\mathcal{S}$ is globally reversal-bounded $\overset{\text{def}}{\Leftrightarrow}$ there is r such that every initialized CA defined from $\mathcal{S}$ is r-reversal-bounded.



Is $(\mathcal{S}, (q_1, \vec{0}))$ reversal-bounded?
For which q, every $(\mathcal{S}, (q, \vec{x}))$ is reversal-bounded?
Show that for every q, $\{\vec{x} \in \mathbb{N}^2 : (\mathcal{S}, (q, \vec{x}))$ is RB$\}$ is semilinear.

Reversal-boundedness detection problem

- Reversal-bounded counter automata are not defined syntactically.

- REVERSAL-BOUNDEDNESS DETECTION PROBLEM

  Input: Initialized CA $(\mathcal{S}, (q, \vec{x}))$.
  Question: Is $(\mathcal{S}, (q, \vec{x}))$ reversal-bounded?

- Reversal-boundedness detection problem is undecidable.
  [Ibarra, JACM 78]

- Checking whether an initialized CA is r-reversal-bounded is undecidable too.

- Restriction to VASS is decidable (and for variants too).
  [Finkel & Sangnier, MFCS'08]

Simple undecidability proof

- Minsky machine $\mathcal{S}$ with halting instruction q_h: `halt`.
- Either $\mathcal{S}$ has a unique infinite run (and never visits q_h) or $\mathcal{S}$ has a finite run (and halts at q_h).
- Counter automaton $\mathcal{S}'$ (dim. 3): replace $t = q_i \xrightarrow{\varphi} q_j$ by

$$q_i \xrightarrow{\mathrm{inc}(3)} q_{1,t}^{new} \xrightarrow{\mathrm{dec}(3)} q_{2,t}^{new} \xrightarrow{\varphi} q_j$$

- We have the following equivalences:
  - $\mathcal{S}$ halts.
  - For $\mathcal{S}'$, q_h can be reached from $(q, \vec{0})$.
  - Unique run of $\mathcal{S}'$ starting by $(q, \vec{0})$ is finite.
  - $\mathcal{S}'$ is reversal-bounded from $(q, \vec{0})$.

Semilinearity

- Theorem: [Ibarra, JACM 78] Let $(\mathcal{S}, (q_0, \vec{x_0}))$ be r-reversal-bounded. For each $q \in Q$, the set $\{\vec{x} \in \mathbb{N}^n : (q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{x})\}$ is effectively semilinear.

- REACHABILITY PROBLEM WITH BOUNDED NUMBER OF REVERSALS

  Input: CA $\mathcal{S}$, $(q, \vec{x})$, $(q', \vec{x'})$ and $r \geq 0$.

  Question: Is there a run $(q, \vec{x}) \xrightarrow{*} (q', \vec{x'})$ s.t. each counter performs during the run a number of reversals bounded by r?

- Corollary: The reachability problem with bounded number reversals is decidable.

Let's prove the corollary from the theorem !

- Instance: $\mathcal{S}$, $(q, \vec{x})$, $(q', \vec{x'})$, $r \geq 0$.
- Let us build $\mathcal{S}' = (Q', n, \delta')$, globally r-reversal bounded by construction.
- $Q' \stackrel{\text{def}}{=} Q \times \{\text{DEC}, \text{INC}\}^n \times [0, r]^n$.
- $(q, \vec{mode}, \sharp\vec{alt}) \xrightarrow{\varphi} (q', \vec{mode}', \sharp\vec{alt}') \in \delta' \stackrel{\text{def}}{\Leftrightarrow} q \xrightarrow{\varphi} q' \in \delta$ and for each $i \in [1, n]$:

φ	$\vec{mode}(i)$	$\vec{mode}'(i)$	$\sharp\vec{alt}'(i)$
$\text{dec}(i)$	DEC	DEC	$\sharp\vec{alt}(i)$
$\text{dec}(i)$	INC	DEC	$\sharp\vec{alt}(i) + 1$ and $\sharp\vec{alt}(i) < r$
$\text{inc}(i)$	INC	INC	$\sharp\vec{alt}(i)$
$\text{inc}(i)$	DEC	INC	$\sharp\vec{alt}(i) + 1$ and $\sharp\vec{alt}(i) < r$
$\text{zero}(i)$	DEC	DEC	$\sharp\vec{alt}(i)$
$\text{zero}(i)$	INC	INC	$\sharp\vec{alt}(i)$

Main equivalence

- By construction, $\mathcal{S}'$ is globally r-reversal-bounded.
- Equivalence:
  1. In $\mathcal{S}$, there is a run $(q, \vec{x}) \xrightarrow{*} (q', \vec{x'})$ such that each counter has at most r reversals,
  2. In $\mathcal{S}'$, $(q, \vec{\mathrm{INC}}, \vec{0}, \vec{x}) \xrightarrow{*} (q', \vec{mode}, \vec{\sharp alt}, \vec{x'})$ for some $\vec{mode}$, $\vec{\sharp alt}$.
- The values $\vec{mode}$, $\vec{\sharp alt}$ belong to finite sets.

Completing the proof

- By the above theorem,

$$X_{(\vec{mode},\sharp\vec{alt})} = \{\vec{y} \in \mathbb{N}^n : (q, \vec{\mathrm{INC}}, \vec{0}, \vec{x}) \xrightarrow{*} (q', \vec{mode}, \sharp\vec{alt}, \vec{y})\}$$

  is effectively semilinear.

- $\mathrm{REL}(\varphi_{(\vec{mode},\sharp\vec{alt})}) = X_{(\vec{mode},\sharp\vec{alt})}$ for some $\varphi_{(\vec{mode},\sharp\vec{alt})}$.

- $\vec{x} \in X_{(\vec{mode},\sharp\vec{alt})}$ is equivalent to satisfaction of

$$(\bigwedge_{i=1}^{i=n} \mathrm{x}_i = \vec{x}(i)) \wedge \varphi_{(\vec{mode},\sharp\vec{alt})}.$$

- Since the satisfiability problem for Presburger arithmetic is decidable, we get an algorithm to solve the reachability problem with bounded number of reversals. (disjunctions of at most $2^n(1+r)^n$ disjuncts).

Encodings of reversal-bounded counter automata

1. r-reversal-bounded initialized CA, reversal-boundedness being established in some unspecified way.
2. r-reversal-bounded initialized VASS (reversal-boundedness detection problem is decidable).
3. Initialized CA $\mathcal{S}$ with a bound $r \geq 0$ and we assume that this encodes the restriction of $\mathcal{S}$ with at most r reversals.

Classes of counter systems with semilinear reachability sets

- $\mathrm{REL}(\varphi) = \{\vec{x} \in \mathbb{N}^n : (q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{x})\}$.
- φ allows to answer questions about the set of configurations reachable from $(q_0, \vec{x_0})$.
- Sometimes, we also need effective semilinearity of reachability relations, for instance for answering more general questions of the form

$$\exists\, \vec{x}, \vec{y}\ (q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{x}) \xrightarrow{*} (q', \vec{y}) \quad \text{and} \quad \vec{x}, \vec{y} \models \psi\ ?$$

- Examples of classes with semilinear reachability sets:
  1. VASS with dimension ≤ 2. [Hopcroft & Pansiot, TCS 79]
  2. Communication-free Petri nets. [Esparza, FI 97]
  3. Flat relational counter systems. [Comon & Jurski, CAV'98]
  4. Flat affine counter systems with finite monoids. [Boigelot, PhD 98; Finkel & Leroux, FST&TCS'02]
  5. See also recent [Bozga & Iosif & Konečný, CAV'10]

Main Proof for Effective Semilinearity

Three main parts

- Reduction to semilinearity for 1-reversal-bounded CA.
- Using finite-state automata to overapproximate reachability sets for 1-reversal-bounded CA.

$$(q_0, \vec{x_0}) \xrightarrow{a_1} (q_1, \vec{x_1}) \xrightarrow{a_2} (q_2, \vec{x_2}) \cdots \xrightarrow{a_k} (q_k, \vec{x_k})$$

$u = a_1 \cdots a_k \in \Sigma^*$ with $\Sigma = \{\mathrm{zero}(i), \mathrm{inc}(i), \mathrm{dec}(i) : i \in [1, n]\}$

  - For $i \in [1, n]$, $\vec{x_k}(i) = \vec{x_0} + \Pi(u)(\mathrm{inc}(i)) - \Pi(u)(\mathrm{dec}(i))$.
  - Parikh image of u uniquely defines $\vec{x_k}$.

- Exact values regained since Parikh images of context-free languages are effectively semilinear (see details later).

Parikh image

- $\Sigma = \{a_1, \ldots, a_k\}$ with ordering $a_1 < \cdots < a_k$.
- Parikh image of $u \in \Sigma^*$: $\Pi(u) \stackrel{\text{def}}{=} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix} \in \mathbb{N}^k$ where each n_j is the number of occurrences of a_j in u.
- Parikh image of $a\ b\ a\ a\ b$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
- Definition for Parikh image extends to languages.
- The Parikh image of any context-free language is semilinear. [Parikh, JACM 66]
- Effective computation from pushdown automata.

Two simple properties

- Control graph of $\mathcal{S}$ allows to perform the sequence of instructions u from q_0 to q_k.
- By 1-reversal-boundedness, the projection of u on $\Sigma_i = \{\mathrm{inc}(i), \mathrm{dec}(i), \mathrm{zero}(i)\}$ belongs to

$$\mathrm{L}_i = \mathrm{zero}(i)^* \cdot \mathrm{inc}(i)^* \cdot \mathrm{dec}(i)^* \cdot \mathrm{zero}(i)^*$$

- Finite-state automaton $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$:
  - $\mathcal{A}_1$ built from $\mathcal{S}$ with symbolic alphabet Σ.
  - $\mathcal{A}_2$ guarantees that projection on Σ_i is in L_i.
- By Parikh's Theorem, the set below is effectively semilinear:

$$\{\vec{x_0} + (\Pi(u)(\mathrm{inc}(1)), \ldots, \Pi(u)(\mathrm{inc}(n))) -$$
$$(\Pi(u)(\mathrm{dec}(1)), \ldots, \Pi(u)(\mathrm{dec}(n))) : u \in \mathrm{L}(\mathcal{A})\}$$

Three more properties

- Counter values are nonnegative:
  - For every prefix v of u_{Σ_i},
    $\vec{x_0}(i) + \Pi(v)(\mathrm{inc}(i)) - \Pi(v)(\mathrm{dec}(i)) \geq 0$.
  - It sufficient to check $\vec{x_0}(i) + \Pi(u)(\mathrm{inc}(i)) - \Pi(u)(\mathrm{dec}(i)) \geq 0$.
- $\vec{x_0}(i) \neq 0$ implies the first letter of u_{Σ_i} is different from $\mathrm{zero}(i)$.
- Last letter of u_{Σ_i} equal to $\mathrm{zero}(i)$ implies

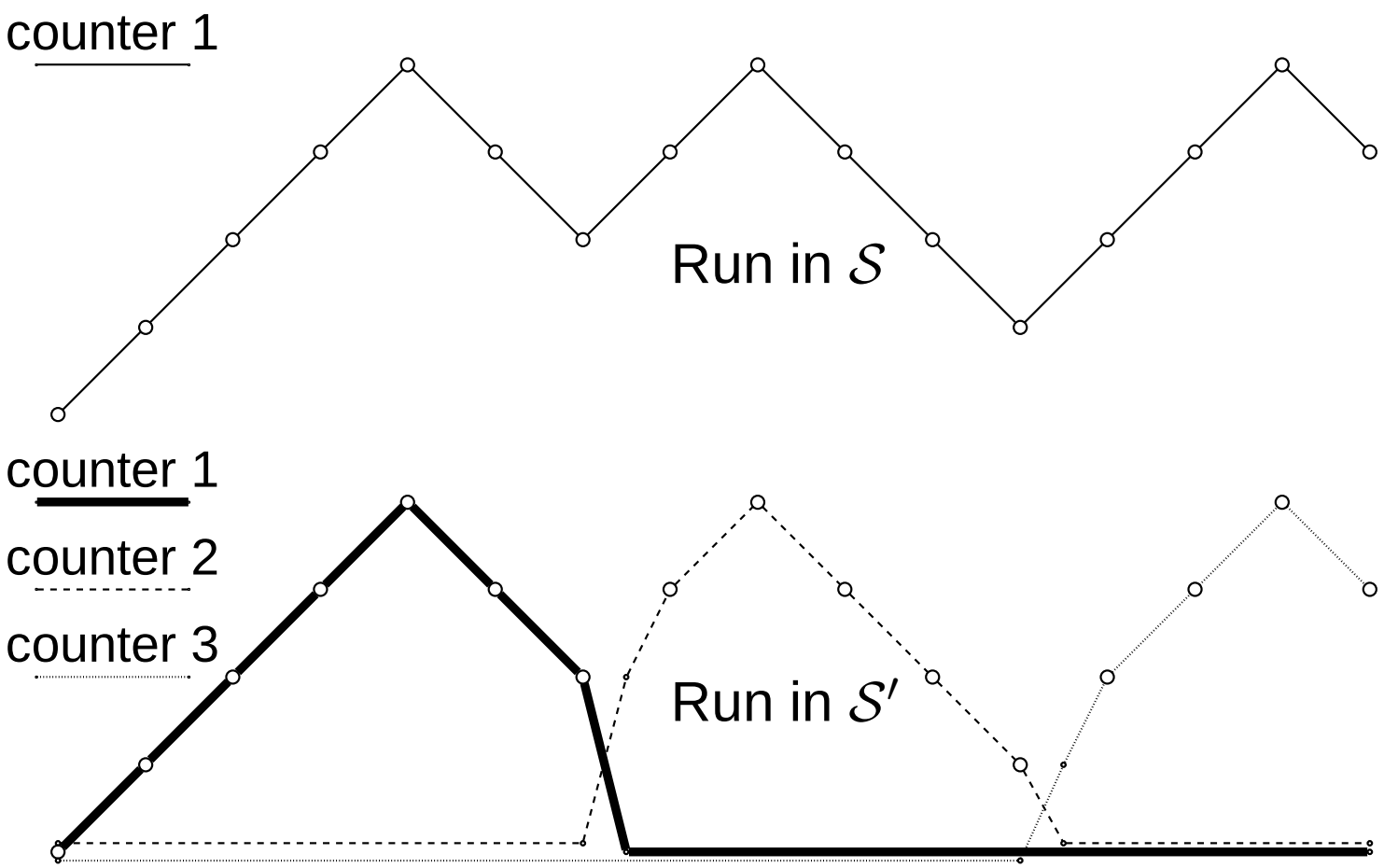
$$\vec{x_0}(i) + \Pi(u)(\mathrm{inc}(i)) - \Pi(u)(\mathrm{dec}(i)) = 0.$$

Initial states of $\mathcal{A}$ and effective semilinearity of $\mathrm{L}(\mathcal{A})$ allow to encode these properties.

Reduction to 1-reversal-boundedness

- Principle as for reversal-bounded multistack systems.

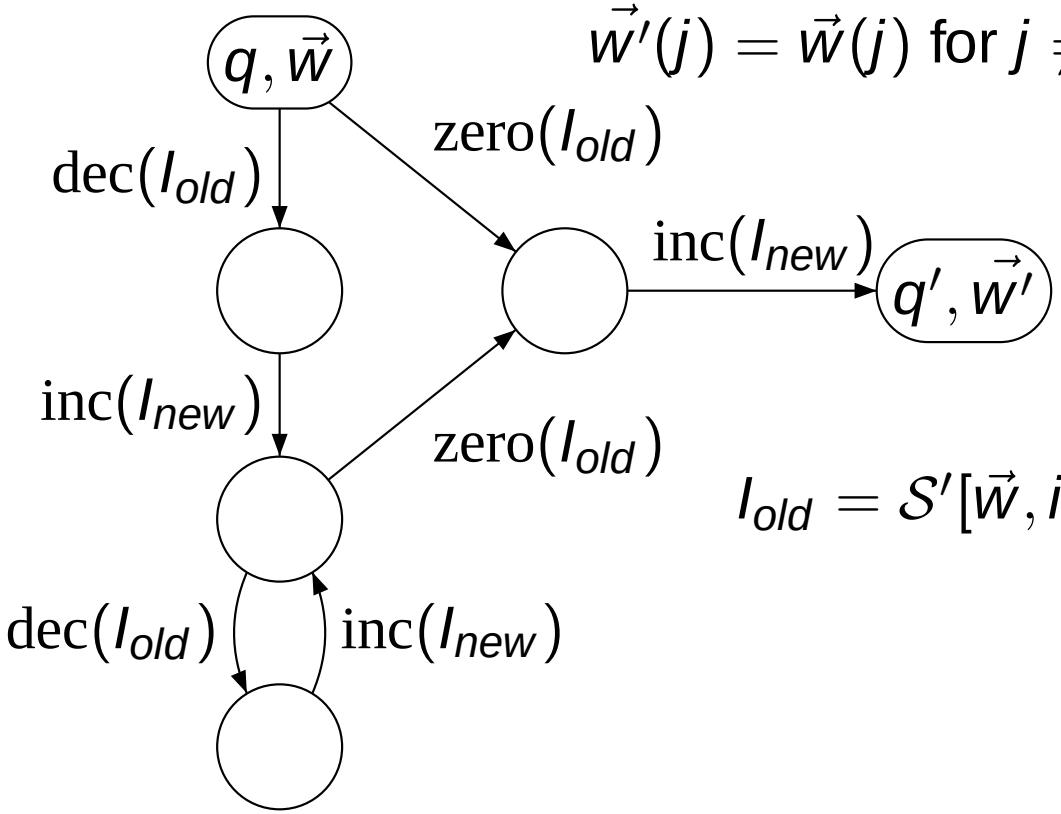
[Baker & Book, JCSS 74]



From $\mathcal{S}$ to $\mathcal{S}'$

- From Q to a superset of $Q \times (\{\mathrm{INC}, \mathrm{DEC}\} \times [0, \frac{r}{2}])^n$.
- From $\vec{w} \in (\{\mathrm{INC}, \mathrm{DEC}\} \times [0, \frac{r}{2}])^n \ni \vec{w}$ and $i \in [1, n]$, we define $\mathcal{S}'[\vec{w}, i] \in [1, n \times (1 + \frac{r}{2})]$.
  ("active" counter in $\mathcal{S}'$ corresponding to counter i in $\mathcal{S}$)
- $q \xrightarrow{\mathrm{inc}(i)} q' \in \delta$, $\vec{w} \in (\{\mathrm{INC}, \mathrm{DEC}\} \times [0, \frac{r}{2}])^n$, $\vec{w}(i) = (\mathrm{DEC}, l)$.

$\vec{w}'(j) = \vec{w}(j)$ for $j \neq i$, $\vec{w}'(i) = (\mathrm{INC}, l + 1)$



$l_{old} = \mathcal{S}'[\vec{w}, i]$, $l_{new} = \mathcal{S}'[\vec{w}', i]$

Semilinearity (I)

- If for $(q, \vec{w}) \in Q \times (\{\mathrm{INC}, \mathrm{DEC}\} \times [0, \frac{r}{2}])^n$, the set

$$\{\vec{x} : ((q_0, (\vec{\mathrm{INC}}, 0)), \vec{x_0'}) \xrightarrow{*} ((q, \vec{w}), \vec{x}) \text{ in } \mathcal{S}'\}$$

  is effectively semilinear, then

$$\{\vec{x} \in \mathbb{N}^n : (q_0, \vec{x}_0) \xrightarrow{*} (q, \vec{x}) \text{ in } \mathcal{S}\}$$

  is effectively semilinear too, for every control state q.

- Formula for configurations with control state $(q, \vec{w})$ reachable from $((q_0, (\vec{\mathrm{INC}}, 0)), \vec{x_0'})$:

$$\varphi_{(q,\vec{w})}(\mathrm{x}_1, \ \ldots, \mathrm{x}_{n'})$$

- Formula for configurations with control state q reachable from $(q_0, \vec{x}_0)$:

$$\bigvee_{\vec{w} \in [0, \frac{r}{2}]^n} (\exists\, \mathrm{y}_1 \cdots \mathrm{y}_{n'} \ \varphi_{(q,\vec{w})}(\mathrm{y}_1, \ \ldots, \mathrm{y}_{n'}) \wedge (\bigwedge_{i \in [1,n]} \mathrm{x}_i = \mathrm{y}_{\mathcal{S}'[\vec{w},i]})).$$

Semilinearity (II)

- If $\mathcal{S}$ is globally r-reversal-bounded and the reachability relation for $\mathcal{S}'$ is Presburger definable, then the reachability relation for $\mathcal{S}$ is Presburger definable too.

- Formula $\varphi_{q,q'}(\mathsf{z}_1, \ldots, \mathsf{z}_{n'}, \mathsf{z}'_1, \ldots, \mathsf{z}'_{n'})$.

- From q to q':

$$\bigvee_{\vec{w},\vec{w}' \in [0,\frac{r}{2}]^n} (\exists\, \mathsf{x}_1 \cdots \mathsf{x}_{n'},\ \mathsf{y}_1 \cdots \mathsf{y}_{n'}\ \varphi_{(q,\vec{w}),(q',\vec{w}')}$$

$$\wedge(\bigwedge_{i \in [1,n]} \mathsf{z}'_i = \mathsf{y}_{\mathcal{S}'[\vec{w},i]}) \wedge (\bigwedge_{i \in [1,n]} \mathsf{z}_i = \mathsf{x}_{\mathcal{S}'[(\vec{\mathrm{INC}},0),i]}) \wedge (\bigwedge_{j \in NA} \mathsf{x}_j = 0)).$$

where $NA = ([1,n'] \setminus \{\mathcal{S}'[(\vec{\mathrm{INC}},0),i] : i \in [1,n]\})$ (set of initial “nonactive” counters).

Completing the part about 1-reversal-bounded CA

- Finite-state automaton $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ over Σ.

- By Parikh Theorem, for $(q, \vec{v}) \in Q'$, one can compute

$$\varphi^{(q_0,\vec{v_0})}_{(q,\vec{v})}(\mathrm{x}^1_{inc}, \mathrm{x}^1_{dec}, \mathrm{x}^1_{zero}, \ldots, \mathrm{x}^n_{inc}, \mathrm{x}^n_{dec}, \mathrm{x}^n_{zero})$$

  s.t. for every $\mathbf{v}$, we have $\mathbf{v} \models \varphi^{(q_0,\vec{v_0})}_{(q,\vec{v})}$ iff there is an accepted u s.t. $\Pi(u) = (\mathbf{v}(\mathrm{x}^1_{inc}), \ldots, \mathbf{v}(\mathrm{x}^n_{zero}))$.

- For $q \in Q$,

$$\bigvee_{\vec{v} \in S^{\vec{v_0}}} \exists \mathrm{x}^1_{inc}, \cdots, \mathrm{x}^n_{zero} \; (\varphi^{(q_0,\vec{v_0})}_{(q,\vec{v})}(\mathrm{x}^1_{inc}, \ldots, \mathrm{x}^n_{zero})$$

$$\wedge(\bigwedge_{i \in [1,n] \text{ s.t. } \vec{v}(i) \in \{\xrightarrow{0}_1, \xrightarrow{0}_2\}} \mathrm{y}_i = 0)) \wedge (\bigwedge_{i \in [1,n]} \mathrm{y}_i = \mathrm{x}^i_{inc} + \vec{x_0}(i) - \mathrm{x}^i_{dec})$$

Global reversal-boundedness

Let $\mathcal{S}$ be a globally 1-reversal-bounded CA. For $q, q' \in Q$, one can effectively compute $\varphi_{q,q'}(\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y}_1, \ldots, \mathrm{y}_n)$ such that for $\mathbf{v}$, we have

$$\mathbf{v} \models \varphi_{q,q'} \text{ iff } (q, (\mathbf{v}(\mathrm{x}_1), \ldots, \mathbf{v}(\mathrm{x}_n))) \xrightarrow{*} (q', (\mathbf{v}(\mathrm{y}_1), \ldots, \mathbf{v}(\mathrm{y}_n))).$$

Parikh image of regular languages

- Directed graph $G = (V, E)$ and $f : E \to \mathbb{N}$.

- f corresponds to a path iff
  1. the subgraph induced by f is connected.
  2. The number of edges entering in a node is equal to the number of edges going out of the node, except possibly for two extremity nodes.
  3. If the initial node is different from final node, the number of edges entering in the the initial node is one less than the number of edges outgoing out the the initial node.
  4. Similar condition for the terminal node, if any.

- These conditions can be expressed as a finite disjunction of equations in Presburger arithmetic.

Building the Presburger formula

- Finite-state automaton $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$.
- Variables x_a for $a \in \Sigma$ and $\mathsf{x}_{t'}$ for $t \in \delta$.
- Presburger formula of the form:

$$\exists \mathsf{x}_{t_1} \cdots \mathsf{x}_{t_{k'}} \Big(\bigwedge_{i=1}^{k} \mathsf{x}_{a_i} = \sum_{\Sigma(t)=a_i} \mathsf{x}_t \Big) \wedge$$

$$\Big(\bigvee_{q_0 \in Q_0, q_f \in F} \quad \bigvee_{\text{connected } (Q',\delta'),\ q_0,q_f \in Q'}$$

$$\varphi_{(Q',q_0,q_f,\delta')} \wedge \Big(\bigwedge_{t \in \delta'} \mathsf{x}_t > 0 \Big) \wedge \Big(\bigwedge_{t \in (\delta \setminus \delta')} \mathsf{x}_t = 0 \Big) \Big)$$

- For instance, if $q_0 = q_f$, then

$$\varphi_{(Q',q_0,q_f,\delta')} \stackrel{\text{def}}{=} \bigwedge_{q'' \in Q'} \Big(\sum_{t \in \delta' \text{ s.t. } end(t)=q''} \mathsf{x}_t - \sum_{t \in \delta' \text{ s.t. } beg(t)=q''} \mathsf{x}_t = 0. \Big)$$

Recapitulation

- Reduction to semilinearity for 1-reversal-bounded CA.
- Approximation of reachability sets for 1-reversal-bounded CA by using FSA.
- Exact values regained since Parikh images of regular languages are effectively semilinear.

Variants

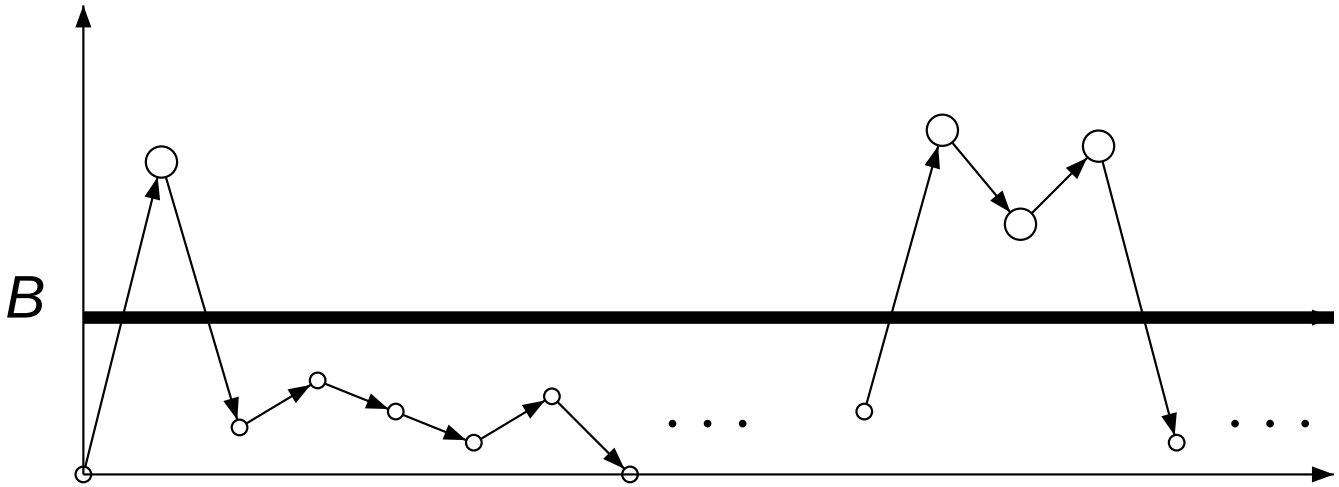
Reversal-boundedness with one free counter

Same results for effective semilinearity apply.

- $\mathcal{S} \mapsto \mathcal{S}'$ that is 1-reversal-bounded with at most one free counter.
- $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ with one-counter automaton $\mathcal{A}_1$.
- $\mathcal{A}_1$ is a pushdown system and Parikh's Theorem applies to context-free languages.
- So, the proof works smoothly by adding one free counter. (and by using Parikh's Theorem)

Weak reversal-boundedness
[Finkel & Sangnier, MFCS'08]

- Reversals are recorded only above a bound B:



- Same results for effective semilinearity apply.

- Whenever a counter value is below B, this can be encoded in the control states.

Decidable Reachability Problems

Decidability

- Control state repeated reachability problem restricted to reversal-bounded initialized counter automata is decidable.
  [Dang & Ibarra & San Pietro, FSTTCS'01]

- $\exists$-PRESBURGER INFINITELY OFTEN PROBLEM

  Input: Initialized CA $(\mathcal{S}, (q, \vec{x}))$ of dimension n that is r-reversal-bounded and a temporal formula of the form $\psi = \mathtt{GF}\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ where φ is a Presburger formula on counters.

  Question: Is there an infinite run from $(q, \vec{x})$ satisfying ψ?

- $\exists$-Presburger infinitely often problem is decidable.
  [Dang & San Pietro & Kemmerer, TCS 03]

Idea of the proof
(for control state repeated reachability problem)

- r-reversal-bounded initialized CA $(\mathcal{S}, (q_0, \vec{x_0}))$ and $q_f \in Q$.
- Property $(\star)$: there is an infinite run from $(q_0, \vec{x_0})$ such that q_f is repeated infinitely often.
- We reduce $(\star)$ to a reachability question for a new reversal-bounded counter automaton $\mathcal{S}'$.
- Property $(\star\star)$: there exist $\rho = (q_0, \vec{x_0}) \xrightarrow{t_1} (q_1, \vec{x_1}) \cdots \xrightarrow{t_l} (q_l, \vec{x_l})$ and $l' \in [0, l-1]$ s.t.
  1. $q_l = q_{l'} = q_f$,
  2. $\vec{x_{l'}} \preceq \vec{x_l}$,
  3. if X is the set of counters tested to zero between $(q_l, \vec{x_l})$ and $(q_{l'}, \vec{x_{l'}})$, then $\vec{x_{l'}}(X) = \vec{x_l}(X) = \vec{0}$.

Equivalence

- $(\star)$ is equivalent to $(\star\star)$.
- $(\star\star)$ implies $(\star)$:
  - $\rho = (q_0, \vec{x_0}) \xrightarrow{t_1} (q_1, \vec{x_1}) \cdots \xrightarrow{t_l} (q_l, \vec{x_l})$.
  - $\rho' s$ is defined with $t_1 \cdots t_{l'} (t_{l'+1} \cdots t_l)^\omega$.
  - q_f is repeated infinitely often.
  - Zero-tests are also successful.

($\star$) implies ($\star\star$)

- $\rho = (q_0, \vec{x_0}) \xrightarrow{t_1} (q_1, \vec{x_1}) \xrightarrow{t_2} (q_2, \vec{x_2}) \cdots$ with q_f repeated infinitely often.

- X: set of counters that are successfully tested to zero in ρ infinitely often.

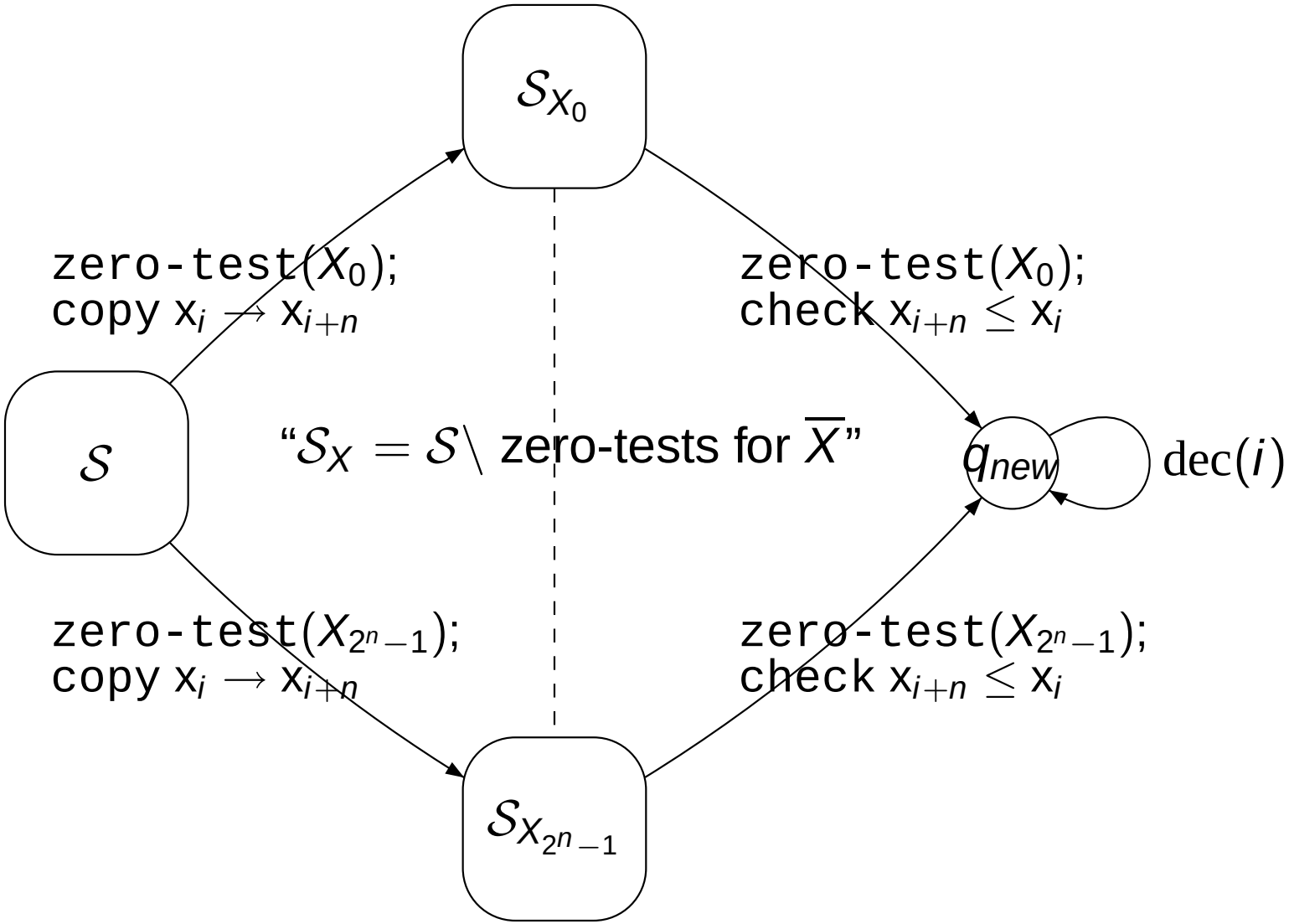
- By reversal-boundedness, there is $I \geq 0$ s.t. for $k \geq I$, we have $\vec{x_k}(X) = \vec{0}$.

- There exists $I \leq k_1 < k_2 < k_3 < \ldots$ s.t. for $1 \leq j < j'$, we have $q_{k_j} = q_f$ and between $(q_{k_j}, \vec{x_{k_j}})$ and $(q_{k_{j'}}, \vec{x_{k_{j'}}})$, exactly the counters in X are tested to zero.

- By Dickson's Lemma, there exists $J < J'$ such that $\vec{x_{k_J}} \preceq \vec{x_{k_{J'}}}$.

Reduction to a reachability question

$\mathcal{S}' = (Q', q_0, 3 \times n, \delta')$ s.t. $(\star\star)$ iff $(q_0, \vec{x_0}) \xrightarrow{*} (q_{new}, \vec{0})$ in $\mathcal{S}'$.



Construction of $\mathcal{S}'$

- Let $\mathcal{S}' = (Q', q_0, 3 \times n, \delta')$ s.t. $(\star\star)$ iff $(q_0, \vec{x_0}) \xrightarrow{*} (q_{new}, \vec{0})$ in $\mathcal{S}'$.

- One can effectively build φ s.t.

$$\mathrm{REL}(\varphi) = \{\vec{x} : (q_0, \vec{x_0}) \xrightarrow{*} (q_{new}, \vec{x})\}$$

- $\mathcal{S}'$ is made of $2^n + 1$ copies of $\mathcal{S}$ plus some extra control states such as q_{new}.

- It includes an initial distinguished copy of $\mathcal{S}$.

- For $X \subseteq [1, n]$, the control states of the X-copy are among $Q \times \{X\} \times \mathcal{P}(X)$.

- Third component records the counters that have been tested to zero since the run has entered in the X-copy.

Entering into the X-copy

- For $X \subseteq [1, n]$, we consider a sequence of transitions from q_f to $(q_f, X, \emptyset)$ whose effect is to perform a zero-test on counters in X and to copy the value of each counter $i \in \overline{X}$ into the counter $n + i$.

- copy $\mathrm{x}_i \rightarrow \mathrm{x}_{i+n}$:
  1. Decrement the counter i until zero and for each decrement, the counters $n + i$ and $2n + i$ are incremented.
  2. When counter i is equal to zero, decrement the counter $2n + i$ until zero while incrementing the counter i at each step.
  3. The number of reversals is at most augmented by 2.

Transitions in the X-copy

- $(q, X, Y) \xrightarrow{\varphi} (q', X, Y')$ is a transition whenever there is a transition $q \xrightarrow{\varphi'} q'$ in $\mathcal{S}$ for which
  - φ performs the same instruction as φ',
  - for $i \in \overline{X}$, φ' is a not a zero-test on i,
  - if $\varphi = \mathrm{zero}(j)$, then $Y' = Y \cup \{j\}$ otherwise $Y' = Y$.
- When all the counters in X have been tested to zero at least once and q_f is reached, we may jump to q_{new}.

Final step

- Consider a sequence of transitions from (q_f, X, X) to q_{new} performing the following tasks:

  1. for $i \in X$, perform a zero-test on counter i,
  2. for $i \in \overline{X}$, test whether the counter value for i is greater or equal to the counter value for $n + i$,
  3. empty all the counters.

- check $\mathrm{x}_{i+n} \leq \mathrm{x}_i$: decrement i and $n + i$ simultaneously and nondeterministically test whether the counter $n + i$ has value zero.

- $(\mathcal{S}', (q_0, \vec{x_0}))$ is $(r + 3)$-reversal-bounded.

Undecidable Model-Checking Problems

Universal problem for one-counter automaton

- One-counter automaton with alphabet: FSA + 1 counter.
- The universal problem for 1-reversal-bounded one-counter automata with alphabet is undecidable [Ibarra, MST 79].
- One-counter automata with alphabet defines context-free languages.

A simple undecidable temporal fragment

- The $\exists$-PRESBURGER-ALWAYS PROBLEM:

  Input: Initialized CA $(\mathcal{S}, (q, \vec{x}))$ that is r-reversal-bounded and a formula $\psi = \mathsf{G}\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ where φ is a Presburger formula on counters.

  Question: Is there an infinite run from $(q, \vec{x})$ satisfying ψ?

- The $\exists$-Presburger-always problem for reversal-bounded counter automata is undecidable.

  [Dang & San Pietro & Kemmerer, TCS 03]

- By reduction from halting problem for Minsky machines: one counter is encoded by two increasing counters, counting the number of increments and decrements, respectively.

Reduction from the halting problem

- Proof analogous to the undecidability of the reachability problem for reversal-bounded CA augmented with guards $x_i = x_{i'}$ and $x_i \neq x_{i'}$. [Ibarra et al., TCS 02]
- Given a Minsky machine $\mathcal{S}$ with halting state q_h, we build a 0-reversal-bounded counter automaton $\mathcal{S}'$ such that
  - counter i in $\mathcal{S}'$ records the increments of counter i in $\mathcal{S}$,
  - counter $i+2$ in $\mathcal{S}'$ records the decrements of counter i in $\mathcal{S}$.
  - zero-test on counter i in $\mathcal{S}$ is simulated by formula $x_i = x_{i+2}$.
- W.l.o.g., we can assume that
  - $\mathcal{S} = (Q, 2, \delta)$ is a deterministic CA,
  - Halting control states in $Q_h \subseteq Q$,
  - $Q_1, Q_2 \subseteq Q$ contains exactly the control states that are reached after zero-tests.

Building $\mathcal{S}'$ by erasing zero-tests

- 0-reversal-bounded CA $\mathcal{S}' = (Q, 5, \delta')$:
  - $q \xrightarrow{\text{inc}(i)} q' \in \delta$ implies $q \xrightarrow{\text{inc}(i)} q' \in \delta'$.
  - $q \xrightarrow{\text{dec}(i)} q' \in \delta$ implies $q \xrightarrow{\text{inc}(i+2)} q' \in \delta'$.
  - $q \xrightarrow{\text{zero}(i)} q' \in \delta$ implies $q \xrightarrow{\text{inc}(5)} q' \in \delta'$.
- No halting control state is reached from $(q, \vec{0})$ in $\mathcal{S}$ iff there is an infinite run from $(q, \vec{0})$ in $\mathcal{S}'$ satisfying

$$\mathtt{G}(\overbrace{\bigwedge_{i\in\{1,2\}} \bigwedge_{q\in Q_i} (q \Rightarrow \mathsf{x}_i = \mathsf{x}_{i+2})}^{\text{simulation of zero-tests}}) \wedge \mathtt{G}(\overbrace{\bigwedge_{i\in\{1,2\}} \mathsf{x}_i \geq \mathsf{x}_{i+2}}^{\text{no negative counter values}}) \wedge \mathtt{G}(\overbrace{\bigwedge_{q\in Q_h} \neg q}^{\text{no halting state reached}})$$

- Control states can be eliminated by adding increasing counters whose differences encode control states.

Final remarks

- Reversal-bounded counter automata has effective semilinear reachability sets.

- Decidability for model-checking $\mathrm{LTL}^{\mathrm{CS}}(\mathrm{PrA})$ fragments as well as undecidability results.

How to define maximal decidable fragments ?

- Decidability results can be extended to variants: one free counter, lower bound to count the reversals, addition of parameters (not presented here), etc..

- Open problem: Characterizing the computational complexity of the reachability problem with bounded number of reversals when integers are encoded in binary.

Conclusion

- Today's lecture:
  - Definitions about reversal-bounded CA.
  - Proof for effective semilinearity.
  - Repeated reachability and related problems.
- Tomorrow's lecture:
  - Repeated reachability problem for several classes of CS.
  - Plain LTL for several classes of counter systems.
  - $\mathrm{LTL}^{\mathrm{CS}}(\mathrm{PrA})$ for admissible counter systems.