Covering and Boundedness Problems for Branching Vector Addition Vectors

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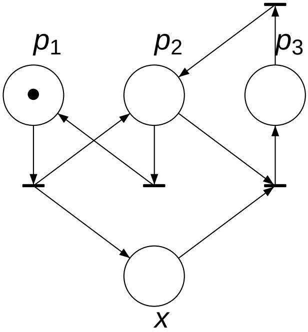
Vector Addition Systems

Petri nets, VASS and VAS
(zero-test free models with ℕ-counters)



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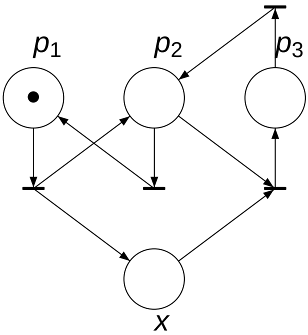

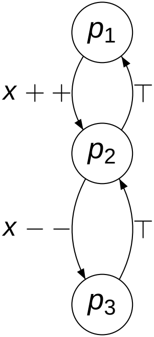

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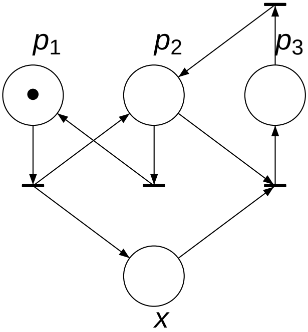

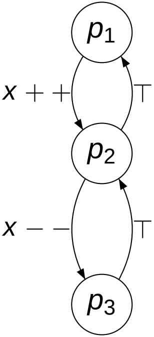

(p_1, p_2, p_3, x)

$(-1, 1, 0, 1)$

$(1, -1, 0, 0)$

$(0, -1, 1, -1)$

$(0, 1, -1, 0)$

Vector Addition Systems

- VAS $\mathcal{V} = (k, \mathbf{a}, R)$:
  - $k \in \mathbb{N}$ (*dimension*), $\mathbf{a} \in \mathbb{N}^k$ (*axiom*).
  - R is a finite subset of $\mathbb{Z}^k$ (*unary rules*).

- Derivation $\mathcal{D}$:

$$\dfrac{\mathbf{v}}{\mathbf{v} + r_1}\ (r_1)$$
$$\vdots$$
$$\dfrac{\mathbf{v} + (r_1 + \cdots + r_{\alpha-1})}{\mathbf{v} + (r_1 + \cdots + r_\alpha)}\ (r_\alpha)$$

- *Initialized* derivation: $\mathbf{v} = \mathbf{a}$.

- *Admissible* derivation: all derived values are in $\mathbb{N}$.

- $\mathcal{V}$ produces/derives configurations in $\mathbb{N}^k$.

Decision problems

- Covering problem: given a VAS $\mathcal{V}$ and $\mathbf{t} \in \mathbb{N}^k$, is there $\mathbf{v}$ produced by $\mathcal{V}$ such that $\mathbf{v} \geq \mathbf{t}$?
  Control state reachability problem with $\mathbf{t} = (1, 0, \ldots, 0)$.

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  Control state reachability problem with $\mathbf{t} = (1, 0, \ldots, 0)$.
- Boundedness problem: given a VAS $\mathcal{V}$, is the set of configurations produced by $\mathcal{V}$ finite?
- Reachability problem: given a VAS $\mathcal{V}$ and $\mathbf{t} \in \mathbb{N}^k$, is $\mathbf{t}$ produced by $\mathcal{V}$?

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  - EXPSPACE upper bound for path sublogic [Faouzi Atig & Habermehl, RP 09], correcting [Yen, IC 92].

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- Reachability problem for VAS is decidable
  [Mayr, STOC 81; Kosaraju, STOC 82; Reutenauer, 89]
  - No primitive recursive algorithm is known.
  - EXPSPACE-hardness [Lipton, TR 76].
  - "Equivalent" to satisfiability for FO2 over data words [Bojańczyk et al., LICS 06].

Branching Vector Addition Systems

From VAS to Branching VAS

- BVAS $\mathcal{B} = \langle k, A_0, R_1, R_2 \rangle$:
  - $k \in \mathbb{N}$, A_0 is a finite subset of $\mathbb{N}^k$ (*axioms*).
  - R_1 is a finite subset of $\mathbb{Z}^k$ (*unary rules*) and R_2 is a finite subset of $\mathbb{Z}^k$ (*binary rules*).

- Derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{(0,0)}}{(0,2)}{\scriptstyle (0,2)}}{(0,4)}{\scriptstyle (0,2)} \qquad \dfrac{\mathbf{(3,4)}}{(10,4)}{\scriptstyle (7,0)}}{(10,5)}{\scriptstyle (0,-3)} \quad \mathbf{(3,4)}}{(13,6)}{\scriptstyle (0,-3)}$$

- We are interested in initialized admissible derivations!

Pioneering works

- BVAS are linear index grammars [Rambow, ACL 94].
- Reachability for BVAS is decidable iff provability in multiplicative exponential linear logic MELL is decidable [de Groote et al., LICS'04].
- Relationships between BVAS and equational tree automata [Verma & Goubault-Larrecq, DMTCS 05].
- If FO2 over finite data trees is decidable then so is reachability for BVAS [Bojańczyk et al., JACM 09].

BVAS: terra incognita?

- Theorem: [Verma & Goubault-Larrecq, DMTCS 05]
  Covering and boundedness for BVAS are decidable.
  (introduction of *covering derivations*)
  ...but non primitive recursive algorithm in the worst case.

- Decidability of reachability problem for BVAS is open.

- This work: covering and boundedness for BVAS are 2EXPTIME-complete.

Solving Covering Problem for BVAS

Sketch of the proof

- **Main property**: existence of a covering of **t** implies existence of a covering of doubly exponential depth.

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- This is already true for VAS.

If a VAS $\langle k, \mathbf{a}, R\rangle$ has an initialised admissible covering of **t**, then it has one whose depth is at most $2^{(3L)^{k+1}}$, where $L = \max\{\mathrm{size}(R), \mathrm{size}(\mathbf{t})\}$ [Rackoff, TCS 78].

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- **Main property**: existence of a covering of $\mathbf{t}$ implies existence of a covering of doubly exponential depth.

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> If a VAS $\langle k, \mathbf{a}, R \rangle$ has an initialised admissible covering of $\mathbf{t}$, then it has one whose depth is at most $2^{(3L)^{k+1}}$, where $L = \max\{\mathrm{size}(R), \mathrm{size}(\mathbf{t})\}$ [Rackoff, TCS 78].

- Computing such a "small" covering can be done with an alternating Turing machine using exponential space.

- 2EXPTIME= AEXPSPACE [Chandra et al., JACM 81].

On shortening the depth of long derivations (or how to minimize our work using [Rackoff78])

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathbf{v}'_5\\ \vdots\\ \mathbf{v}_5\end{matrix} \quad \begin{matrix}\vdots\\ \mathbf{v}'_4\end{matrix}}{\mathbf{v}_4}\,(r_4) \quad \begin{matrix}\vdots\\ \mathbf{v}'_3\end{matrix}}{\mathbf{v}_3}\,(r_3) \quad \begin{matrix}\vdots\\ \mathbf{v}'_2\end{matrix}}{\mathbf{v}_2}\,(r_2) \quad \begin{matrix}\vdots\\ \mathbf{v}'_1\end{matrix}}{\mathbf{v}_1 \geq \mathbf{t}}\,(r_1)$$

$$\begin{matrix}\mathbf{v}'_5\\ \vdots\\ \mathbf{v}_5\\ \hline \mathbf{v}_4\\ \hline \mathbf{v}_3\\ \hline \mathbf{v}_2\\ \hline \mathbf{v}_1\end{matrix}\ \begin{matrix}(r_4 + \mathbf{v}'_4)\\ (r_3 + \mathbf{v}'_3)\\ (r_2 + \mathbf{v}'_2)\\ (r_1 + \mathbf{v}'_1)\end{matrix} \quad \xrightarrow{\textit{Rackoff}} \quad \begin{matrix}\vdots\\ \hline \mathbf{v}''_3\\ \hline \mathbf{v}''_2\\ \hline \mathbf{v}''_1 \geq \mathbf{t}\end{matrix}\ \begin{matrix}(r_1 + \mathbf{v}'_1)\\ (r_3 + \mathbf{v}'_3)\\ (r_2 + \mathbf{v}'_2)\end{matrix} \quad \xrightarrow{\textit{Delin.}} \quad \dfrac{\dfrac{\dfrac{\vdots \quad \begin{matrix}\vdots\\ \mathbf{v}'_1\end{matrix}}{\mathbf{v}''_3}\,(r_1) \quad \begin{matrix}\vdots\\ \mathbf{v}'_3\end{matrix}}{\mathbf{v}''_2}\,(r_3) \quad \begin{matrix}\vdots\\ \mathbf{v}'_2\end{matrix}}{\mathbf{v}''_1}\,(r_2)$$

Two obstacles

- No bound on $\mathrm{size}(R_1 \cup R_1')$ in new VAS $\mathcal{V}'$ in terms of $\mathrm{size}(\mathcal{B})$ and $\mathrm{size}(\mathbf{t})$.

- In rearranged derivation, there is no guarantee that there are fewer excessively high leaves.

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- Let us revisit Rackoff's proof !

Two obstacles

- No bound on $\mathrm{size}(R_1 \cup R'_1)$ in new VAS $\mathcal{V}'$ in terms of $\mathrm{size}(\mathcal{B})$ and $\mathrm{size}(\mathbf{t})$.
- In rearranged derivation, there is no guarantee that there are fewer excessively high leaves.
- Let us revisit Rackoff's proof !
- If a VAS $\langle k, \mathbf{a}, R \rangle$ has an initialised admissible covering $\mathcal{D}$ of $\mathbf{t}$, then it has one which is a contraction of $\mathcal{D}$ and whose depth is at most $(\max(R^-) + \max(\mathbf{t}) + 2)^{(3k)!}$.
  - *Contraction*: derivation as a subsequence of rules ($\sqsubseteq$).
  - $\max(R^-)$: absolute value of the min. negative value in R.

Coming back to the proof schema

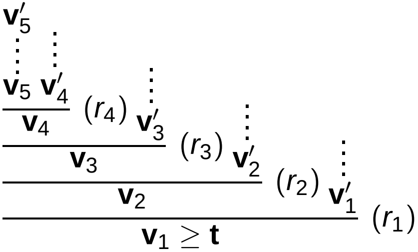
$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\mathbf{v}'_5\\ \vdots\\ \mathbf{v}_5\end{array} \quad \begin{array}{c}\vdots\\ \mathbf{v}'_4\end{array}}{\mathbf{v}_4}\,(r_4) \quad \begin{array}{c}\vdots\\ \mathbf{v}'_3\end{array}}{\mathbf{v}_3}\,(r_3) \quad \begin{array}{c}\vdots\\ \mathbf{v}'_2\end{array}}{\mathbf{v}_2}\,(r_2) \quad \begin{array}{c}\vdots\\ \mathbf{v}'_1\end{array}}{\mathbf{v}_1 \geq \mathbf{t}}\,(r_1)$$

$$\begin{array}{cl} \mathbf{v}'_5 & \\ \vdots & \\ \mathbf{v}_5 & \\ \hline \mathbf{v}_4 & (r_4 + \mathbf{v}'_4) \\ \hline \mathbf{v}_3 & (r_3 + \mathbf{v}'_3) \\ \hline \mathbf{v}_2 & (r_2 + \mathbf{v}'_2) \\ \hline \mathbf{v}_1 & (r_1 + \mathbf{v}'_1) \end{array} \quad \xrightarrow{\textit{Rackoff}} \quad \begin{array}{cl} \vdots & \\ \hline \mathbf{v}''_3 & (r_4 + \mathbf{v}'_4) \\ \hline \mathbf{v}''_2 & (r_3 + \mathbf{v}'_3) \\ \hline \mathbf{v}''_1 & (r_2 + \mathbf{v}'_2) \end{array} \quad \xrightarrow{\textit{Delin.}} \quad \cfrac{\cfrac{\cfrac{\vdots \quad \begin{array}{c}\vdots\\ \mathbf{v}'_4\end{array}}{\mathbf{v}''_3}\,(r_4) \quad \begin{array}{c}\vdots\\ \mathbf{v}'_3\end{array}}{\mathbf{v}''_2}\,(r_3) \quad \begin{array}{c}\vdots\\ \mathbf{v}'_2\end{array}}{\mathbf{v}''_1 \geq \mathbf{t}}\,(r_2)$$

$$\max((R_1 \cup R'_1)^-) \leq \max((R_1 \cup R_2)^-)$$

2EXPTIME Upper Bound

Solving covering in A2EXPSPACE



- depth $\leq (\max((R_1 \cup R_2)^-) + \max(\mathbf{t}) + 2)^{(3k)!}$.
- Universal states perform the verification on both children.
- Vectors can be encoded with double exponential space. (a doubly exponential depth covering can have triple exponential number of nodes)

B-truncated derived vectors

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{(0,0)}}{(0,2)}{\scriptstyle (0,2)}}{(0,4)}{\scriptstyle (0,2)} \qquad \dfrac{\mathbf{(3,4)}}{(10,4)}{\scriptstyle (7,0)}}{(10,5)}{\scriptstyle (0,-3)} \quad \mathbf{(3,4)}}{(13,6)}{\scriptstyle (0,-3)}$$

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ATM M with exponential space

- $N = \mathrm{size}(\mathcal{B}) + \mathrm{size}(\mathbf{t})$,
  $D = (\max((R_1 \cup R_2)^-) + \max(\mathbf{t}))^{3k!} \leq 2^{2^{C_1 N \log N}}$,
  $B = D^2$.

- M guesses B-truncated derived vectors from a init. admi. covering of **t** of depth at most D.
  …and truncated vectors are fine !

- Nondeterministic states guess the rules labelling the current node and its children and their B-truncated derived vectors.

- M uses universal states to proceed with the guessing and verification process to both children.

- Counting until B (and D) requires only exponential space.

Solving the Boundedness Problem for BVAS

Self-covering derivations

- *Self-covering* derivation: for some derived $\mathbf{v}'$, $\mathbf{v} > \mathbf{v}'$

$$\dfrac{\begin{array}{cc} \vdots & \\ \mathbf{v}' & \\ \vdots & \vdots \end{array}}{\mathbf{v} > \mathbf{v}'}$$

- A BVAS produces infinitely many vectors iff it has an initialized admissible self-covering derivation.

[Verma & Goubault-Larrecq, DMTCS 05]

Sketch of the proof for VAS

- Existence of an adm. init. self-covering implies the existence of a self-covering of doubly exponential depth.

> If a VAS $\langle k, \mathbf{a}, R \rangle$ has an initialised admissible self-covering, then it has one whose depth is at most $2^{2^{c_2\ N\ log\ N}}$, where $N = \mathrm{size}(R)$ [Rackoff, TCS 78].

- Computing such a small self-covering can be done with a nondeterministic Turing machine using exponential space.

- NExpSpace= ExpSpace [Savitch, JCSS 70].

Structure of the proof [Rackoff, TCS 78]

If a VAS $\langle k, \mathbf{a}, R \rangle$ has an initialised admissible self-covering, then it has one whose depth is at most $2^{2^{C_2\ N\ log\ N}}$, where $N = \mathrm{size}(R)$.

$\Uparrow$ (induction on dimension)

KEY TECHNICAL LEMMA

$\Uparrow$ (elimination of loops)

Let $\mathbf{A} \in (-m, m)^{k \times n}$ and $\mathbf{b} \in (-m, m)^k$, where $k, n, m \in \mathbb{N}$. If there exists $\mathbf{x} \in \mathbb{N}^n$ such that $\mathbf{Ax} \geq \mathbf{b}$, then there exists $\mathbf{y} \in [0, (\max\{n, m\})^{C_4 k}]^n$ such that $\mathbf{Ay} \geq \mathbf{b}$.
[Borosh & Treybig, AMS 76]

Sketch of the proof for BVAS

- Existence of an adm. init. self-covering implies the existence of a self-covering of doubly exponential depth.

- Computing such a “small” self-covering can be done with an alternating Turing machine using exponential space.
  1. *B*-truncated vectors are involved again.
  2. Small coverings are used in some places.

- 2EXPTIME= AEXPSPACE [Chandra et al., JACM 81].

A key improvement somewhere in the proof (or how to relax hypotheses from [BT76])

Let $\mathbf{A} \in (-m,\infty)^{k \times n}$ and $\mathbf{b} \in (-\infty, m)^k$, where $k, n, m \in \mathbb{N}$. If there exists $\mathbf{x} \in \mathbb{N}^n$ such that $\mathbf{A}\mathbf{x} \geq \mathbf{b}$, then there exists $\mathbf{y} \in [0, L]^n$ such that $\mathrm{card}(\mathrm{supp}(\mathbf{y})) \leq L$ and $\mathbf{A}\mathbf{y} \geq \mathbf{b}$, where $L = m^{2^{C_5 k^2}}$.

instead of

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Conclusion

- Covering and boundedness problems for BVAS are 2EXPTIME-complete.
- 2EXPTIME-hardness is obtained by adapting Lipton's proof with alternating counter machines.

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- Covering and boundedness problems for BVAS are 2EXPTIME-complete.
- 2EXPTIME-hardness is obtained by adapting Lipton's proof with alternating counter machines.
- Open problem: decidability of the reachability problem for BVAS.