Deciding regular grammar logics with converse through GF2

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Outline

1. Modal logics.
2. Guarded fragment GF.
3. Regular grammar logics with converse.
4. Translation into GF2 by simulating NDFA.

Modal languages

Simple and sufficiently expressive to talk about relational structures

Local view for the description of structures

Application domains:

- Computer Science: temporal logics . . .
- Knowledge Representation
- Mathematics: arithmetics . . .
- Linguistics

Basic modal languages and structures

Language:

$$\phi ::= \overbrace{\mathrm{p} \mid \neg\phi \mid \phi \wedge \phi' \mid \phi \vee \phi'}^{\text{propositional calculus}} \mid \overbrace{\langle a \rangle \phi \mid [a]\phi}^{\text{modal extension}}$$

$$a \in \Sigma \quad \mathrm{p} \in \mathrm{PROP}$$

Kripke models: $\mathcal{M} = \langle W, (R_a)_{a \in \Sigma}, m \rangle$:

- W: non-empty set
- R_a: binary relation on W
- $m : \mathrm{PROP} \rightarrow \mathcal{P}(W)$: meaning function

Possible worlds semantics

$$\mathcal{M}, w \models \mathrm{p} \quad \overset{\text{def}}{\Leftrightarrow} \quad w \in m(\mathrm{p}).$$

$$\mathcal{M}, w \models \neg\phi \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathcal{M}, w \not\models \phi.$$

$$\mathcal{M}, w \models [a]\phi \quad \overset{\text{def}}{\Leftrightarrow} \quad \text{for every } w' \in R_a(w), \; \mathcal{M}, w' \models \phi.$$

$$\mathcal{M}, w \models \langle a \rangle\phi \quad \overset{\text{def}}{\Leftrightarrow} \quad \text{there is } w' \in R_a(w) \text{ such that } \mathcal{M}, w' \models \phi.$$

Standard decision problems

- $\mathcal{C}$-satisfi ability:

  input: formula ϕ,

  question: $\exists\ \mathcal{M} \in \mathcal{C}$ & $w \in \mathcal{M}$ s.t. $\mathcal{M}, w \models \phi$?

  ($\mathcal{C}$: class of models depending on application domain)

- Model-checking.

- $\mathcal{C}$-validity, global $\mathcal{C}$-satisfi ability.

Decision procedures

- Direct method:
  - filtration (abstraction),
  - proof-theoretical analysis (sequents),
  - automata (emptiness problem).
- Translation into
  - richer modal logics (PDL, μ-calculus, . . .),
  - first-order fragments (FO2, GF, LGF, . . .),
  - second-order logics (S2S, μLGF).

Translations into FOL

- **Syntactic translation** (Morgan 76)

  Hilbert-style system $\rightarrow$ reifi cation

  provability predicate

- **Relational translation** (van Benthem 76 + ...)

  Exact encoding of the semantics

- **Functional translation** (Ohlbach 88, Herzig 89)

  Path terms and equational theories

- **+ other (sometimes ad hoc) translations**

Relational translation

$$
\begin{aligned}
t(s\mathrm{p}, \alpha) &= \text{s}\ \mathbf{P}(\alpha) \quad s \in \{\epsilon, \neg\} \\
t(\psi_1 \wedge \psi_2, \alpha) &= t(\psi_1, \alpha) \wedge t(\psi_2, \alpha) \\
t(\psi_1 \vee \psi_2, \alpha) &= t(\psi_1, \alpha) \vee t(\psi_2, \alpha) \\
t(\langle a \rangle \psi, \alpha) &= \exists\ \beta\ \mathbf{R}_a(\alpha, \beta) \wedge t(\psi, \beta), \text{ new } \beta \\
t([a]\psi, \alpha) &= \forall\ \beta\ (\mathbf{R}_a(\alpha, \beta) \Rightarrow t(\psi, \beta)), \text{ new } \beta
\end{aligned}
$$

$\mathcal{M}, w \models \phi$ iff $\mathcal{M}, v[\alpha \leftarrow w] \models_{\mathrm{FOL}} t(\phi, \alpha)$

(v is a valuation)

Recycling of variables

$$\{\alpha, \beta\} = \{x_0, x_1\}$$

$$\begin{aligned}
t(\mathrm{p}, \alpha, \beta) &= \mathbf{P}(\alpha) \\
t(\neg\mathrm{p}, \alpha, \beta) &= \neg\mathbf{P}(\alpha) \\
t(\psi_1 \wedge / \vee \psi_2, \alpha, \beta) &= t(\psi_1, \alpha, \beta) \wedge / \vee\, t(\psi_2, \alpha, \beta) \\
t(\langle a \rangle \psi, \alpha, \beta) &= \exists\, \beta\ \mathbf{R}_a(\alpha, \beta) \wedge t(\psi, \beta, \alpha) \\
t([a]\psi, \alpha, \beta) &= \forall\, \beta\ (\mathbf{R}_a(\alpha, \beta) \Rightarrow t(\psi, \beta, \alpha))
\end{aligned}$$

$\mathcal{M}, w \models \phi$ **iff** $\mathcal{M}, v[x_0 \leftarrow w] \models_{\mathrm{FOL}} t(\phi, x_0, x_1)$

($v : \{x_0, x_1\} \to W$)

How to simply translate a large class of modal logics into the decidable GF2?

Guarded fragment (GF)

- Restriction of FOL introduced in (Andreka & van Benthem & Nemeti 98) to identify the “modal fragment of FOL”.
- $$\forall \overline{x}\ (R \Rightarrow \phi), \quad \exists \overline{x}\ (R \wedge \phi).$$

  R: atomic & FreeVar$(\phi) \subseteq$ FreeVar(R).
- Relational translation of modal formulae falls into GF.

GF - Complexity

GF: 2EXPTIME-complete (Grädel, JSL 99)

GF_k: EXPTIME-complete (Grädel, JSL 99)

Other decidable fragments of FOL:

- GF + equality + constants.
- GF2 with transitive guards.
  2EXPTIME-complete
  (Szwast & Tendera 01, Kieronski 03).

GF - Theorem provers

Resolution-based decision procedure for GF (Ganzinger & de Nivelle 99).

Tableaux-based decision procedure for GF (Hadlik 02) and for FO2 (Marx et al. 00).

Main defect of GF

Simple frame conditions such as transitivity can't be expressed in GF.

$$\forall x, y, z\ \mathbf{R}(x, y) \Rightarrow (\mathbf{R}(y, z) \Rightarrow \mathbf{R}(x, z)) \notin \mathrm{GF}.$$

Relation inclusions as rules

transitivity: $R_a \circ R_a \subseteq R_a \;\approx\; a \to a \cdot a$

symmetry: $R_a^{-1} \subseteq R_a \;\approx\; a \to \overline{a}$ (plus $\overline{a} \to a$)

euclideanity: $R_a^{-1} \circ R_a \subseteq R_a \;\approx\; a \to \overline{a} \cdot a$

(plus $\overline{a} \to \overline{a} \cdot a$)

CF condition:

$$R_{a_1} \circ \cdots \circ R_{a_n} \subseteq R_a \approx\; a \to a_1 \cdot \ldots \cdot a_n$$

(plus $\overline{a} \to \overline{a_n} \cdot \ldots \cdot \overline{a_1}$)

Semi-Thue systems

Semi-Thue system S: subset of $\Sigma^* \times \Sigma^*$.

$\langle u, v \rangle \in \mathrm{S}$ read as a rule $u \to v$.

One-step derivation relation $\Rightarrow_{\mathrm{S}}$:

$u \Rightarrow_{\mathrm{S}} v$ iff there exist

1. $u_1, u_2 \in \Sigma^*$, and
2. $u' \to v' \in \mathrm{S}$,

such that $u = u_1 \cdot u' \cdot u_2$ and $v = u_1 \cdot v' \cdot u_2$.

Semi-Thue systems - Languages

Full derivation relation $\Rightarrow_{\mathrm{S}}^{*}$: reflexive and transitive closure of $\Rightarrow_{\mathrm{S}}$.

Language $\mathrm{L}_{\mathrm{S}}(u) = \{v \in \Sigma^* : u \Rightarrow_{\mathrm{S}}^{*} v\}$ for $u \in \Sigma^*$.

Example:

$\mathrm{S} = \{a \to \epsilon, a \to \overline{a}, a \to a \cdot a, \overline{a} \to \overline{a} \cdot \overline{a}, \overline{a} \to a\}$.

$\mathrm{L}_{\mathrm{S}}(a) = \mathrm{L}_{\mathrm{S}}(\overline{a}) = \{a, \overline{a}\}^*$.

Regular semi-Thue sys. with converse

$\mathrm{S} \subseteq \Sigma^* \times \Sigma^*$ such that

1. $\Sigma = \Sigma^+ \cup \Sigma^-$ (disjoint subsets) based on

   $\bar{\cdot} : \Sigma \to \Sigma$ satisfying $\bar{\cdot}(\Sigma^+) = \Sigma^-$, $\bar{\cdot}(\Sigma^-) = \Sigma^+$,

   and $\overline{\overline{a}} = a$ for every $a \in \Sigma$,
2. S is fi nite and $\mathrm{S} \subseteq \Sigma \times \Sigma^*$ (context-freeness),
3. $u \to v \in \mathrm{S}$ iff $\overline{u} \to \overline{v} \in \mathrm{S}$

   ($\overline{a \cdot u} = \overline{u} \cdot \overline{a}$ and $\overline{\epsilon} = \epsilon$),
4. each $\mathrm{L}_{\mathrm{S}}(a)$ is a regular language.

Example of semi-Thue systems

$$\mathrm{S} = \{b \to a, b \to \overline{a}, b \to b \cdot b, b \to \overline{b}, b \to \epsilon\}$$

plus the converse rules:

$$\{\overline{b} \to \overline{a}, \overline{b} \to a, \overline{b} \to \overline{b} \cdot \overline{b}, \overline{b} \to b, \overline{b} \to \epsilon\}$$

$\mathrm{L}_{\mathrm{S}}(a) = \{a\}$, $\quad \mathrm{L}_{\mathrm{S}}(b) = \{a, \overline{a}, b, \overline{b}\}^*$.

$\mathrm{L}_{\mathrm{S}}(\overline{a}) = \{\overline{a}\}$, $\quad \mathrm{L}_{\mathrm{S}}(\overline{b}) = \{a, \overline{a}, b, \overline{b}\}^*$.

$\mathrm{S} \approx$ modal axioms for

K$_t$ + universal modality $[U]$.

$\langle \Sigma, \bar{\cdot} \rangle$-frames

- $\langle \Sigma, \bar{\cdot} \rangle$-frame $\mathcal{F} = \langle W, R \rangle$:

  for $a \in \Sigma$, $R_a \subseteq W \times W$ and $R_{\overline{a}} = R_a^{-1}$.

- $\mathcal{F}$ satisfi es $\mathrm{S} \overset{\text{def}}{\Leftrightarrow}$ for $u \to v \in \mathrm{S}$, $R_v \subseteq R_u$

  with $R_\epsilon = \{\langle w, w \rangle : w \in W\}$, $R_{a \cdot u} = R_a \circ R_u$.

Regular grammar logics with converse

Def. Logic characterized by a class $\mathcal{C}$ of $\langle \Sigma, \bar{\cdot} \rangle$-frames defined by a regular semi-Thue sys. with converse.

$\rightarrow$ They are Sahlqvist's modal logics with frame conditions in Π_1.

$\rightarrow$ Relational translation works but not into GF.

Standard modal logics

logic	$\mathrm{L_S}(a)$	frame condition
K	$\{a\}$	(none)
KT	$\{a, \epsilon\}$	reflexivity
KB	$\{a, \overline{a}\}$	symmetry
KTB	$\{a, \overline{a}, \epsilon\}$	refl. and sym.
K4	$\{a\} \cdot \{a\}^*$	transitivity
KT4 = S4	$\{a\}^*$	refl. and trans.
KB4	$\{a, \overline{a}\} \cdot \{a, \overline{a}\}^*$	sym. and trans.
K5	$(\{\overline{a}\} \cdot \{a, \overline{a}\}^* \cdot \{a\}) \cup \{a\}$	euclideanity
KT5 = S5	$\{a, \overline{a}\}^*$	equivalence rel.
K45	$(\{\overline{a}\}^* \cdot \{a\})^*$	trans. and eucl.

Numerous other logics in the class

- Description logics (with role hierarchy, transitive roles),
- Knowledge logics, e.g. $\mathrm{S5}_m$(DE),
- Bimodal logics for intuitionistic modal logics **IntK**$_{\Box} + \Gamma$,
- Extensions with the universal modality,
- Information logics.

From frames to S*-frames*

$C_{\mathrm{S}}(\mathcal{F}) \stackrel{\text{def}}{=} \langle W, R' \rangle$ with $R'_a = \bigcup_{u \in \mathrm{L}_{\mathrm{S}}(a)} R_u$.

$C_{\mathrm{S}} =$ inverse substitution h^{-1} (Caucal 96) with $h : a \in \Sigma \mapsto \mathrm{L}_{\mathrm{S}}(a)$.

Lemma. $C_{\mathrm{S}}(\mathcal{F})$ is the smallest $\langle \Sigma, \bar{\cdot} \rangle$-frame including $\mathcal{F}$ and satisfying S (CF semi-Thue sys. with converse).

Corollary. $\mathcal{F}$ satisfi es S iff $C_{\mathrm{S}}(\mathcal{F}) = \mathcal{F}$.

Principles of the translation

Previous work: S4, S5, K5 into GF2 with ad hoc method (de Nivelle 2001).

Simulation of the propagation of formulae as in analytic proof systems.

Differences with the relational translation:

1. not an exact encoding of the semantics;
2. built-in encoding of the frame conditions.

Input formula

- W.l.o.g. ϕ in NNF ($\neg$ only in front of p).

  For $a \in \Sigma$, $\mathrm{L}_{\mathrm{S}}(a)$ is regular.

- Finite state automaton $\mathcal{A}_a = \langle Q, q_0, q_f, \delta \rangle$,

  $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathcal{P}(Q)$ (with ϵ-transitions)

  $\mathrm{L}(\mathcal{A}_a) = \mathrm{L}_{\mathrm{S}}(a)$.

- $\mathcal{A}_a^q = \langle Q, q, q_f, \delta \rangle$ ($q \in Q$).

Predicate symbols

accessibility relation: binary $\mathbf{R}_a$ ($a \in \Sigma^+$).

proposition: unary P for $\mathrm{p} \in \phi$.

automaton predicates: unary $\mathbf{q}_\psi$ for $[a]\psi \in sub(\phi)$ and q in $\mathcal{A}_a$.

If q_0 initial state of $\mathcal{A}_a$, $\mathbf{q}_{0,\psi}(\alpha)$ intends to mean:

"$[a]\psi$ holds in the state assigned to α"

$\mathbf{q}_{0,\psi}(\alpha) \approx$ renaming of $t([a]\psi, \alpha, \beta)$.

Interpreting $\mathbf{q}_\psi$

$[a]\psi \in sub(\phi)$, q in $\mathcal{A}_a$.

$\mathbb{S}$-model $\mathcal{M}$ with

$$\mathcal{M}, v[\alpha \leftarrow w_0] \models_{\mathrm{FOL}} \mathbf{q}_\psi(\alpha)$$

iff

whenever $w_0 R_{a_1} w_1 R_{a_2} \ldots R_{a_n} w_n$ &

$a_1 \ldots a_n \in \mathrm{L}(\mathcal{A}_a^q)$, $\mathcal{M}, v[\alpha \leftarrow w_n] \models \psi$.

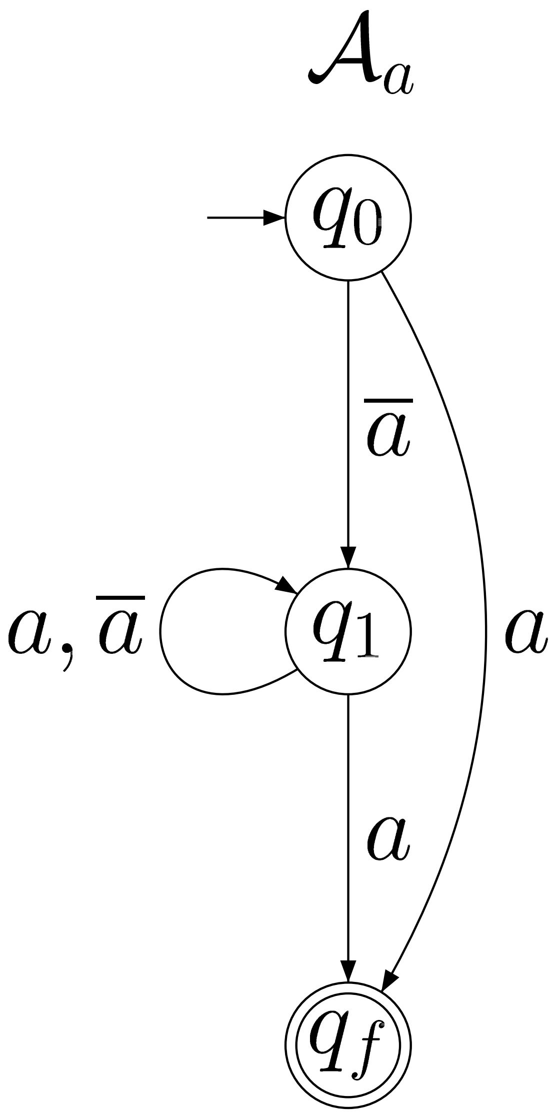
Encoding the automata

$[a]\psi \in sub(\phi)$ & $\mathcal{A}_a = \langle Q, q_0, q_f, \delta \rangle$.

$\mathrm{FSA}_{[a]\psi}$ is the conjunction of ($\{\alpha, \beta\} = \{x_0, x_1\}$)

- $\forall\ \alpha\ \beta\ \mathbf{R}_a(\alpha, \beta) \Rightarrow (\mathbf{q}_\psi(\alpha) \Rightarrow \mathbf{r}_\psi(\beta))$ if $q \stackrel{a}{\rightarrow} r$ and $a \in \Sigma^+$,
- $\forall\ \alpha\ \beta\ \mathbf{R}_a(\beta, \alpha) \Rightarrow (\mathbf{q}_\psi(\alpha) \Rightarrow \mathbf{r}_\psi(\beta))$ if $q \stackrel{\overline{a}}{\rightarrow} r$ and $a \in \Sigma^+$,
- $\forall\ \alpha\ \mathbf{q}_\psi(\alpha) \Rightarrow \mathbf{r}_\psi(\alpha)$ if $q \stackrel{\epsilon}{\rightarrow} r$,
- $\forall\ \alpha\ \mathbf{q}_{f,\psi}(\alpha) \Rightarrow t(\psi, \alpha, \beta)$ ($t(.,.,.)$ to be defi ned).

$\mathrm{FSA}_{[a]\mathrm{p}}$ *for K5*



$$\forall \alpha\beta\ \mathbf{R}(\alpha, \beta) \to \mathbf{q}_{0,\mathrm{p}}(\alpha) \to \mathbf{q}_{f,\mathrm{p}}(\beta)$$
$$\forall \alpha\beta\ \mathbf{R}(\beta, \alpha) \to \mathbf{q}_{0,\mathrm{p}}(\alpha) \to \mathbf{q}_{1,\mathrm{p}}(\beta)$$

$$\forall \alpha\beta\ \mathbf{R}(\alpha, \beta) \to \mathbf{q}_{1,\mathrm{p}}(\alpha) \to \mathbf{q}_{1,\mathrm{p}}(\beta)$$
$$\forall \alpha\beta\ \mathbf{R}(\beta, \alpha) \to \mathbf{q}_{1,\mathrm{p}}(\alpha) \to \mathbf{q}_{1,\mathrm{p}}(\beta)$$

$$\forall \alpha\beta\ \mathbf{R}(\alpha, \beta) \to \mathbf{q}_{1,\mathrm{p}}(\alpha) \to \mathbf{q}_{f,\mathrm{p}}(\beta)$$

$$\forall \alpha\ \mathbf{q}_{f,\mathrm{p}}(\alpha) \to \mathbf{P}(\alpha)$$

Local propagation of constraints

$$\forall\alpha\beta\ \mathbf{R}(\alpha,\beta) \to \mathbf{q}_{0,\mathrm{p}}(\alpha) \to \mathbf{q}_{f,\mathrm{p}}(\beta)$$

$$\forall\alpha\beta\ \mathbf{R}(\beta,\alpha) \to \mathbf{q}_{0,\mathrm{p}}(\alpha) \to \mathbf{q}_{1,\mathrm{p}}(\beta)$$

. . .

$$\forall\alpha\beta\ \mathbf{R}(\alpha,\beta) \to \mathbf{q}_{1,\mathrm{p}}(\alpha) \to \mathbf{q}_{f,\mathrm{p}}(\beta)$$

$$\forall\alpha\ \mathbf{q}_{f,\mathrm{p}}(\alpha) \to \mathbf{P}(\alpha)$$

If $\mathcal{M}, v \models_{\mathrm{FOL}} \mathrm{FSA}_{[a]\mathrm{p}} \wedge \mathbf{q}_{0,\mathrm{p}}(\alpha)$ and,

$v(\alpha)R_{a_1}w_1 \ldots R_{a_n}w_n$ & $a_1 \ldots a_n \in \mathrm{L}_{\mathrm{S}}(a)$

then $\mathcal{M}, v[\alpha \leftarrow w_n] \models_{\mathrm{FOL}} \mathbf{P}(\alpha)$.

Translation

$$t(s\ \mathrm{p}, \alpha, \beta) = s\ \mathbf{P}(\alpha) \quad s \in \{\epsilon, \neg\}$$

$$t(\psi_1 \wedge / \vee\ \psi_2, \alpha, \beta) = t(\psi_1, \alpha, \beta) \wedge / \vee\ t(\psi_2, \alpha, \beta)$$

$$t(\langle a \rangle \psi, \alpha, \beta) = \exists\ \beta\ \mathbf{R}_a(\alpha, \beta) \wedge t(\psi, \beta, \alpha)$$

for $a \in \Sigma^+$ (forward witness)

$$t(\langle a \rangle \psi, \alpha, \beta) = \exists\ \beta\ \mathbf{R}_a(\beta, \alpha) \wedge t(\psi, \beta, \alpha)$$

for $a \in \Sigma^-$ (backward wit.)

$$t([a]\psi, \alpha, \beta) = \mathbf{q}_{\mathbf{0},\psi}(\alpha)$$

q_0 initial state of $\mathcal{A}_a$

Translation of ϕ

$$T(\phi) = (\bigwedge_{[a]\psi \in sub(\phi)} \mathrm{FSA}_{[a]\psi}) \wedge t(\phi, x_0, x_1).$$

- α is the only free variable in $t(\psi, \alpha, \beta)$.
- $T(\phi)$ is in GF with variables $\{x_0, x_1\}$.
- Size of $T(\phi)$ in $\mathcal{O}(|\phi| + m)$.

  $T(\phi)$ can be computed in logspace in

  $|\phi| + m$ (m is the size of the largest $\mathcal{A}_a$).

Satisfiability preservation

Theorem. ϕ is $\mathbb{S}$-satisfi able iff $T(\phi)$ is satisfi able in GF2.

Important points in the proof:

- regularity of the languages $\mathrm{L}_{\mathbb{S}}(a)$s,
- existence of a closure operator on $\langle \Sigma, \bar{\cdot} \rangle$-frames.

Extensions: nominals, universal modality, intuitionistic (modal) logics.

Regular languages and FOL

- $\mathrm{S} = \{a \rightarrow bba, a \rightarrow \epsilon, \overline{a} \rightarrow \overline{bb}\overline{a}, \overline{a} \rightarrow \epsilon\}$

  $\mathrm{L}_{\mathrm{S}}(a) = (bb)^{*}(a \cup \epsilon)$ is not star-free.

- There is no CF semi-Thue sys. S such that $\mathrm{L}_{\mathrm{S}}(a) = (ab)^{*}$ and $(ab)^{*}$ is star-free.

- No encoding of regular languages into GF2 but rather encoding of modal logics whose frame conditions satisfy some regularity conditions.

Some open questions

1. Extension to decidable CF grammar logics (undecidability in full generality).

   $\mathrm{S}_n = \{a \rightarrow \overline{a}^n a, \overline{a} \rightarrow \overline{a} a^n\}$

   $\mathrm{L}_{\mathrm{S}_n}(a)$ is not regular for $n > 1$ but

   S_n-satisfi ability is decidable (Gabbay 75).

2. Characterize the complexity of S-satisfi ability by analyzing the $\mathrm{L}_{\mathrm{S}}(a)$s?

3. Design a PSPACE fragment of GF.