# Chapter 3

# Termination

## 3.1 Wqos

A quasi-ordering is a transitive and reflexive relation. The equivalence relation $\simeq$ associated with a quasi-ordering $\geq$ is defined as $x \simeq y$ iff $x \geq y$ and $y \geq x$. The strict ordering associated with a quasi-ordering $\geq$ is the relation $\geq \setminus \simeq$.

**Definition 4** *A quasi-ordering $\geq$ is* well-founded *if there is no infinite sequence $\{s_i\}_{i \in \mathbb{N}}$ sur that, for every $i$, $s_i > s_{i+1}$.*

**Definition 5** *A* well quasi ordering *(wqo in short) is a quasi ordering $\geq$ such that, for every infinite sequence $\{s_i\}_{i \in \mathbb{N}}$, there are two indices $i < j$ such that $s_j \geq s_i$.*

**Proposition 1** *If $\geq$ is a wqo on the set $D$, then every infinite subset of $D$ contains finitely many minimal elements, up to $\simeq$.*

**Proof:**
By contradiction: if there was infinitely many minimal elements, we could construct an infinite sequence of pairwise incomparable elements. □

**Proposition 2** *Any wqo is well-founded.*

**Proposition 3** *Any quasi-ordering that contains a wqo is well-founded.*

**Proof:**
Any infinite decreasing sequence does not contain two elements $i < j$ such

that $s_j \geq s_i$. □

**Lemma 1** *If $\geq$ is well-founded, then for every $d$ there is a $d'$ such that $d \geq d'$ and, for every $d''$, $d' \not> d''$ ($d'$ is minimal).*

**Proof:**
let us construct a strictly decreasing sequence as follows: $d_0 = d$ and, if $d_n$ is not minimal, then $d_n > d_{n+1}$. By well-foundedness this sequence is finite, hence there is a $n$ such that $d_n$ is minimal. Furthermore, $d_0 \geq d_n$ by transitivity. □

**Proposition 4** *A quasi-ordering $\geq$ is a wqo iff*

1. *It is well-founded*
2. *Every infinite sequence contains two comparable elements*

**Proof:**
The only if direction is a consequence of the two previous propositions.

Consider now a well founded quasi-ordering such that any infinite sequence contains two comparable elements. Let $\{s_i\}_{i \in \mathbb{N}}$ be an infinite sequence. If there are two indices $i, j$ such that $s_i \simeq s_j$, then the proof is completed. Assume now it is not the case.

Let $M = \{s_i | i \in \mathbb{N}, \forall j . s_i \not> s_j\}$ (minimal elements). By well-foundedness and lemma 1,

$$\{s_i \mid i \in \mathbb{N}\} = \bigcup_{m \in M} \{s_j \mid j \in \mathbb{N}, s_j \geq m\}$$

Since every infinite sequence contains two comparable elements, $M$ is finite, hence there is a $s_{i_0} = m \in M$ such that $\{s_j | j \in \mathbb{N}, s_j \geq m\}$ is infinite. In particular it contains a $s_{j_0}$ with $j_0 > i_0$. This shos that there are two indices $i_0 < j_0$ such that $s_{i_0} \leq s_{j_0}$. $\geq$ is therefore a wqo. □

**Proposition 5** *If $\geq$ is a wqo, then from every infinite sequence $\{s_i\}_{i \in \mathbb{N}}$ it is possible to extract a subsequence $\{s_{i_j}\}_{j \in \mathbb{N}}$ such that, for every $j$, $s_{i_{j+1}} \geq s_{i_j}$.*

**Proof:**
We construct by induction on $j$ an increasing subsequence $s_{i_j}$ such that the sets $E_j = \{s_k \mid k \geq i_j, s_k \geq s_{i_j}\}$ is infinite. $E_0 = \mathbb{N}$ and, for every $j$, we let

$M_j$ be the set of minimal elements of $E_j$, up to $\simeq$: $\forall e \in E_j, (\forall e' \in E_j . e \not> e') \Rightarrow (\exists e'' \in M_j . e \simeq e'')$ and two elements in $M_j$ are incomparable.

By proposition 3 and lemma 1,

$$E_j = \bigcup_{m \in M_j} \{s_k \in E_j \mid s_k \geq m\}$$

Since $E_j$ is infinite, there is a $m = s_{i_{j+1}} \in M$ such that $E_{j+1} = \{s_k \in E_j \mid s_k \geq m\}$ is infinite.

By construction, $s_{i_{j+1}} \geq s_{i_j}$ for every $j$. □

## 3.2 Construction of orderings

**Definition 6** *If* $(D_1, \geq_1), \ldots (D_n, \geq_n)$ *are quasi-ordered sets, then the* product quasi-ordering $\geq_\times = (\geq_1, \ldots, \geq_n)$ *is defined on* $D_1 \times \cdots \times D_n$ *by*

$$(d_1, \ldots, d_n) \geq_\times (d'_1, \ldots, d'_n) \quad \text{iff} \quad \forall i.\ d_i \geq_i d'_i$$

**Proposition 6** *A product quasi-ordering is a wqo (resp. is well-founded) iff each of its components is a wqo (resp. well-founded).*

Example: the product ordering on $\mathbb{N}^k$ is a wqo.

**Definition 7** *If* $(D_1, \geq_1), \ldots (D_n, \geq_n)$ *are quasi-ordered sets, then the* lexicographic composition $\geq = (\geq_1, \ldots, \geq_n)_{lex}$ *is defined on* $D_1 \times \cdots \times D_n$ *by*

$$(d_1, \ldots, d_n) >_{lex} (d'_1, \ldots, d'_n) \quad \text{iff} \quad \exists j.\ (\forall i < j.\ d_i \simeq_i d'_i) \ \wedge \ d_j >_j d'_j$$

$$(d_1, \ldots, d_n) >_{lex} (d'_1, \ldots, d'_n) \quad \text{iff} \quad \forall i.\ d_i \simeq_i d'_i$$

**Proposition 7** *The lexicographic composition of quasi-orderings is a wqo (resp. is well-founded) iff each of its components is a wqo (resp. well-founded).*

A (finite) multiset on $D$ is a mapping from $D$ to $\mathbb{N}$, which is 0, except on a finite subset of $D$. $M + N$ is defined by $(M + N)(k) = M(k) + N(k)$. $\emptyset$ is the multiset mapping every element to 0. $\{\{x_1, \ldots, x_n\}\}$ is the multiset mapping $x_i$ to $|\{j \in \{1, ..., n\} \mid x_j = x_i\}|$ and 0 otherwise.

**Definition 8** *The* multiset extension *of a quasi-ordering* $\geq$ *on* $D$ *is the least quasi-ordering* $\geq_{mul}$ *on the multisets such that:*

1. $M \geq_{mul} \emptyset$

2. *for every* $M, M, N$,

$$M \geq_{mul} M' \quad \Rightarrow \quad M + N \geq_{mul} M' + N$$

3. *for every* $n \in \mathbb{N}$, *for every* $M, x, x_1, \ldots, x_n$,

$$(\forall i.\ x >_D x_i) \quad \Rightarrow \quad M + \{\{x\}\} \geq_{mul} M + \{\{x_1, \ldots, x_n\}\}$$

**Proposition 8** *The multiset extension of a quasi-ordering* $\geq$ *is well-founded (res. is a wqo) iff* $\geq$ *is well-founded (resp. is a wqo).*

## 3.3 Embedding

**Definition 9** *Let* $(D, \leq)$ *be a quasi-ordered set. The embedding extension* $\trianglelefteq^w_\leq$ *of* $\leq$ *on* $D^*$ *is the least relation on* $D^*$ *such that*

1. $\epsilon \trianglelefteq^w_\leq \epsilon$

2. *for every* $u, v \in D^*$, *for every* $a \in D$, $u \trianglelefteq^w_\leq v \Rightarrow u \trianglelefteq^w_\leq a \cdot v$

3. *for every* $a, b \in D$ *and every* $u, v \in D^*$,

$$u \trianglelefteq^w_\leq v \wedge a \leq b \quad \Rightarrow \quad au \trianglelefteq^w_\leq bv$$

**Lemma 2** $\trianglelefteq^w_\leq$ *is a well-founded quasi-ordering if* $\leq$ *is a well-founded quasi-ordering.*

**Lemma 3 (Higman)** $\trianglelefteq^w_\leq$ *is a wqo iff* $\leq$ *is a wqo.*

**Proof:**
By contradiction: assume there is an infinite sequence $\{w_i\}_{i \in \mathbb{N}}$ such thar, for every $i < j$, $w_i \not\trianglelefteq w_j$. Then the set $\mathcal{E} = \{(w_i)_{i \in \mathbb{N}} \mid \forall i < j. w_i \not\trianglelefteq^w_\leq w_j\}$ is not empty. We construct by induction a minimal counter-example $(v_i)_{i \in \mathbb{N}}$ and non-empty sets of counter-examples $\mathcal{E}_i$ as follows: $\mathcal{E}_0 = \mathcal{E}$. Let $(w_i)_{i \in \mathbb{N}} \in \mathcal{E}_j$ be such that $w_0 = v_0, \ldots w_{j-1} = v_{j-1}$ and $|w_j|$ is minimal. We let then $v_j = w_j$ and $E_{j+1} = \{(w_i)_{i \in \mathbb{N}} \in E_j \mid w_0 = v_0, \ldots, w_j = v_j\}$. $E_{j+1}$ is non empty by construction.

Consider then the sequence $a_i$ of the first letters of $v_i$. Since $\leq$ is a wqo, thanks to proposition 5 there is an infinite increasing subsequence $\{a_{i_j}\}_{i \in \mathbb{N}}$.

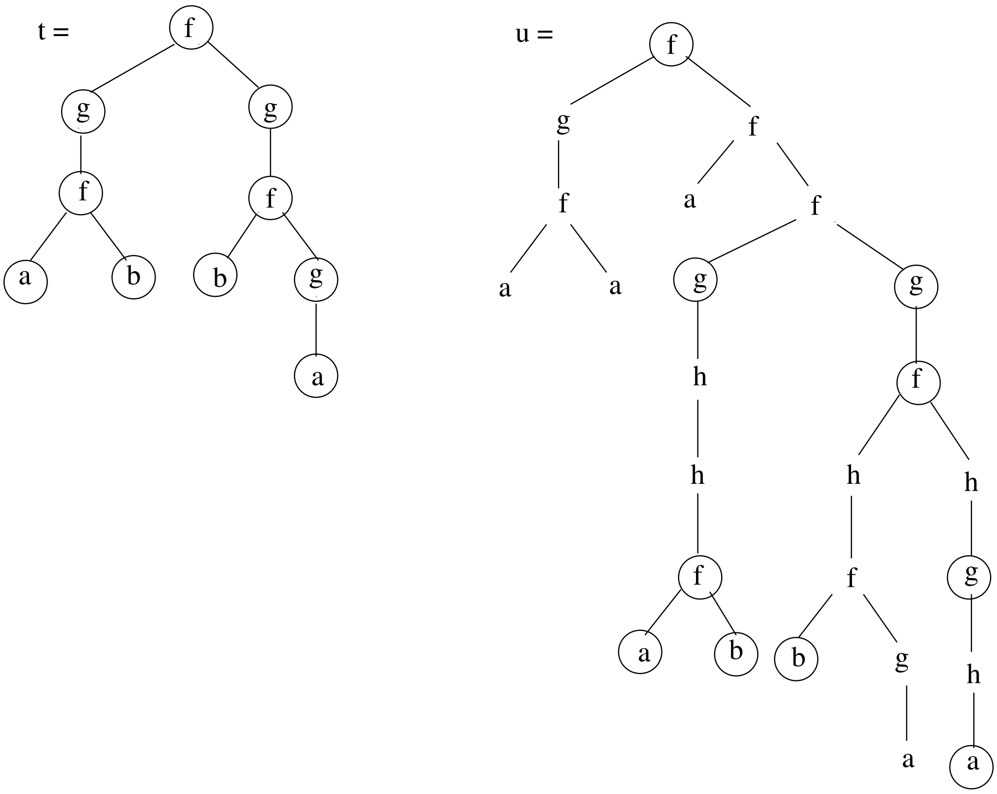


Figure 3.1: Example of embedding: $t \trianglelefteq u$

Consider then the sequence $\{x_i\}_{i\in\mathbb{N}}$: $v_1, \ldots, v_{i_0-1}, v'_{i_0}, v'_{i_1}, \ldots v'_{i_n}, \ldots$ where $v'_{i_j}$ is obtained from $v_{i_j}$ by removing the first letter $a_{i_j}$. By minimality assumption on the counter example, there are two indices $j < k$ such thar $x_j \trianglelefteq^w_\leq x_k$. By construction of the sequence $v_i$, $j \geq i_0$ (otherwise $v_i$ is not a counter-example sequence): there are two indices $m < n$ such that $v'_{i_m} \trianglelefteq^w_\leq v'_{i_n}$. But, since $a_{i_m} \leq a_{i_n}$, thanks to the last point of the definiton, $v_{i_m} = a_{i_m} \cdot v'_{i_m} \trianglelefteq^w_\leq a_{i_n} \cdot v'_{i_n} = v_{i_n}$. A contradiction.

□

**Definition 10** *Assuming a quasi-ordering on $\mathcal{F}$, embedding $\trianglelefteq$ is the least relation on $T(\mathcal{F})$ such that*

1. *for every* $u \in T(|calF)$, $u \trianglelefteq u$

2. *for every* $f \in \mathcal{F}$, $i \in [1..a(f)]$, $u_1, \ldots, u_{a(f)}, v \in T(\mathcal{F})$, $v \trianglelefteq u_i \Rightarrow v \trianglelefteq f(u_1, \ldots, u_{a(f)})$

3. *for every* $f, g \in \mathcal{F}$ *such that* $a(f) = m$ *and* $a(g) = n \geq m$, *for every increasing index sequence* $j_1 < \ldots < j_m$, *if, for every* $k$, $v_k \trianglelefteq u_{j_k}$, *then* $f(v_1, \ldots, v_k) \trianglelefteq g(u_1, \ldots, u_n)$.

An example of embedding is displayed in the figure 3.1, when $\geq_{\mathcal{F}}$ is the equality.

**Proposition 9** *The tree embedding* $\trianglelefteq$ *is well-founded iff* $\geq_{\mathcal{F}}$ *is well-founded.*

**Proposition 10** *Assume that* $\mathcal{F}$ *is finite and* $\geq_{\mathcal{F}}$ *is the equality. Then the tree embedding is simply the rewrite relation on* $T(\mathcal{F})$, *associated with the rewriting system*

$$f(x_1, \ldots, x_n) \to x_i \qquad f \in \mathcal{F}, i \in [1..n]$$