Probabilistic Aspects of Computer Science: Exercise 5

Exercise 1 *Given an alphabet* Σ*, a PFA* $\mathcal{A}$ *on* Σ*, and two numbers* $x, y \in (0, 1)$*, show that there is a* $\mathcal{B}$ *such that* $\mathcal{L}_{>x}(A) = \mathcal{L}_{>y}(\mathcal{B})$ *and* $\mathcal{L}_{\geqslant x}(A) = \mathcal{L}_{\geqslant y}(\mathcal{B})$*.*

Exercise 2 *Given an alphabet* Σ *and a PFA* $\mathcal{A}$ *on* Σ*:*

1. *For any regular language* $L \subset \Sigma^*$*, there is a PFA* $\mathcal{A}$ *such that* $L = \mathcal{L}_{=1}(\mathcal{A})$*.*
2. *The problem of checking emptiness of* $\mathcal{L}_{=1}(\mathcal{A})$ *is* `PSPACE`*-complete.*
3. *The problem of checking universality of* $\mathcal{L}_{=1}(\mathcal{A})$ *is decidable in polynomial time.*

Exercise 3 *Let* $\Sigma = \{\mathbf{0}, \mathbf{1}\}$*. Consider the PFA* $\mathcal{A} = (Q, q_s, Q_f, \delta)$ *where the set* $Q = (q_s, q_a, q_r)$*,* $Q_f = \{q_a\}$ *and* δ *is defined as follows.*

- $\delta(q_s, \mathbf{0}, q_s) = \delta(q_s, \mathbf{0}, q_r) = \frac{1}{2}$.
- $\delta(q_s, \mathbf{1}, q_s) = \delta(q_s, \mathbf{1}, q_a) = \frac{1}{2}$.
- $\delta(q_a, \mathbf{0}, q_a) = \delta(q_a, \mathbf{1}, q_a) = 1$.
- $\delta(q_r, \mathbf{0}, q_r) = \delta(q_r, \mathbf{1}, q_r) = 1$.

1. *Show that for any word* $u = a_1 \cdots a_n \in \Sigma^*$*. the probability of* $\mathcal{A}$ *accepting* u *is*

   $$\sum_{i=1}^{n} \frac{num(a_i)}{2^i}$$

   where $num(a_i)$ *is the number* 0 *if* a_i *is* $\mathbf{0}$ *and is number* 1 *one when* a_i *is* $\mathbf{1}$.
2. *Compute* $\mathcal{L}_{>\frac{1}{4}}(\mathcal{A})$*. Is it regular?*

Exercise 4 *Given any PFA* $\mathcal{A}$*, show that there is another automata* $\mathcal{B}$ *such that:*

1. $\mathcal{L}_{>\frac{1}{2}}(\mathcal{B}) = \mathcal{L}_{>\frac{1}{\sqrt{2}}}(\mathcal{A})$.
2. *In general for any* $x \in [0, 1]$ *,* $\mathcal{L}_{>x^2}(\mathcal{B}) = \mathcal{L}_{>x}(\mathcal{A})$.