On the Expressive Power of 2-Stack Visibly Pushdown Automata

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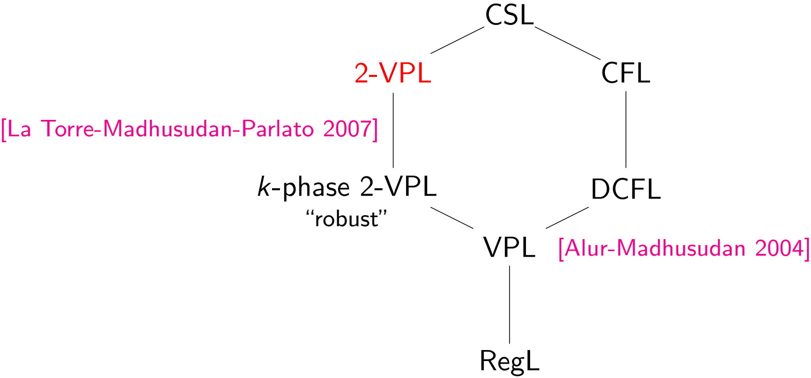
2-Stack Visibly Pushdown Automata (2-VPA)



CSL

2-VPL

CFL

[La Torre-Madhusudan-Parlato 2007]

k-phase 2-VPL

"robust"

DCFL

VPL

[Alur-Madhusudan 2004]

RegL

Visibly Pushdown Automaton (VPA)

Input alphabet is partitioned into push, pop, and internal actions:

$$\widetilde{\Sigma} = (Push, Pop, Int)$$

A VPA

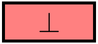
- pushes a symbol onto the stack when reading a letter from *Push*
- pops a symbol from the stack when reading a letter from *Pop*
- does not touch the stack when reading a letter from *Int*

Example of a VPA

$$\widetilde{\Sigma} = (\{a\}, \{b\}, \{c\})$$

Transitions of VPA $\mathcal{A}$ with initial state q_0 and final state q_1:

δ_{push}	δ_{pop}	δ_{int}
(q_0, a, A, q_0)	(q_0, b, A, q_0)	(q_0, c, q_0)
	$(q_0, b, \bot, q_1)$	



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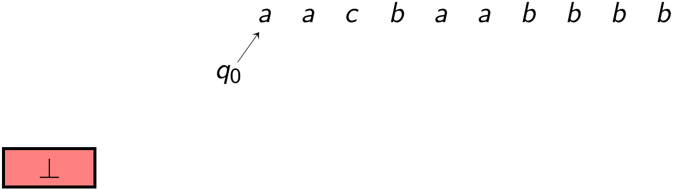

$a \quad a \quad c \quad b \quad a \quad a \quad b \quad b \quad b \quad b$

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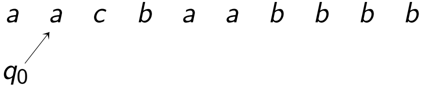

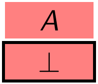

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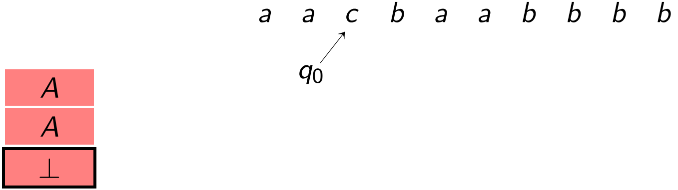

$a \quad a \quad c \quad b \quad a \quad a \quad b \quad b \quad b \quad b$

q_0

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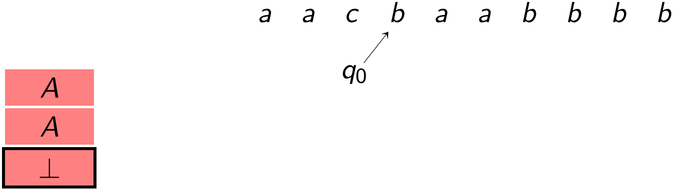

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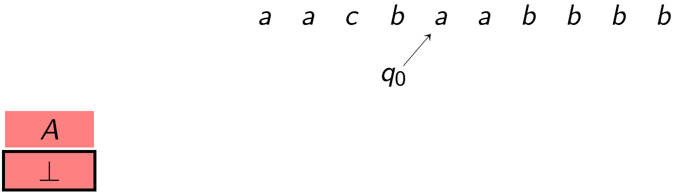

$a \quad a \quad c \quad b \quad a \quad a \quad b \quad b \quad b \quad b$

q_0

A

$\perp$

Example of a VPA

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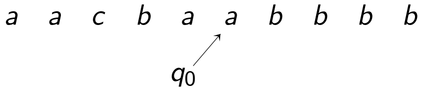
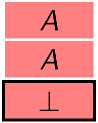


$a \quad a \quad c \quad b \quad a \quad a \quad b \quad b \quad b \quad b$

q_0



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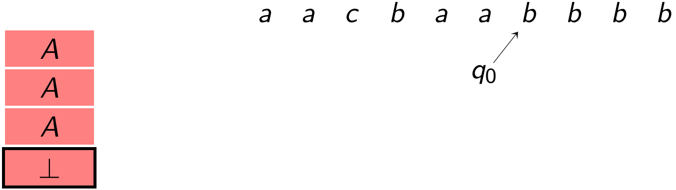

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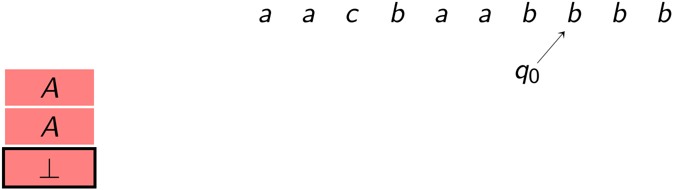

$a \quad a \quad c \quad b \quad a \quad a \quad b \quad b \quad b \quad b$

q_0

A

A

$\bot$

Example of a VPA

$$\widetilde{\Sigma} = (\{a\}, \{b\}, \{c\})$$

Transitions of VPA $\mathcal{A}$ with initial state q_0 and final state q_1:

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(q_0, a, A, q_0)	(q_0, b, A, q_0)	(q_0, c, q_0)
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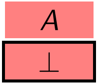
$a \quad a \quad c \quad b \quad a \quad a \quad b \quad b \quad b \quad b$

q_0



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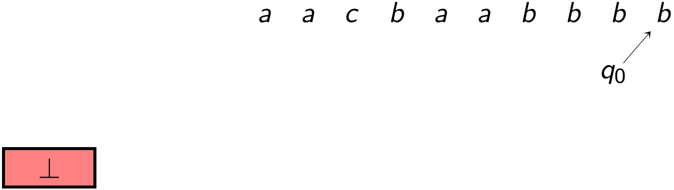

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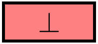
a	a	c	b	a	a	b	b	b	b
									q_1



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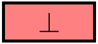
	$(q_0, b, \bot, q_1)$	

$$L(\mathcal{A}) = \{wb \in \{a, b, c\}^* \mid |w|_a = |w|_b \text{ and any prefix of } w \text{ has at least as many } a\text{'s as } b\text{'s}\}$$



Visibly Pushdown Languages

- $\{a^n b^n \mid n \in \mathbb{N}\}$ is a VPL over $(\{a\}, \{b\}, \emptyset)$
- $\{a^n b^n \mid n \in \mathbb{N}\}$ is not a VPL over $\widetilde{\Sigma} \neq (\{a\}, \{b\}, \emptyset)$
- $\{a^n b a^n \mid n \in \mathbb{N}\}$ is not a VPL (no matter over which partition)
- Every regular language is a VPL (independently of the partition)

Closure properties

	$\cup$	$\cap$	$\bar{\cdot}$	Det.	Empt.	Incl.	Logic
RegL	✓	✓	✓	✓	NLOG	PSPACE	MSO
DCFL	✗	✗	✓	✓	PTIME	undecidable	?
CFL	✓	✗	✗	✗	PTIME	undecidable	MSO
VPL	✓	✓	✓	✓	PTIME	EXPTIME	MSO
k-phase MVPL	✓	✓	✓	✗	2EXPTIME	3EXPTIME	MSO
2-VPL	✓	✓	✗	✗	undecidable	undecidable	EMSO
CSL	✓	✓	✓	not known	undecidable	undecidable	?

2-Stack Visibly Pushdown Automata (2-VPA)

Input alphabet is partitioned into push, pop, and internal actions:

$$\widetilde{\Sigma} = (Push_1, Pop_1, Push_2, Pop_2, Int)$$

A 2-VPA

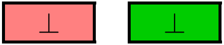
- pushes a symbol onto i'th stack when reading a letter from $Push_i$
- pops a symbol from i'th stack when reading a letter from Pop_i
- does not touch the stacks when reading a letter from Int

Example of a 2-VPA

$$\widetilde{\Sigma} = (\{a\}, \{\overline{a}\}, \{b\}.\{\overline{b}\}, \emptyset)$$

Transitions of 2-VPA $\mathcal{A}$ with initial state q_0 and final state q_5:

$$\begin{array}{llcll}
\delta_{\text{push}}: & (q_0, a, A, q_2) & \quad & \delta_{\text{pop}}: & (q_3, \overline{a}, A, q_3) \\
 & (q_1, a, A, q_2) & & & (q_3, \overline{a}, \perp, q_4) \\
 & (q_2, b, B, q_1) & & & (q_4, \overline{b}, B, q_4) \\
 & (q_2, b, B, q_3) & & & (q_4, \overline{b}, \perp, q_5)
\end{array}$$



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Transitions of 2-VPA $\mathcal{A}$ with initial state q_0 and final state q_5:

$$\begin{array}{llll}
\delta_{\text{push}} : & (q_0, a, A, q_2) & \delta_{\text{pop}} : & (q_3, \overline{a}, A, q_3) \\
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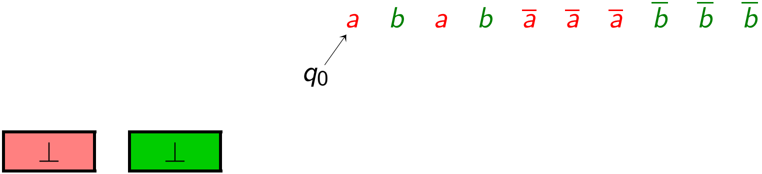

$$a \quad b \quad a \quad b \quad \overline{a} \quad \overline{a} \quad \overline{a} \quad \overline{b} \quad \overline{b} \quad \overline{b}$$

q_0

$\bot$ $\bot$

Example of a 2-VPA

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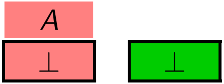
 & (q_2, b, B, q_1) & & & (q_4, \overline{b}, B, q_4) \\
 & (q_2, b, B, q_3) & & & (q_4, \overline{b}, \perp, q_5)
\end{array}$$

$$a \quad \overset{}{b} \quad a \quad b \quad \overline{a} \quad \overline{a} \quad \overline{a} \quad \overline{b} \quad \overline{b} \quad \overline{b}$$

q_2 ↗



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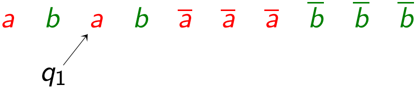
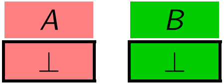


$a \quad b \quad a \quad b \quad \overline{a} \quad \overline{a} \quad \overline{a} \quad \overline{b} \quad \overline{b} \quad \overline{b}$

q_1



Example of a 2-VPA

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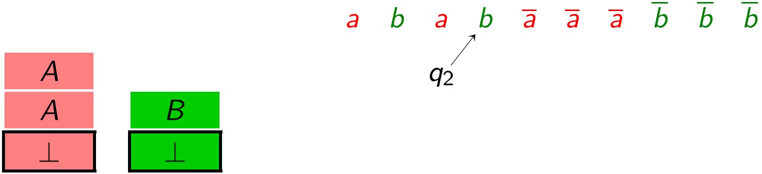

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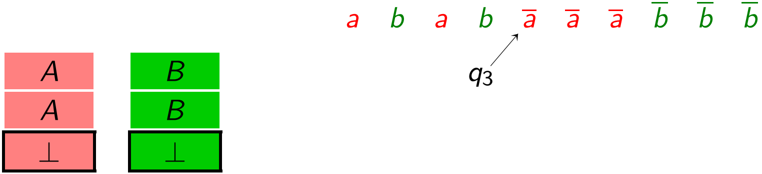

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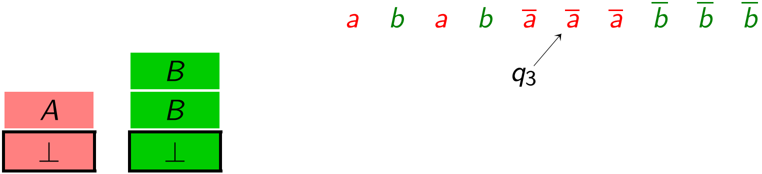

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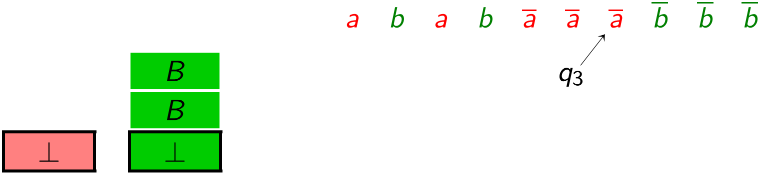

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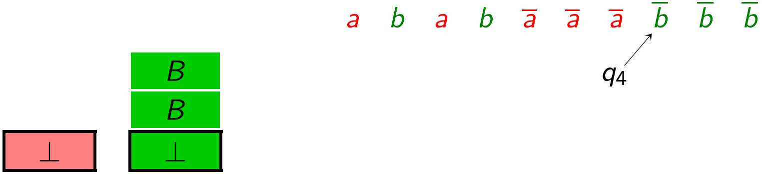

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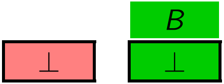
	(q_1, a, A, q_2)		$(q_3, \overline{a}, \perp, q_4)$
	(q_2, b, B, q_1)		$(q_4, \overline{b}, B, q_4)$
	(q_2, b, B, q_3)		$(q_4, \overline{b}, \perp, q_5)$

$$a \quad b \quad a \quad b \quad \overline{a} \quad \overline{a} \quad \overline{a} \quad \overline{b} \quad \underset{\nearrow q_4}{\overline{b}} \quad \overline{b}$$



Example of a 2-VPA

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Transitions of 2-VPA $\mathcal{A}$ with initial state q_0 and final state q_5:

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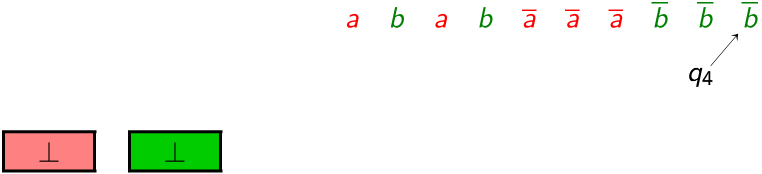

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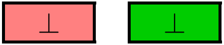
$a \quad b \quad a \quad b \quad \overline{a} \quad \overline{a} \quad \overline{a} \quad \overline{b} \quad \overline{b} \quad \overline{b}$

q_5



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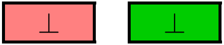
$$L(\mathcal{A}) = \{(ab)^n \; \overline{a}^{\,n+1} \; \overline{b}^{\,n+1} \mid n \geq 1\}$$



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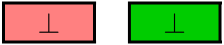
2-phase words $\implies$ 2-phase 2-VPA



k-phase 2-VPA vs. 2-VPA [La Torre-Madhusudan-Parlato 2007]

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A k-phase 2-VPA is a 2-VPA with an additional parameter $k \in \mathbb{N}$.
The domain of input words is restricted to k-phase words.

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k-phase 2-VPA are “robust”

- Decidable emptiness, inclusion, and equivalence problem
- Closed under all boolean operations
- Not determinizable
- MSO characterization

k-phase 2-VPA vs. 2-VPA

2-VPA

- Undecidable emptiness problem
- Closed under union and intersection
- Not closed under complementation
- EMSO characterization

Key technique: Nested words [Alur-Madhusudan 2006]

Word over $\Sigma = \{a, \overline{a}, b, \overline{b}\}$:

$$a \longrightarrow b \longrightarrow a \longrightarrow b \longrightarrow \overline{a} \longrightarrow \overline{a} \longrightarrow \overline{a} \longrightarrow \overline{b} \longrightarrow \overline{b} \longrightarrow \overline{b}$$

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Nested word over $\widetilde{\Sigma} = (\{a\}, \{\overline{a}\}, \{b\}.\{\overline{b}\}, \emptyset)$:

$$\begin{tikzcd}
a \arrow[r] & b \arrow[r] & a \arrow[r] \arrow[rr, bend left] & b \arrow[r] & \overline{a} \arrow[r] & \overline{a} \arrow[r] & \overline{a} \arrow[r] & \overline{b} \arrow[r] & \overline{b} \arrow[r] & \overline{b}
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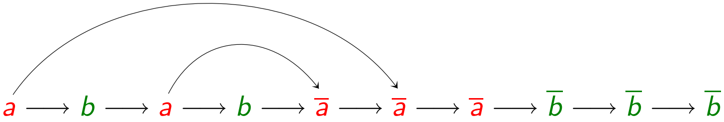

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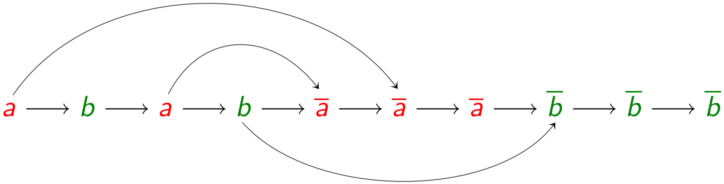

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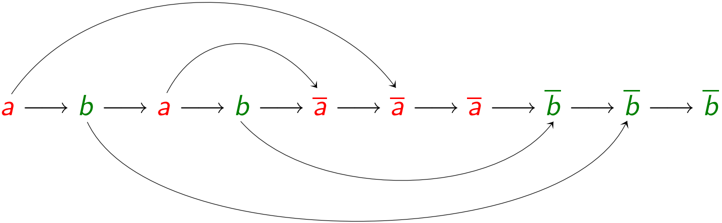

2-Stack Nested Word Automata (2-NWA)

$$\widetilde{\Sigma} = (\{a\}, \{\overline{a}\}, \{b\}.\{\overline{b}\}, \emptyset)$$

Transitions of a 2-NWA $\mathcal{A}$:

$$\begin{array}{llcll}
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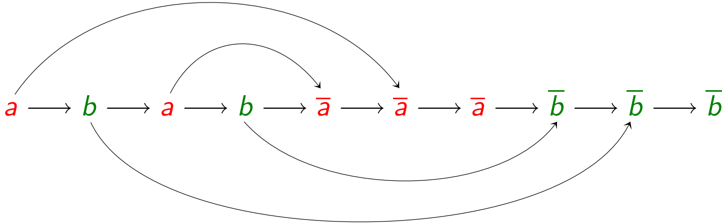

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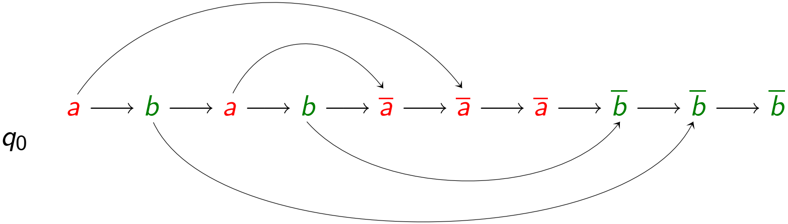

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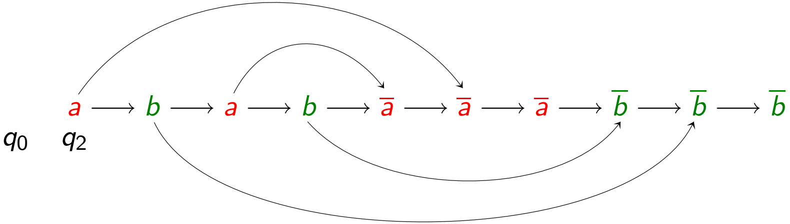

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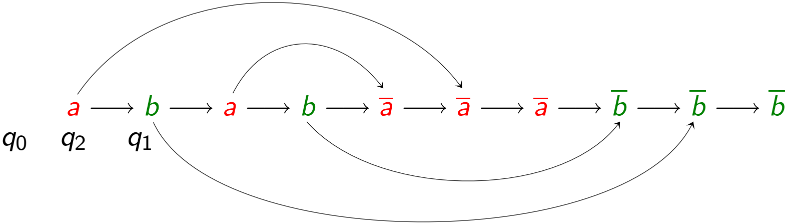

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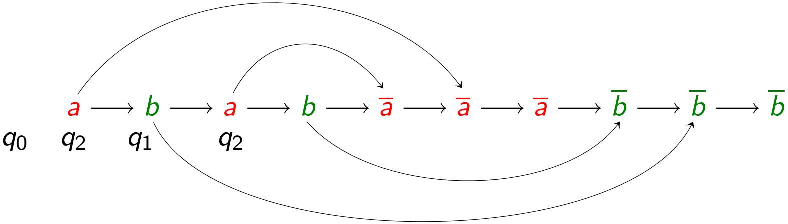

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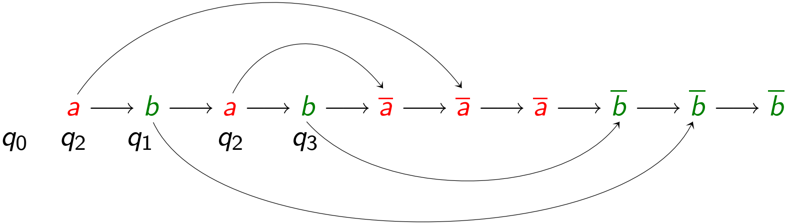

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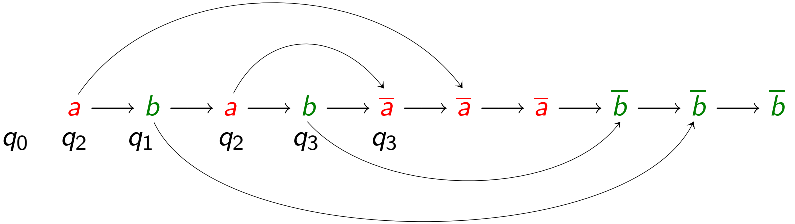

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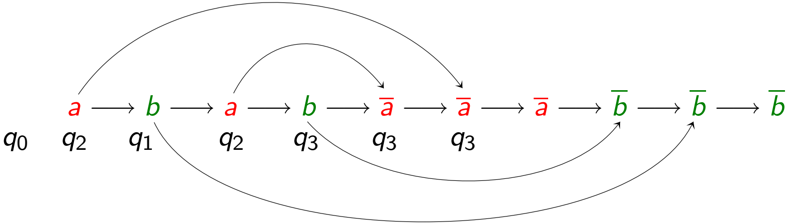

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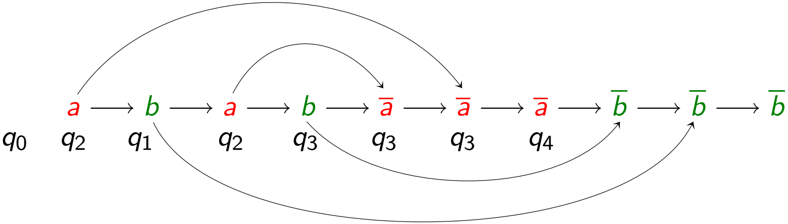

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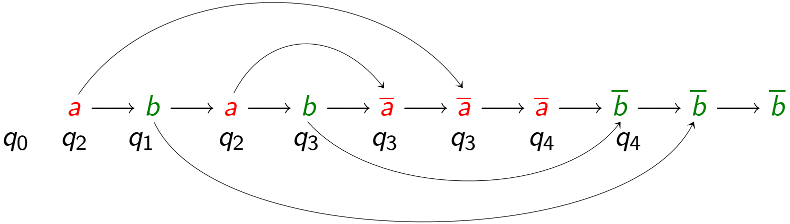

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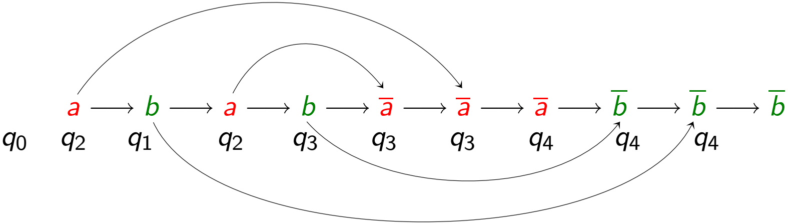

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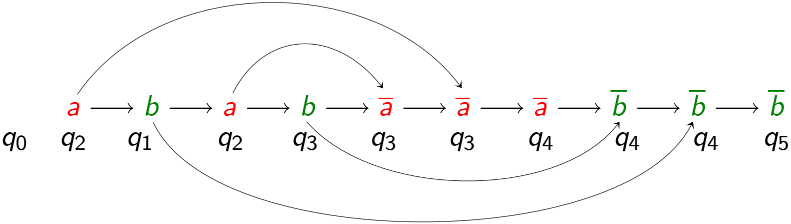
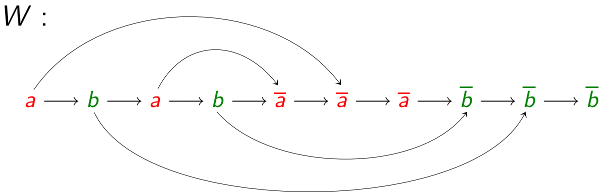


A Logical Characterization of 2-VPA

Logical characterization of 2-NWA/2-VPA

MSO Logic

$$\varphi ::= \mathrm{label}_a(x) \mid \mathrm{succ(x,y)} \mid \mathrm{match(x,y)} \mid x = y \mid x \in X$$
$$\neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists x \varphi \mid \exists X \varphi$$



Example

- $W \models \forall x.\mathrm{label}_a(x) \rightarrow \exists y.\mathrm{match}(x,y))$
- $W \not\models \exists x.\exists y.(\mathrm{succ}(x,y) \wedge \mathrm{match}(x,y))$

Fragments of MSO

Fragments of MSO

An existential MSO (EMSO) formula is of the form

$$\exists X_1 \ldots \exists X_n \ \varphi$$

with first-order kernel φ.

In general, Σ_k shall contain formulas of the form

$$\exists \overline{X_1} \ \forall \overline{X_2} \ \exists \overline{X_3} \ \forall \overline{X_4} \ \ldots \ \exists/\forall \overline{X_k} \ \varphi$$

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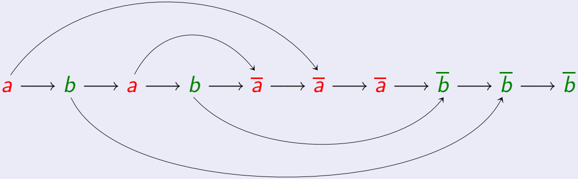
Logical characterization of 2-NWA/2-VPA

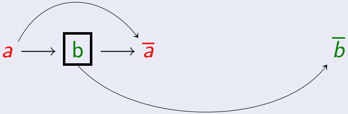
Theorem

2-NWA are expressively equivalent to EMSO logic.

Proof.

Construct 2-NWA that detects spheres, then apply Hanf's Theorem.





1-sphere around
4th position

□

Hanf's Theorem and the sphere automaton

In the context of structures of *bounded degree*, it is sufficient to count spheres of a structure up to some threshold to know if an FO formula is satisfied.

Definition

Let $r, t \in \mathbb{N}$. We write $W \approx_{r,t} W'$ if, for any r-sphere S,

$$|W|_S = |W'|_S \quad \text{or} \quad t < |W|_S \text{ and } t < |W'|_S .$$

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Theorem (Hanf 1965)

For any $k \in \mathbb{N}$, there are $r, t \in \mathbb{N}$ such that

$$W \approx_{r,t} W' \quad \textit{implies} \quad \begin{array}{c} \textit{for any } \varphi \in \mathrm{FO}[k], \\ W \models \varphi \;\; \textit{iff} \;\; W' \models \varphi \end{array}$$

for any nested words W and W'.

Hanf's Theorem and the sphere automaton

The key connection between 2-NWA and FO logic:

There is a 2-NWA that can detect spheres!

Proposition

Let $r \in \mathbb{N}$. There is a 2-NWA $\mathcal{A}$ and a mapping η from the set of states to an the set of r-spheres such that, for any nested word W, the following hold:

- *$\mathcal{A}$ accepts W*
- *for any accepting run ρ of $\mathcal{A}$ on W and any position i of W,*

$$\eta(\rho(i)) \text{ is the } r\text{-sphere of } W \text{ around } i$$

The sphere automaton – First attempt

First attempt:

“Make each state guess its sphere and relate the guesses of neighboring positions.”

The sphere automaton – First attempt

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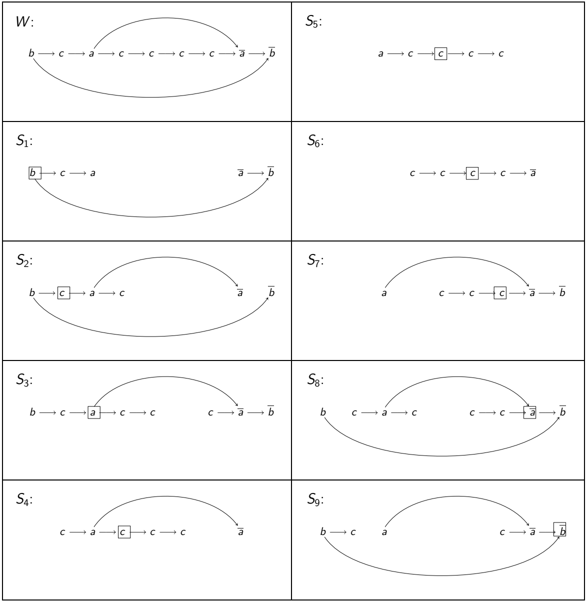
This approach will fail!

The sphere automaton – First attempt



The sphere automaton – The solution

The solution:

“In any state, the 2-NWA will guess the current sphere as well as spheres of nodes nearby and the current position in these additional spheres.”

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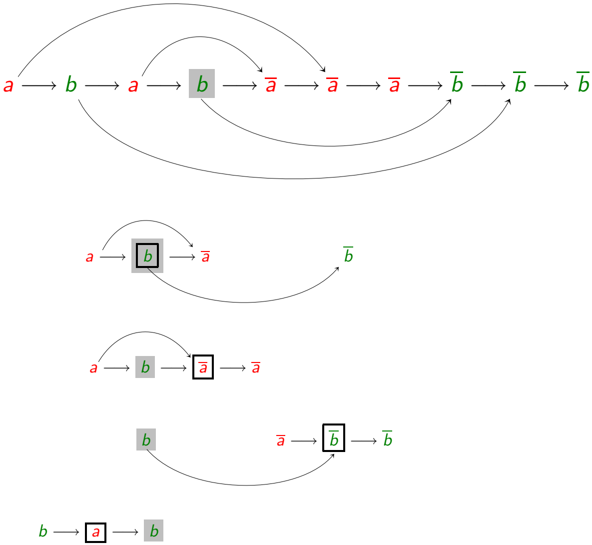
A local state holds a set of extended *r*-spheres

$$(S, \alpha, col)$$

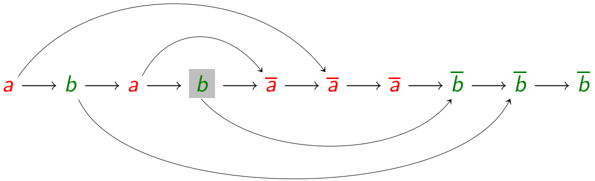
where

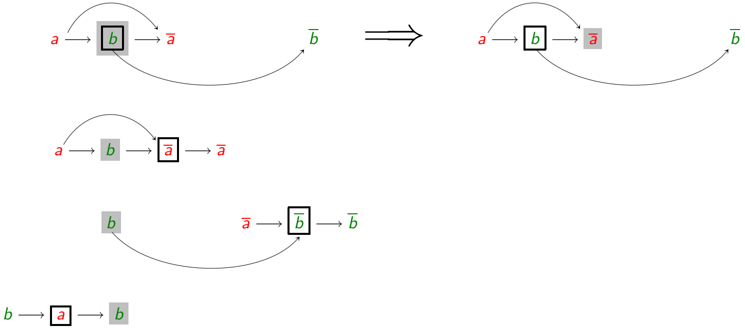
- S is an r-sphere
- α is a node in S, called active node
- i is a coloring from a finite set of colors that depends on $\widetilde{\Sigma}$ and r

The sphere automaton



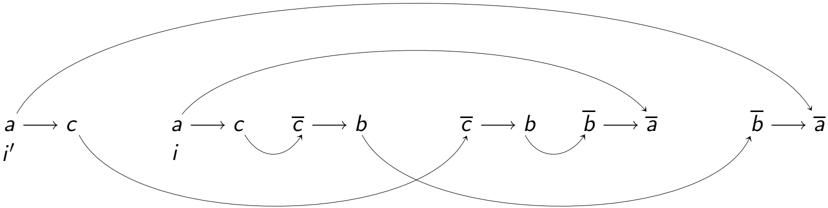
The sphere automaton





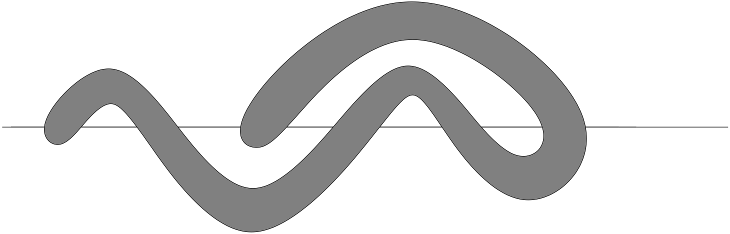
Simulating circles ...

... fails over three stacks:



Simulating circles ...

... works over two stacks:



Topological circles behave aperiodically.

Beyond Recognizability

Second-order quantifier alternation

Theorem

The monadic second-order quantifier alternation hierarchy over nested words is infinite: for any $k \in \mathbb{N}$, Σ_{k+1} is more expressive than Σ_k.

Proof

FO reduction from grids to nested words, then apply [Matz-Thomas 1997]:

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Second-order quantifier alternation

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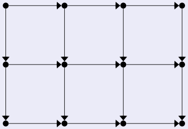
2-NWA and complementability

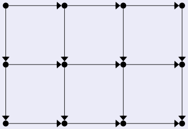
Proof (contd.)



☐

2-NWA and complementability

Proof (contd.)

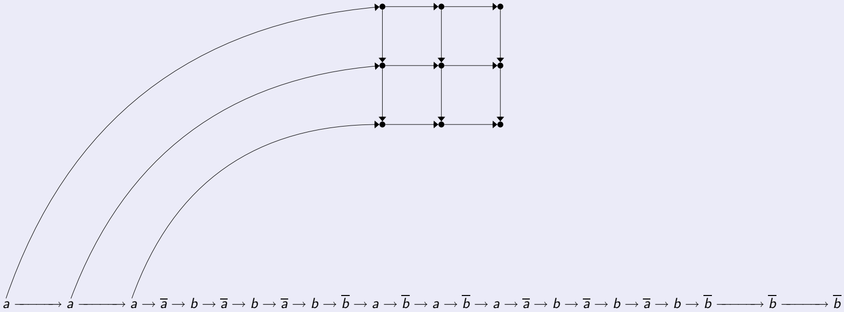


$$a \longrightarrow a \longrightarrow a \rightarrow \overline{a} \rightarrow b \rightarrow \overline{a} \rightarrow b \rightarrow \overline{a} \rightarrow b \rightarrow \overline{b} \rightarrow a \rightarrow \overline{b} \rightarrow a \rightarrow \overline{b} \rightarrow a \rightarrow \overline{a} \rightarrow b \rightarrow \overline{a} \rightarrow b \rightarrow \overline{a} \rightarrow b \rightarrow \overline{b} \longrightarrow \overline{b} \longrightarrow \overline{b}$$

□

2-NWA and complementability

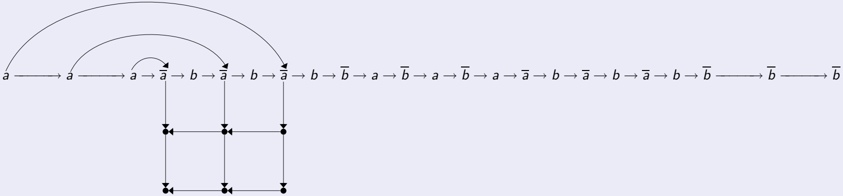
Proof (contd.)



□

2-NWA and complementability

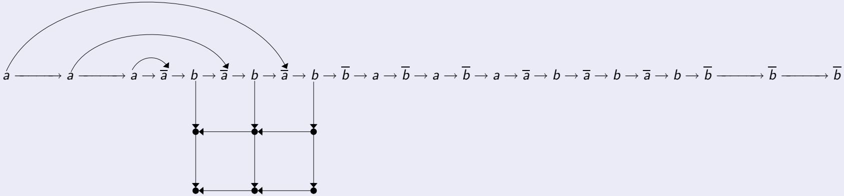
Proof (contd.)



□

2-NWA and complementability

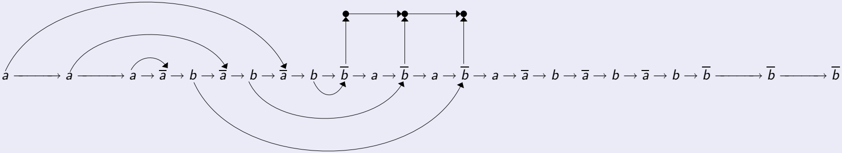
Proof (contd.)



□

2-NWA and complementability

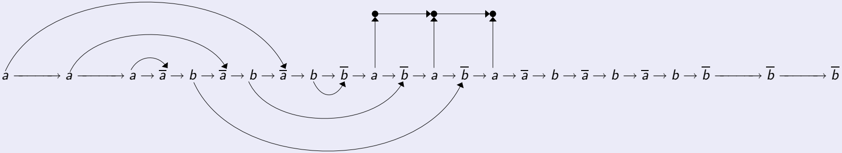
Proof (contd.)



□

2-NWA and complementability

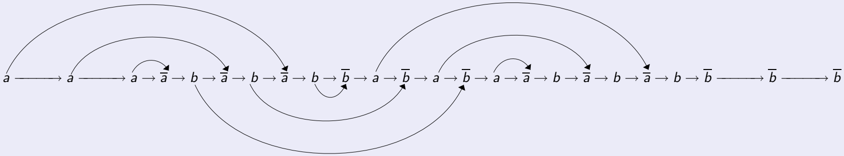
Proof (contd.)



□

2-NWA and complementability

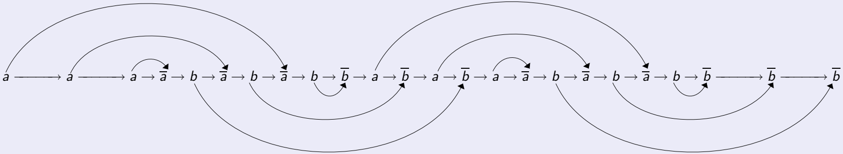
Proof (contd.)



□

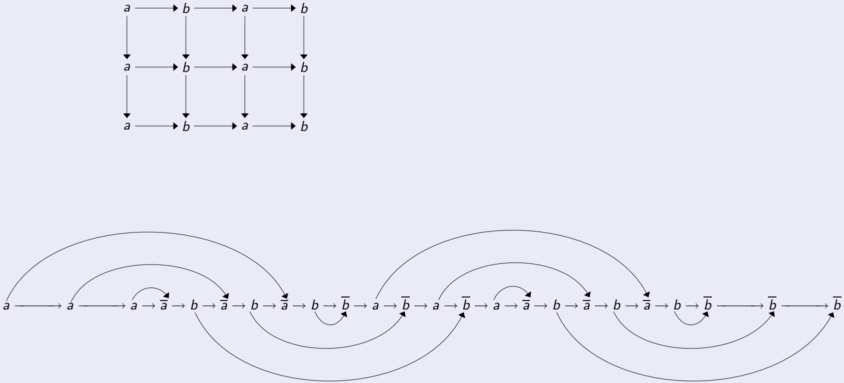
2-NWA and complementability

Proof (contd.)



2-NWA and complementability

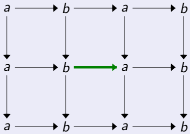
Proof (contd.)



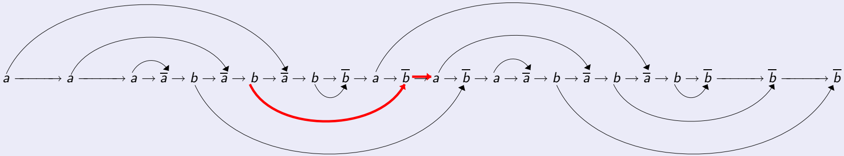
□

2-NWA and complementability

Proof (contd.)



$$\text{h-succ}(x, y) \quad \equiv \quad \exists z(\text{match}(x, z) \wedge \text{succ}(z, y))$$



□

Complementability

We have shown:

- 2-VPA are expressively equivalent to EMSO logic.
- MSO is strictly more expressive than EMSO (over nested words).

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We conclude:

Theorem

2-VPA cannot be complemented.

Determinizability

Corollary

2-VPA are strictly more expressive than deterministic 2-VPA.

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2-VPA are strictly more expressive than deterministic 2-VPA.

But this can be shown more easily [La Torre-Madhusudan-Parlato 2007]:

$$\widetilde{\Sigma} = (\{a\}, \{c, d\}, \{b\}, \{x, y\}, \emptyset)$$

The language

$$\{(ab)^m c^n d^{m-n} x^n y^{m-n} \mid m \in \mathbb{N},\ n \in \{1, \ldots, m\}\}$$

is accepted by some 2-VPA but not by any deterministic 2-VPA over $\widetilde{\Sigma}$.

Future work

- Logical characterization of general multi-stack VPA?
- How about more general visibly pushdown alphabets? [Carotenuto-Murano-Peron 2007]
- How about $\leq$ in EMSO logic?