On Analyticity in Deep Inference

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Abstract

In this note, we discuss the notion of analyticity in deep inference and propose a formal definition for it. The idea is to obtain a notion that would guarantee the same properties that analyticity in Gentzen theory guarantees, in particular, some reasonable starting point for algorithmic proof search. Given that deep inference generalises Gentzen proof theory, the notion of analyticity discussed here could be useful in general, and we offer some reasons why this might be the case.

This note is dedicated to Dale Miller on the occasion of his 60th birthday. Dale has been for both of us a friend and one of our main sources of scientific inspiration. Happy birthday Dale!

1 Introduction

It seems that the notion of analyticity in proof theory has been defined by Smullyan, firstly for natural-deduction systems in [Smu65], then for tableaux systems in [Smu66]. Afterwards, analyticity appeared as a core concept in his influential textbook [Smu68b]. There is no widely accepted and general definition of analytic proof system; however, the commonest view today is that such a system is a collection of inference rules that all possess some form of the subformula property, as Smullyan argued. Since analyticity in the sequent calculus is conceptually equivalent to analyticity in the other Gentzen formalisms, in this note we focus on the sequent calculus as a representative of all formalisms of traditional proof theory [Gen69].

Normally, the only rule in a sequent system that does not possess the subformula property is the cut rule. Therefore, proof theorists look for ways of eliminating from sequent proofs all instances of cut, because they are the sources of nonanalyticity. As is well known, the procedural way to get analytic proofs is called 'cut elimination', and it is the cornerstone of structural proof theory. Thus, the following three properties of sequent systems are usually considered equivalent: 1) *with the subformula property*, 2) *cut-free*, and 3) *analytic*.

In the last few years, the *deep inference* methodology has emerged. It generalises Gentzen's methodology and allows the design of systems for logics for which the Gentzen theory fails. In brief, we can define deep inference as the theory of proof systems where

- *we compose derivations with the same connectives that compose formulae*, or, equivalently,
- *we apply inference rules at any depth inside formulae.*

Deep inference is an attempt at purifying and unifying syntax for the widest array of logics, with a view of improving proof semantics, proof complexity and computational interpretations of logics. We will only assume the very basics of deep inference: the paper [BG09] provides sufficient details to follow the discussion here, and the web page [Gug] gives a complete overview of the state of the art in deep inference. We will represent proofs in the notation called 'open deduction' [GGP10]. A further accessible introduction to deep inference is in [Gug15].

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A typical deep inference rule is something like

$$\mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \quad .$$

In a rule like this the premiss is not a subformula of the conclusion, but a rearrangement of its subformulae. Therefore, deep inference does not guarantee the subformula property and the nondeterminism in proof search becomes huge compared to the sequent calculus.

However, since deep inference is a generalisation of Gentzen theory, it is indeed possible to restrict its proof systems in such a way that the subformula property, or some suitable variation, is guaranteed, and that both proof search and proof normalisation behave as in Gentzen theory. This means that one can design a proof system with the freedom guaranteed by deep inference, and only at a later time restrict it in such a way that nondeterminism in proof search is reduced, much like focussing works for the sequent calculus [MNPS91, And92]. This approach has been pursued by Bruscoli and Kahramanoğulları [Bru02, Kah06b, Kah06a, Kah14], and we believe that much remains to be done in order to unlock the full potential of focussing in deep inference.

Another argument in favour of exploring focussing in deep inference is provided by some recent results by Anupam Das and ourselves. The papers [BG09, Das11] prove that in fact one can use only a small additional amount of nondeterminism compared to the sequent calculus and obtain in return an exponential speed-up on the size of proofs. Ongoing work suggests that the speedup could even be nonelementary in the case of classical predicate logic proofs.

Despite the differences concerning the subformula property, the whole normalisation theory of deep inference resembles (and generalises) Gentzen normalisation theory and its applications are the same. For example, a consequence of cut elimination in deep inference is consistency. This means that the main reason for the introduction of a notion of analyticity in the sequent calculus is still valid in deep inference.

So, what is going on here? It seems that the tight relation between cut-freeness and low nondeterminism in proof search that we observe in the sequent calculus is broken in deep inference. On the other hand, we observe in deep inference enormous speed-ups in the size of proofs, which of course offer opportunities for an overall more efficient proof search. Is there a way to understand this phenomenon at a more abstract level? Is there any guiding principle? In this note, we argue that a certain notion of analyticity seems to capture the essence of the problem and indeed can guide the design of efficient proof systems.

We present our definition of analyticity in Section 2. Then we argue about its relation with the subformula property in Section 3 and with cut elimination in Section 4. We comment on all the relevant deep inference literature so that this note can serve as a mini-survey. We do not present here any new result and every claim in this note can be readily verified by the interested reader. Our objective is exclusively to present and motivate the definition of analyticity, which has been obtained by trial and error by simply observing many proof systems for many logics and many normalisation procedures. We should mention that Brünnler in [Brü06b] uses the concept of analyticity in deep inference in a rather different way than we do (although his approach is not incompatible with ours).

For the reason explained here and many others that arise from deep inference, we believe that some technical notion of analyticity should replace both the subformula property and cut elimination as a guiding principle in proof theory. Of course, we cannot be sure that our definition is a valid one, therefore we welcome all feedback, and especially from the proof search community, so that we do not invest too many energies in exploring the consequences of our definition in case it presents some unforeseen flaws. We do not have any strong ideology backing it: this definition simply seems to make sense and to stimulate interesting research hypotheses, one of which we are currently pursuing as we explain in Section 4.

2 Definition of Analyticity

In [BG09]–Section 6.4, we write that it would be tempting to call 'analytic' a restriction of a deep inference proof system that polynomially simulates Frege systems and nonanalytic sequent calculus systems. This could be done by adopting the following, natural notion of analyticity. An inference rule would be called analytic if, given an instance of its conclusion, the set of possible instances of the premiss is finite. Let us call this definition 'naif' and let us see what would happen if we adopted the naif definition.

Consider the *finitely generating atomic cut* rule $\mathsf{fai}\!\uparrow$, defined in [BG04]:

$$K\left\{\mathsf{fai}\!\uparrow\frac{p(\vec{x}) \wedge \bar{p}(\vec{x})}{\mathsf{f}}\right\} \quad \text{, where } p \text{ appears in } K\{\ \}.$$

Under the naif definition, the finitary version of the atomic cut rule would be analytic, and replacing the generic cut rule with $\mathsf{fai}\!\uparrow$ would only entail a polynomial cost for the size of proofs. In fact, as [BG04] shows, transforming a deep inference proof with generic cut rules into a proof only employing cuts of the $\mathsf{fai}\!\uparrow$ variety can be done by a series of independent proof transformations that only inflate the size of the proof linearly. Moreover, these transformations are local, in the sense that, contrary to cut elimination procedures in Gentzen's theory, no copying of unbounded portions of the proof takes place. This dissuades us from deeming $\mathsf{fai}\!\uparrow$ analytic, basically because we do not believe that there should be a free lunch here: reducing nondeterminism cannot be too cheap in terms of the size of proofs. In other words, we believe that a productive and inspiring notion of analyticity needs to exclude the $\mathsf{fai}\!\uparrow$ rule.

Let us call 'finitely generating' the rules that only allow a finite choice of premisses given a conclusion. The cited paper [BG04] basically shows that, in deep inference, transforming generic proofs into ones only adopting finitely generating rules is cheap: it can be done in linear time. The idea of the definition of analyticity proposed here is to further raise the bar, and move it from a finite set of premisses to a bounded set of premisses. This is our proposal:

Definition 1. For every formula B, context $K\{\ \}$ and rule r, we define the set of *premisses of B in $K\{\ \}$ via r*:

$$\mathsf{pr}(B, K\{\ \}, r) = \left\{ A \;\middle|\; K\left\{ r\,\frac{A}{B} \right\} \right\} \quad .$$

Given a rule r:

1. if, for every B and $K\{\ \}$, the set $\mathsf{pr}(B, K\{\ \}, r)$ is finite, then we say that r is *finitely generating*;
2. if, for every B, there is a natural number n such that for every context $K\{\ \}$ we have $|\mathsf{pr}(B, K\{\ \}, r)| < n$, then we say that r is *analytic*.

Obviously, an analytic rule is finitely generating, and, crucially, the finitely generating atomic cut rule is not analytic, as desired.

We can apply this definition to the natural translation of the sequent calculus rules into deep inference. As desired, all inference rules of a standard sequent calculus system are analytic, except for the cut rule, which is not finitely generating. On the contrary, and as desired, Smullyan's analytic cut rule, as defined in [Smu68a], is indeed analytic according to Definition 1.

$$
\mathsf{i}{\downarrow}\,\frac{\mathsf{t}}{A \vee \bar{A}} \qquad \mathsf{w}{\downarrow}\,\frac{\mathsf{f}}{A} \qquad \mathsf{c}{\downarrow}\,\frac{A \vee A}{A} \qquad \mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}
$$

Figure 1: Rules of system KSg: *identity*, *weakening*, *contraction*, and *switch*.

As defined in [BG09], the inference rule $=$ of system SKS for classical logic is not analytic, because an arbitrary number of units can occur in the premiss, for any given conclusion. However, we can ask for the rule $=$ to apply only to unit-canonical formulae, where unit-canonical formulae are defined as those formulae with the minimum number of units in each equivalence class modulo $=$. This way, the $=$ rule would be analytic. Another possibility is not to close the equivalence $=$ for transitivity and allow for an arbitrary number of $=$ rules between non-$=$ inference steps, in a derivation. Both solutions do not alter any of the properties of the deep inference systems involved, and they can be applied to all deep inference systems where equations are used. All this applies not just to SKS but to virtually every deep inference systems where units are controlled by equations.

3 Analyticity and the Subformula Property

In this section, we argue that in the analytic deep inference formalisms, despite their apparently weaker subformula property, we can actually recover the subformula property as in Gentzen. This is possible thanks to *splitting theorems*, which allow us to suitably constrain the search space for proofs. In fact, we could design analytic sequent systems as special applications of the splitting theorems of deep inference systems.

Examples of analytic deep inference rules are in Figure 1. These four rules, together with De Morgan laws and equations for the commutativity and associativity of $\vee$ and $\wedge$, and for the units f and t, form a complete system for classical propositional logic, called KSg [BT01, Brü06c].

Our main reason for deeming KSg and other deep inference systems analytic is that their cut elimination is similar to cut elimination in the sequent calculus. It turns out that sequent systems (or other systems based on the Gentzen subformula property) can be recovered from the deep inference analytic systems, as applications of *splitting theorems*, a class of similar theorems that are individually proved for each proof system, similarly to what happens with cut-elimination theorems. The point here is that whatever guarantees cut elimination also guarantees something conceptually equivalent to the subformula property. In other words, the link between cut elimination and the subformula property is still present in deep inference, albeit in a more general form.

Let us start from a weak, or 'shallow', version of the splitting theorem for the classical propositional logic system KSg:

Theorem 2. (Shallow splitting for KSg) *For every formulae A, B and C, if there exists a* KSg *proof of $(A \wedge B) \vee C$ then there exist* KSg *proofs of $A \vee C$ and $B \vee C$.*

This theorem allows us to restrict the search space for KSg proofs to an isomorphic space to the one generated by a sequent system with the already mentioned rule

$$
\wedge\,\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \wedge B, \Gamma} \quad ,
$$

which has the Gentzen subformula property. The formula C in the statement of the theorem plays the role of Γ in the rule $\wedge$. In fact, given any formula D to prove, we can invoke the theorem on any external conjunction of D and proceed recursively until no more conjunctions are present; at that point, only identity and weakening rules in shallow contexts are needed (exactly as in a sequent proof). So, in practice, we can use a deep inference system as we would use a sequent system. In other words, if we can prove a shallow splitting theorem, we have the Gentzen subformula property. For example, we can design a proof search algorithm for KSg based on its shallow splitting theorem, and this would be the same as the one for a sequent system for propositional classical logic.

As a matter of fact, we can prove much more general splitting theorems. The following is probably the most easy-to-state variant for KSg.

Theorem 3. (Splitting for KSg) *For every formulae A, B and context $K\{\ \}$, if there exists a proof*

$$\begin{array}{c} \Vert\,\mathsf{KSg} \\ K\{A \wedge B\} \end{array}$$

then there exist a formula C and derivations

$$\begin{array}{c} C \vee \{\ \} \\ \Vert\,\mathsf{KSg} \\ K\{\ \} \end{array} \quad , \qquad \begin{array}{c} \Vert\,\mathsf{KSg} \\ A \vee C \end{array} \qquad \textit{and} \qquad \begin{array}{c} \Vert\,\mathsf{KSg} \\ B \vee C \end{array} \quad .$$

Here,

$$\begin{array}{c} C \vee \{\ \} \\ \Vert\,\mathsf{KSg} \\ K\{\ \} \end{array}$$

stands for a derivation where a special atom $\{\ \}$ appears, which is a hole that can be filled by any formula.

This theorem gives us great flexibility in modelling the search space of proofs. If we are given the formula D to prove in KSg, we can arbitrarily choose A, B and $K\{\ \}$ such that $D = K\{A \wedge B\}$, and the theorem guarantees that if a proof exists, then one can be found by looking in the smaller search spaces of $K\{\ \}$, $A \vee C$ and $B \vee C$. (We leave it as an exercise to find that in the case of KSg, a suitable C can be easily obtained from $K\{\ \}$.)

The splitting theorem allows us to trade between the depth and the breadth of the proof-search space tree. In fact, the shallower the chosen context $K\{\ \}$ is, the fewer choices there are for C, and so the smaller is the breadth of the proof-search space tree. The special case of shallow splitting corresponds to the smallest breadth, that of the sequent calculus, but it is interesting to allow for larger breadth. For example, by allowing A and B to be chosen, inside $K\{\ \}$, deeper than in the outermost conjunctions (so making real use of deep inference), we could access exponentially shorter proofs, so significantly reducing the depth of the proof-search space tree. In [BG09] there is an example of such an exponential speed-up for Statman tautologies. In that paper, we show that we can prove Statman tautologies with polynomially growing KSg proofs by just going one level deeper into the context than the sequent calculus, which only can prove them with exponentially growing proofs. A recent result by Das in [Das11] shows that by going deeper than that we only improve polynomially on the size of proofs, for every class of propositional classical logic tautologies. In other words, just a very small amount of deep inference, and consequential nondeterminism, buys a very large speed-up.

The pleasant situation that we have just described is not peculiar to KSg, but it holds for all (well-designed) proof systems in deep inference; the various instances of splitting present some variations, but they have the same meaning and impact on the status of the subformula property and the search space for proofs. In particular there exist splitting theorems for classical predicate logic [Brü06a], for linear logic and for several extensions of linear logic that do not admit complete and analytic sequent systems [Gug07, GS02, SG11, GS11, Kah06b].

4 Eliminating Nonanalytic Rules

In much of the deep inference literature, a distinction is made between 'up' and 'down' rules. The general idea is that each logical law comes in two dual forms, and the 'down' form is the one retained in the analytic fragment. For example, the identity rule is 'down' and its dual, the cut, is 'up'. According to our definition of analyticity, it turns out that indeed all the rules in the 'down' fragments of the deep inference systems in the literature are analytic, and so are some 'up' rules, like cocontraction. When both dual versions of a rule are analytic (as for contraction/cocontraction) the choice of the label 'up' or 'down' is based on other criteria, for example, analogy with the sequent calculus, or proof complexity properties (as is the case for cocontraction).

Normally, in the propositional case, the only 'up' rules that are nonanalytic are cut and coweakening. Obviously, cut elimination takes care of eliminating cut instances. Many of the same methods eliminate other 'up' rule instances, some of which are analytic. Several general deep inference techniques can take care of eliminating coweakening as well, but, in this case, the preferred elimination technique is simply transforming any coweakening instance

$$\mathsf{w}{\uparrow}\,\frac{A}{\mathsf{t}}$$

into several atomic instances

$$\mathsf{w}{\uparrow}\,\frac{a}{\mathsf{t}} \quad ,$$

and then permuting them up in the proof until they hit matching weakenings or hit identity axioms. In both cases, the offending instances disappear and the whole process actually reduces the size of a proof. This technique is a very simple instance of a general normalisation theory based on 'atomic flows' [GG08, GGS10].

In the predicate calculus, the situation is more interesting. Consider the ∃-introduction rule, which is the mechanism by which witnesses are created:

$$\exists\,\frac{A[x/\tau]}{\exists x A} \quad .$$

As is argued in [BG04] this rule, which has so far been included in the 'down' fragment, can be replaced by a finitely generating one. The idea is, essentially, that instead of performing a substitution all in one go, one does it piece by piece. Since the number of function symbols can be made finite without losing provability, the choice of premiss is finite. However, even in its finitely generating version, this rule is not analytic.

The situation is not very different from the ∃-introduction rule in the sequent calculus, which does not really possess the subformula property and which in fact causes troubles for proof search, as is well known. Reasonable proof search methods in Gentzen theory aim at delaying the application of this rule until the proof context provides a witness.

In deep inference, Guglielmi and Benjamin Ralph are currently exploring the possibility of simply permuting up instances of the ∃ rule in a proof until they reach an identity axiom, where the following proof transformation takes place:

$$\mathsf{i}{\downarrow}\frac{\mathsf{t}}{\boxed{\exists\frac{A[x/\tau]}{\exists xA}}\vee\bar{A}[x/\tau]}\quad\rightarrow\quad\mathsf{i}{\downarrow}\frac{\mathsf{t}}{\exists xA\vee\boxed{\forall\frac{\forall x\bar{A}}{\bar{A}[x/\tau]}}}\quad.$$

This is very similar to what happens when permuting up coweakening: we transform a nonanalytic rule instance into an analytic one – in this case the $\forall$ rule. If all goes well this would be the first application of the definition of analyticity given in this note. In order to make the whole thing work, Ralph is reformulating some of the original Herbrand ideas in deep inference, in a manner that would be impossible in the sequent calculus.

As a matter of fact, there are currently independent attempts at using similar ideas in the sequent calculus, in particular by Juan P. Aguilera and Matthias Baaz [AB16]. By allowing locally unsound rules, the authors manage to obtain a nonelementary speed-up in predicate calculus proofs. It seems that we can obtain the same speedup in deep inference, without any need for unsound rules. The problem with unsound rules is that one has to guarantee the overall soundness by a global criterion on a proof. This, of course, is not very friendly to proof search, which by necessity has to proceed by local, incremental steps.

The main idea can be illustrated very simply by noting that the following proof of the drinker's formula is available in deep inference, and can be considered the result of pushing up an instance of $\exists$ as we argued above:

$$=\frac{=\frac{\mathsf{i}{\downarrow}\frac{\mathsf{t}}{\exists x\bar{A}(x)\vee\forall yA(y)}}{\exists x\big(\bar{A}(x)\vee\forall yA(y)\big)}}{\exists x\forall y\big(\bar{A}(x)\vee A(y)\big)}\quad.$$

This proof is analytic and involves no contraction. In the sequent calculus, as is well known and as can be readily verified, any proof of the drinker's formula requires a contraction and an instance of $\exists$-introduction. Both are sources on nondeterminism in proof search. Needless to say, the perspective of accessing immensely shorter proofs and with much less proof-search nondeterminism is rather exciting.

5 Conclusions

In this note, we provided a notion of analyticity for deep inference and we argued that it is analogous in its consequences and applications to the analyticity of the sequent calculus. In particular, our notion supports similar notions of subformula property and cut elimination.

Much work still needs to be done in order to develop viable proof search methods in deep inference, but we provided some evidence that making efforts in that direction is both possible, because of the splitting theorems, and rewarding, because of the huge speed-ups available in deep inference.

This kind of research typically benefits from general principles that guide the design of proof systems. Hopefully, the definition of analyticity provided here could become one of those principles.

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