Substructural Cut Elimination

Taus Brock-Nannestad

PDC A/S
tausbn@gmail.com

Abstract

In the paper "Structural Cut Elimination", Pfenning gives a proof of the admissibility of cut for intuitionistic and classical logic. The proof is remarkable in that it does not rely on difficult termination metrics, but rather a nested lexicographical induction on the structure of formulas and derivations. A crucial requirement for this proof to go through is that contraction is not an inference rule of the system. Because of this, it is necessary to change the inference rules so that contraction becomes an admissible rule rather than an inference rule. This change also requires that weakening is admissible, hence it is not directly applicable to logics in which only contraction is admissible (e.g. relevance logic).

We present a sequent calculus which admits a unified structural cut elimination proof that encompasses Intuitionistic MALL and its affine, strict and intuitionistic extensions. A nice feature of the calculus is that, for instance, moving from linear to strict logic is as simple as allowing the presence of a rule corresponding to contraction.

Finally, based on the insights we obtain from this design, we present a strongly focused sequent calculus for strict logic (i.e. Intuitionistic MALL with free contraction).

1 Introduction

The most important theorem about any sequent calculus is that of *cut elimination*. This theorem usually ensures that the sequent calculus is consistent and internally sound. Additionally, the admissibility of the cut rule usually implies a host of useful inversion properties.

There are many different proofs of the admissibility of cut, but among these one of the most straightforward and elegant is the *structural cut admissibility* proof due to Pfenning [Pfe00]. For instance, given the following rules — which correspond to the implicational fragment of Intuitionistic MALL [Gir87],

$$\frac{}{a \Longrightarrow a}\,\text{init} \qquad \frac{\Delta \Longrightarrow A \qquad \Delta', B \Longrightarrow C}{\Delta, \Delta', A \multimap B \Longrightarrow C}\multimap\text{L} \qquad \frac{\Delta, A \Longrightarrow B}{\Delta \Longrightarrow A \multimap B}\multimap\text{R}$$

the cut rule is

$$\frac{\Delta \Longrightarrow A \qquad \Delta', A \Longrightarrow C}{\Delta, \Delta' \Longrightarrow C}\,\text{cut}$$

and the statement of *cut admissibility* is

Theorem 1. *Given derivations $\mathcal{D} :: \Delta \Longrightarrow A$ and $\mathcal{E} :: \Delta', A \Longrightarrow C$ there exists a derivation $\mathcal{F} :: \Delta, \Delta' \Longrightarrow C$.*

Pfenning's proof of this theorem proceeds by a lexicographic induction on the structure of the cut formula A and the two input derivations $\mathcal{D}$ and $\mathcal{E}$. To see how this induction ordering naturally arises, we will consider two cases:

- First, we consider a case where the cut formula is not the principal formula of the last inference rule in $\mathcal{E}$:

$$\frac{\begin{array}{c}\mathcal{D}\\ \Delta \Longrightarrow A\end{array} \qquad \overset{\mathcal{E}}{\dfrac{\Delta', A, B \Longrightarrow C}{\Delta', A \Longrightarrow B \multimap C}\multimap\text{R}}}{\Delta, \Delta' \Longrightarrow B \multimap C}\,\text{cut}$$

Here, the thick rule for cut is a reminder that this is the rule we are going to prove admissible. In particular, this means that both $\mathcal{D}$ and $\mathcal{E}$ are assumed to be cut-free derivations.

In this case, we argue as follows:

$$\begin{array}{ll} \Delta, \Delta', B \Longrightarrow C & \text{by cut on } A, \mathcal{D}, \mathcal{E}. \\ \Delta, \Delta' \Longrightarrow B \multimap C & \text{by} \multimap \text{R.} \end{array}$$

Thus, in this case we appeal to the induction hypothesis on a subderivation, but keeping the cut formula the same. Note that although we are appealing to the induction hypothesis, we will write it as an appeal to the rule we are proving admissible.

- In the principal case:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Delta, A \Longrightarrow B\end{matrix}}{\Delta \Longrightarrow A \multimap B} \multimap \mathrm{R} \qquad \dfrac{\begin{matrix}\mathcal{E}_1\\ \Delta_1 \Longrightarrow A\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ \Delta_2, B \Longrightarrow C\end{matrix}}{\Delta_1, \Delta_2, A \multimap B \Longrightarrow C} \multimap \mathrm{L}}{\Delta, \Delta_1, \Delta_2 \Longrightarrow C} \text{cut}$$

we reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Delta, \Delta_1 \Longrightarrow B & \text{by cut on } A, \mathcal{E}_1, \mathcal{D}. \\ & \Delta, \Delta_1, \Delta_2 \Longrightarrow C & \text{by cut on } B, \mathcal{D}', \mathcal{E}_2. \end{array}$$

Here, although the derivation $\mathcal{D}'$ may be much larger than the two input derivations, the second use of the cut rule is justified by the fact that the cut formula itself got strictly smaller.

If we add the contraction rule

$$\dfrac{\Delta, A, A \Longrightarrow C}{\Delta, A \Longrightarrow C} \text{contract}$$

to the above system, the above proof does not go through. When cutting against an contraction:

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Delta \Longrightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}\\ \Delta', A, A \Longrightarrow C\end{matrix}}{\Delta, A \Longrightarrow C} \text{contract}}{\Delta, \Delta' \Longrightarrow C} \text{cut}$$

we would like to reason as follows:

$$\begin{array}{lll} \mathcal{E}' :: & \Delta, \Delta', A \Longrightarrow C & \text{by cut on } A, \mathcal{D}, \mathcal{E}. \\ & \Delta, \Delta, \Delta' \Longrightarrow C & \text{by cut on } A, \mathcal{D}, \mathcal{E}'.(*) \\ & \Delta, \Delta' \Longrightarrow C & \text{by contract (repeated).} \end{array}$$

But the line marked (*) is not a smaller cut according to our induction ordering.

For intuitionistic logic, Pfenning circumvents this problem by baking the rule of contraction into every single rule of inference in the system. Thus, the left implication rule would look as follows:

$$\dfrac{\Delta, A \supset B \Longrightarrow A \qquad \Delta, A \supset B, B \Longrightarrow C}{\Delta, A \supset B \Longrightarrow C} \supset \mathrm{L}$$

With the above rule, contraction becomes an admissible rule of inference. Note, however, that the above rule requires *weakening* to be present in the system as well. Because of this, the above solution cannot directly be applied to logics that do not admit weakening, unless they also forego contraction, as in linear logic.

We will present three different but related calculi in this paper, all having the desired structural cut admissibility property. The first one will have a straightforward motivation, but fails to be sound for strict logic. With a small tweak, we get the second calculus which is sound and complete for the four substructural fragments of intuitionistic logic that we study, but which has less desirable inversion properties where the additive connectives are concerned. The final system is a strongly focused [And92] calculus for strict logic.

The reason for keeping the "unsound" calculus around is twofold. First of all, the unsoundness is subtle and easy to miss, and hence it is useful to draw attention to exactly where it appears. Secondly, the calculus itself is *internally* sound and complete (in the sense of having the identity expansion and cut admissibility properties), and hence it is by most standards a well-defined calculus for *some* logic, although it is not entirely clear what logic that is.

2 Cut Elimination

From a proof search perspective, the rule of contraction is problematic, as it requires the search procedure to guess in advance whether a formula will be required several times or not. When weakening is allowed, this is not a problem as one can simply make extra copies whenever assumptions are consumed or the context is split. Any superfluous copies can then be discarded by means of weakening.

Rather than require a proof search procedure to be prescient, we do the obvious thing: whenever an assumption is used, it is not *forgotten*, but merely put aside so that it may potentially be used again further up in the proof. With this in mind, it makes sense to divide the assumptions into two contexts corresponding to whether they have already been used or not. Thus, the judgments of our system have two contexts on the left hand side of the sequent arrow. The difference between these contexts can be summed up as follows:

- The context Δ is a multiset of assumptions that *must* be used in the derivation of the sequent.
- The context Γ is a multiset of assumptions that *may* be used in the derivation of the sequent.

The sequent arrows themselves have an annotation to show which structural rules are present in the derivation of the sequent in question. These annotations are elements of the free semilattice over the set $\{\mathbf{w}, \mathbf{c}\}$, in other words they must be one of either $\cdot$, $\mathbf{w}$, $\mathbf{c}$, or $\mathbf{w} + \mathbf{c}$. We use $+$ for the join operation on this structure.

The syntax of formulas is standard:

$$A, B ::= a \mid A \,\&\, B \mid A \otimes B \mid A \multimap B \mid A \oplus B \mid \top \mid \mathbf{0} \mid \mathbf{1}$$

The inference rules can be seen in Figure 1.

To see how these inference rules arise, consider the behaviour of the $\multimap$ L rule in *strict* logic (i.e. with contraction but not weakening)[1]. A first attempt at a suitable $\multimap$ L rule would be the following

$$\frac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \Gamma; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C} \multimap \mathrm{L}$$

[1] One could also refer to this logic as either *relevance* logic or *relevant* logic, but as these names refer to a host of related logics, we prefer to use the word *strict* for the logic arising from disallowing the structural rule of weakening.

$$\frac{}{\Gamma; a \xrightarrow{x} a}\text{init} \qquad \frac{}{\Gamma; \Delta \xrightarrow{x} \top}\top\text{R} \qquad \frac{}{\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C}\mathbf{0}\text{L} \qquad \frac{}{\Gamma; \cdot \xrightarrow{x} \mathbf{1}}\mathbf{1}\text{R}$$

$$\frac{\Gamma, A_1 \mathbin{\&} A_2; \Delta, A_i \xrightarrow{x} C}{\Gamma; \Delta, A_1 \mathbin{\&} A_2 \xrightarrow{x} C}\&\text{L}_i \qquad \frac{\Gamma; \Delta \xrightarrow{x} A \qquad \Gamma; \Delta \xrightarrow{y} B}{\Gamma; \Delta \xrightarrow{x+y} A \mathbin{\&} B}\&\text{R} \qquad \frac{\Gamma, \mathbf{1}; \Delta \xrightarrow{x} C}{\Gamma; \Delta, \mathbf{1} \xrightarrow{x} C}\mathbf{1}\text{L}$$

$$\frac{\Gamma, A \otimes B; \Delta, A, B \xrightarrow{x} C}{\Gamma; \Delta, A \otimes B \xrightarrow{x} C}\otimes\text{L} \qquad \frac{\Gamma, \Delta_2; \Delta_1 \xrightarrow{x} A \qquad \Gamma, \Delta_1; \Delta_2 \xrightarrow{y} B}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} A \otimes B}\otimes\text{R}$$

$$\frac{\Gamma, A \oplus B; \Delta, A \xrightarrow{x} C \qquad \Gamma, A \oplus B; \Delta, B \xrightarrow{y} C}{\Gamma; \Delta, A \oplus B \xrightarrow{x+y} C}\oplus\text{L} \qquad \frac{\Gamma; \Delta \xrightarrow{x} A_i}{\Gamma; \Delta \xrightarrow{x} A_1 \oplus A_2}\oplus\text{R}_i$$

$$\frac{\Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x} A \qquad \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C}\multimap\text{L} \qquad \frac{\Gamma; \Delta, A \xrightarrow{x} B}{\Gamma; \Delta \xrightarrow{x} A \multimap B}\multimap\text{R}$$

$$\frac{\Gamma; \Delta, A \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x+\mathbf{c}} C}\text{promote} \qquad \frac{\Gamma, A; \Delta \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{w}} C}\text{demote}$$

Figure 1: Sequent calculus for IMALL with promote and demote rules.

This rule is too strict, however, as we might want to use the assumption $A \multimap B$ again. To enable this, we copy this formula into both of the "may use" contexts:

$$\frac{\Gamma, A \multimap B; \Delta_1 \xrightarrow{x} A \qquad \Gamma, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C}\multimap\text{L}$$

This is better, but still too restrictive; if both subderivations want to use a hypothesis in Δ_1, say, the proof can't go through. To fix this, we additionally give each of the premisses a copy of the "must use" context of the other premiss. Thus, the final rule looks as follows:

$$\frac{\Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x} A \qquad \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C}\multimap\text{L}$$

Transferring hypotheses between the "must use" and "may use" contexts is done using the structural rules for promotion and demotion. The promote rule moves a hypothesis from the "may use" context to the "must use" context, thus corresponding to a "delayed" application of contraction. Conversely, the demote rule moves a hypothesis from the "must use" context to the "may use" context, corresponding to the action of weakening away the hypothesis in question. Thus, depending on the annotation of the sequent, we get the following four systems:

- If the label is $\cdot$, then so is the label on all sequents in the derivation. Thus, the only way to get rid of assumptions is to decompose them using the corresponding inference rule, and there is no way to reintroduce assumptions from the "may use" context into the "must use" context. The resulting system is thus simply IMALL.

- If the label is $\mathbf{w}$, we can apply the demote rule freely, thus allowing us to discard assumptions at will. This corresponds to an affine extension of IMALL.
- If the label is $\mathbf{c}$, we can move assumptions back into the "must use" context, thus corresponding to using an assumption more than once. We are not able, however, to discard assumptions — every assumption must be used at least once, corresponding to a strict extension of IMALL.
- Finally, if the label is $\mathbf{w} + \mathbf{c}$, we can move assumptions freely between the two contexts. This essentially collapses the two contexts into one, and the resulting system is equivalent to LJ.

As an example of a derivation in this system, here is a proof of $a \multimap (a \otimes a)$ using contraction:

$$\dfrac{\dfrac{\dfrac{}{\cdot; a \xrightarrow{\cdot} a}\,\text{init} \qquad \dfrac{\dfrac{}{\cdot; a \xrightarrow{\cdot} a}\,\text{init}}{a; \cdot \xrightarrow{\mathbf{c}} a}\,\text{promote}}{\cdot; a \xrightarrow{\mathbf{c}} a \otimes a}\,\otimes\text{R}}{\cdot; \cdot \xrightarrow{\mathbf{c}} a \multimap (a \otimes a)}\,\multimap\text{R}$$

Note that although the context Γ above is not the usual persistent context seen in e.g. DILL [BP96], it admits weakening nonetheless:

Theorem 2 (Weakening)**.** *The following rule is admissible:*

$$\dfrac{\Gamma; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\,\text{weaken}$$

Furthermore, it is strongly *admissible, in the sense that it does not change the shape of the resulting derivation.*

Proof. By structural induction on the derivation of $\Gamma; \Delta \xrightarrow{x} C$. □

Conversely, contraction is also admissible. Because of the dyadic context and the rules for moving across this context, proving this requires a mutual induction between three related forms of contraction:

Theorem 3 (Contraction)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma, A, A; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\,\text{ucontract} \qquad \dfrac{\Gamma, A; \Delta, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x} C}\,\text{contract} \qquad \dfrac{\Gamma; \Delta, A, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C}\,\text{pcontract}$$

Proof. By structural induction on the given derivations. We will show a few of the more interesting cases:

- pcontract, principal case for $\multimap$ L:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2, A \multimap B; \Delta_1, A \multimap B \xrightarrow{x} A} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{y} C}}{\Gamma; \Delta_1, \Delta_2, A \multimap B, A \multimap B \xrightarrow{x+y} C}\,\multimap\text{L}}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y+\mathbf{c}} C}\,\text{pcontract}$$

$$\begin{array}{lll} & \Gamma, \Delta_2; \Delta_1, A \multimap B \xrightarrow{x} A & \text{by contract on } \mathcal{D}. \\ \mathcal{D}' :: & \Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x+\mathbf{c}} A & \text{by promote.} \\ & \Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y+\mathbf{c}} C & \text{by} \multimap \text{L on } \mathcal{D}', \mathcal{E}. \end{array}$$

- contract, principal case for promote:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \Delta, A, A \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A \xrightarrow{x+\mathbf{c}} C}\text{promote}}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C}\text{contract}$$

$\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C$ by pcontract on $\mathcal{D}$.

□

With contraction and weakening in place, we can now prove the central cut admissibility result. Again, because we have two contexts, we must have two cuts corresponding to which context the cut formula is in. The second cut rule, ucut, is a curious generalisation of the cut rule for the persistent context one sees in presentations of ILL.

Theorem 4 (Cut admissibility)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma; \Delta \xrightarrow{x} A \qquad \Gamma; \Delta, A \xrightarrow{y} C}{\Gamma; \Delta, \Delta \xrightarrow{x+y} C}\text{cut} \qquad \dfrac{\Gamma; \Delta \xrightarrow{x} A \qquad \Gamma, A; \Delta \xrightarrow{y} C}{\Gamma, \Delta; \Delta \xrightarrow{x+y} C}\text{ucut}$$

Proof. By structural induction on the cut formula and the given derivations. Again, we show some of the more interesting cases:

- Principal cut for $\multimap$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \Delta, A \xrightarrow{x} B\end{array}}{\Gamma; \Delta \xrightarrow{x} A \multimap B}\multimap\text{R} \qquad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{y} A\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{z} C\end{array}}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{y+z} C}\multimap\text{L}}{\Gamma; \Delta, \Delta_1, \Delta_2 \xrightarrow{x+y+z} C}\text{cut}$$

  Let $\mathcal{D}'$ be the entire first premiss of this cut, i.e. the proof of $\Gamma; \Delta \xrightarrow{x} A \multimap B$. We now reason as follows (with implicit appeals to the weakening lemma):

  $\mathcal{E}_1' :: \Gamma, \Delta_2, \Delta; \Delta_1 \xrightarrow{x+y} A$ by ucut on $A \multimap B, \mathcal{D}', \mathcal{E}_1$.
  $\mathcal{E}_2' :: \Gamma, \Delta_1, \Delta; \Delta_2, B \xrightarrow{x+z} C$ by ucut on $A \multimap B, \mathcal{D}', \mathcal{E}_2$.
  $\mathcal{F} :: \Gamma, \Delta_2, \Delta; \Delta, \Delta_1 \xrightarrow{x+y} B$ by cut on $A, \mathcal{E}_1', \mathcal{D}$.
  $\Gamma, \Delta, \Delta_1, \Delta_2; \Delta, \Delta_1, \Delta_2 \xrightarrow{x+y+z} C$ by cut on $B, \mathcal{F}, \mathcal{E}_2'$.
  $\Gamma; \Delta, \Delta_1, \Delta_2 \xrightarrow{x+y+z} C$ by contract (repeated).

- Cutting against a promotion:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \Delta \xrightarrow{x} A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{E}\\ \Gamma; \Delta', A \xrightarrow{y} C\end{array}}{\Gamma, A; \Delta' \xrightarrow{y+\mathbf{c}} C}\text{promote}}{\Gamma, \Delta; \Delta' \xrightarrow{x+y+\mathbf{c}} C}\text{ucut}$$

  $\Gamma; \Delta', \Delta \xrightarrow{x+y} C$ by cut on $A, \mathcal{D}, \mathcal{E}$.
  $\Gamma, \Delta; \Delta' \xrightarrow{x+y+\mathbf{c}} C$ by promote (repeated).

□

Observe that this has exactly the structure we would have liked to see for reducing a cut against a contraction: the cut is permuted above the structural rule, which may therefore need to be replaced with several instances of said structural rule. Because the promote rule does not duplicate any formulas, however, the induction ordering is not violated.

$$\frac{}{a \overset{x}{\Longrightarrow} a}\text{init} \qquad \frac{}{\Delta \overset{x}{\Longrightarrow} \top}\top\text{R} \qquad \frac{}{\Delta, \mathbf{0} \overset{x}{\Longrightarrow} C}\mathbf{0}\text{L} \qquad \frac{}{\cdot \overset{x}{\Longrightarrow} \mathbf{1}}\mathbf{1}\text{R}$$

$$\frac{\Delta, A_i \overset{x}{\Longrightarrow} C}{\Delta, A_1 \,\&\, A_2 \overset{x}{\Longrightarrow} C}\&\text{L}_i \qquad \frac{\Delta \overset{x}{\Longrightarrow} A \qquad \Delta \overset{y}{\Longrightarrow} B}{\Delta \overset{x+y}{\Longrightarrow} A \,\&\, B}\&\text{R} \qquad \frac{\Delta \overset{x}{\Longrightarrow} C}{\Delta, \mathbf{1} \overset{x}{\Longrightarrow} C}\mathbf{1}\text{L}$$

$$\frac{\Delta, A, B \overset{x}{\Longrightarrow} C}{\Delta, A \otimes B \overset{x}{\Longrightarrow} C}\otimes\text{L} \qquad \frac{\Delta_1 \overset{x}{\Longrightarrow} A \qquad \Delta_2 \overset{y}{\Longrightarrow} B}{\Delta_1, \Delta_2 \overset{x+y}{\Longrightarrow} A \otimes B}\otimes\text{R}$$

$$\frac{\Delta, A \overset{x}{\Longrightarrow} C \qquad \Delta, B \overset{y}{\Longrightarrow} C}{\Delta, A \oplus B \overset{x+y}{\Longrightarrow} C}\oplus\text{L} \qquad \frac{\Delta \overset{x}{\Longrightarrow} A_i}{\Delta \overset{x}{\Longrightarrow} A_1 \oplus A_2}\oplus\text{R}_i$$

$$\frac{\Delta_1 \overset{x}{\Longrightarrow} A \qquad \Delta_2, B \overset{y}{\Longrightarrow} C}{\Delta_1, \Delta_2, A \multimap B \overset{x+y}{\Longrightarrow} C}\multimap\text{L} \qquad \frac{\Delta, A \overset{x}{\Longrightarrow} B}{\Delta \overset{x}{\Longrightarrow} A \multimap B}\multimap\text{R}$$

$$\frac{\Delta, A, A \overset{x}{\Longrightarrow} C}{\Delta, A \overset{x+\mathbf{c}}{\Longrightarrow} C}\text{contract} \qquad \frac{\Delta \overset{x}{\Longrightarrow} C}{\Delta, A \overset{x+\mathbf{w}}{\Longrightarrow} C}\text{weaken}$$

Figure 2: Sequent calculus for IMALL with tracked contraction and weakening.

3 Soundness and Completeness

In this section, we will show that the sequent calculus introduced in the previous section is complete and (almost) sound with regard to the standard presentation of IMALL and the structural rules of contraction and weakening. Again, we annotate the sequent arrow to keep track of which structural rules are used. The inference rules can be seen in Figure 2.

Theorem 5 (Completeness)**.** *If* $\Delta \overset{x}{\Longrightarrow} C$ *then* $\Gamma; \Delta \overset{x}{\longrightarrow} C$ *for any* Γ*.*

Proof. By induction over the structure of $\Delta \overset{x}{\Longrightarrow} C$. In the case of contract:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \Delta, A, A \overset{x}{\Longrightarrow} C\end{matrix}}{\Delta, A \overset{x+\mathbf{c}}{\Longrightarrow} C}\text{contract}}{\Gamma; \Delta, A \overset{x+\mathbf{c}}{\longrightarrow} C}\text{(to show)}$$

we appeal to the contraction lemma:

$\Gamma; \Delta, A, A \overset{x}{\longrightarrow} C$ by the induction hypothesis on $\mathcal{D}$.
$\Gamma; \Delta, A \overset{x+\mathbf{c}}{\longrightarrow} C$ by pcontract.

□

Stating the soundness theorem is, alas, not quite as straightforward. A problem in this regard is that we need to account for the fact that Γ can be weakened and thus can contain assumptions that are irrelevant to the given derivation. A more serious issue is that the calculus is in fact not sound for all fragments. To account for this, we state the theorem as follows:

Theorem 6 (Soundness). *If $\Gamma; \Delta \xrightarrow{x} C$, and there are no occurrences of the $\oplus$L or &R rules with the sequent annotation* $\mathbf{c}$*, then $\Gamma^?, \Delta \xRightarrow{x} C$ for some $\Gamma^? \subseteq \Gamma$. If $\mathbf{c} \nleq x$ i.e. there are no occurrences of the* promote *rule in the given derivation, then $\Gamma^?$ may be chosen to be empty.*

Proof. To make the proof more intuitive, we will use $(-)^?$ to indicate that the given formula may occur 0 or 1 times. This is extended to contexts in such a way that e.g. $(A, B)^?$ is either (A, B), (A), (B) or $(\cdot)$. When we apply the contraction rule in the following, we will allow the contraction of $A^?$ and A into just A. If $A^? = A$ this is just contraction, and if $A^? = \cdot$, the contraction rule simply disappears.

Similarly, given $\Gamma^?, \Gamma^?$, we can always contract this together into $\Gamma^?$ by repeated uses of the contract rule.

The proof proceeds by induction on the derivation of $\Gamma; \Delta \xrightarrow{x} C$:

- *Case* $\multimap$L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma, \Delta_2, A_1 \multimap A_2; \Delta_1 \xrightarrow{x} A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma, \Delta_1, A_1 \multimap A_2; \Delta_2, A_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2 \xrightarrow{x+y} C} \multimap \mathrm{L}}{\Gamma^?, \Delta_1, \Delta_2, A_1 \multimap A_2 \xRightarrow{x+y} C} \text{ (to show)}$$

If $\mathbf{c} \nleq x + y$ then $\mathbf{c} \nleq x$ and $\mathbf{c} \nleq y$, and we reason as follows:

$$\begin{array}{ll}\mathcal{D}_1' :: \Delta_1 \xRightarrow{x} A_1 & \text{by i.h. on } \mathcal{D}_1.\\ \mathcal{D}_2' :: \Delta_2, A_2 \xRightarrow{y} C & \text{by i.h. on } \mathcal{D}_2.\\ \quad \Delta_1, \Delta_2, A_1 \multimap A_2 \xRightarrow{x+y} C & \text{by } \multimap \text{L.}\end{array}$$

If $\mathbf{c} \leq x + y$, then $x + y = x + y + \mathbf{c}$ and we reason as follows:

$$\begin{array}{ll}\mathcal{D}_1' :: \Gamma^?, \Delta_2^?, (A_1 \multimap A_2)^?, \Delta_1 \xRightarrow{x} A_1 & \text{by i.h. on } \mathcal{D}_1.\\ \mathcal{D}_2' :: \Gamma^?, \Delta_1^?, (A_1 \multimap A_2)^?, \Delta_2, A_2 \xRightarrow{y} C & \text{by i.h. on } \mathcal{D}_2.\\ \quad \Gamma^?, \Gamma^?, \Delta_1^?, \Delta_2^?, (A_1 \multimap A_2)^?, (A_1 \multimap A_2)^?, \Delta_1, \Delta_2, A_1 \multimap A_2 \xRightarrow{x+y} C & \\ & \text{by } \multimap \text{L.}\\ \quad \Gamma^?, \Delta_1, \Delta_2, A_1 \multimap A_2 \xRightarrow{x+y+\mathbf{c}} C & \text{by contract.}\end{array}$$

$\square$

The extra restrictions on occurrences of the $\oplus$L and & R rules are crucial in order for the proof to go through.

To see where the proof would otherwise break down, this, let us consider the case for &R:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma; \Delta \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma; \Delta \xrightarrow{y} C_2\end{array}}{\Gamma; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2} \&\mathrm{R}}{\Gamma^?, \Delta \xRightarrow{x+y} C_1 \mathbin{\&} C_2} \text{ (to show)}$$

If $\mathbf{c} \nleq x$ we reason as follows:

$$\begin{array}{ll}\mathcal{D}_1' :: \Delta \xRightarrow{x} C_1 & \text{by i.h. on } \mathcal{D}_1.\\ \mathcal{D}_2' :: \Delta \xRightarrow{y} C_2 & \text{by i.h. on } \mathcal{D}_2.\\ \quad \Delta \xRightarrow{x+y} C_1 \mathbin{\&} C_2 & \text{by \&R.}\end{array}$$

If $\mathbf{c} \leq x$, then $x = x + \mathbf{c}$ and we reason as follows:

$\mathcal{D}_1' :: \Gamma^?, \Delta \overset{x}{\Longrightarrow} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma^?, \Delta \overset{y}{\Longrightarrow} C_2$ by i.h. on $\mathcal{D}_2$.

At this point, it would be tempting to attempt to reapply the &R rule in the hopes of deriving $\Gamma^?, \Delta \overset{x+y}{\Longrightarrow} C_1 \mathbin{\&} C_2$. Unfortunately, to apply the rule we need the context in both premises to be exactly the same, and this is not necessarily the case. Recall that the notation $\Gamma^?$ indicates *some* subset of Γ. In particular, there is no guarantee that the two occurrences of $\Gamma^?$ in $\mathcal{D}_1'$ and $\mathcal{D}_2'$ in fact denote the same subset.

If we have $\mathbf{w} \leq x+y$, then we can simply apply weakening to both contexts, turning $\Gamma^?$ into Γ, and then reapply the &R rule to get the desired result. Thus, the only case in which the above proof does not go through is when the derivation is of the form $\Gamma; \Delta \xrightarrow{\mathbf{c}} C$, where there is an occurrence of &R or $\oplus$L somewhere in the derivation. For this reason, we will in the following focus on the case where the sequent arrow annotation is $\mathbf{c}$, i.e. the strict logic fragment, and elide the annotation.

Before we show how to fix this problem, we note that it is not simply a matter of suitably reformulating the soundness theorem to get the proof to go through. To see this, observe that we have the following derivation in our system:

$$\dfrac{\dfrac{}{\cdot; a \longrightarrow a}\,\text{init} \qquad \dfrac{\dfrac{}{a; b \longrightarrow b}\,\text{init} \qquad \dfrac{\dfrac{\dfrac{}{b; a \longrightarrow a}\,\text{init}}{a, b; \cdot \longrightarrow a}\,\text{promote} \qquad \dfrac{\dfrac{}{a; b \longrightarrow b}\,\text{init}}{a, b; \cdot \longrightarrow b}\,\text{promote}}{a, b; \cdot \longrightarrow a \mathbin{\&} b}\,\&\text{R}}{a; b \longrightarrow b \otimes (a \mathbin{\&} b)}\,\otimes\text{R}}{\cdot; a, b \longrightarrow a \otimes (b \otimes (a \mathbin{\&} b))}\,\otimes\text{R}$$

but there is clearly no way to derive $a, b \Longrightarrow a \otimes (b \otimes (a \mathbin{\&} b))$ no matter how contraction is applied.

The problem in this case is that by allowing the context Γ to be copied additively to each premise, we allow the above type of "asymmetric contraction" which is unsound with regard to ordinary contraction.

A similar problem makes itself apparent with the $\oplus$L rule. To fix the above problem, we replace the &R and $\oplus$L rules with the following rules:

$$\dfrac{\cdot; \Delta \xrightarrow{x} A \qquad \cdot; \Delta \xrightarrow{y} B}{\Gamma; \Delta \xrightarrow{x+y} A \mathbin{\&} B}\,\&\text{R} \qquad \dfrac{A \oplus B; \Delta, A \xrightarrow{x} C \qquad A \oplus B; \Delta, B \xrightarrow{y} C}{\Gamma; \Delta, A \oplus B \xrightarrow{x+y} C}\,\oplus\text{L}$$

Intuitively, these rules force any contraction (i.e. use of the promote rule) to take place before the next additive connective is decomposed. By doing this, we effectively prevent the problem that caused the previous unsoundness. That aside, let us now show that with the altered &R and $\oplus$L rules above, the system is sound with regard to MALL. The situation in the &R case is now as follows:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots \\ \cdot; \Delta \xrightarrow{x} C_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \vdots \\ \cdot; \Delta \xrightarrow{y} C_2\end{matrix}}{\Gamma; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\,\&\text{R}}{\Gamma^?, \Delta \overset{x+y}{\Longrightarrow} C_1 \mathbin{\&} C_2}\,\text{(to show)}$$

We reason as follows:

$\mathcal{D}_1' :: (\cdot)^?, \Delta \overset{x}{\Longrightarrow} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: (\cdot)^?, \Delta \overset{y}{\Longrightarrow} C_2$ by i.h. on $\mathcal{D}_2$.

Because $(\cdot)^? = \cdot$ the two contexts are equal, and we may now apply the &R rule to get the desired derivation.

In the case of $\oplus$L:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots \\ A_1 \oplus A_2; \Delta, A_1 \overset{x}{\longrightarrow} C\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots \\ A_1 \oplus A_2; \Delta, A_2 \overset{y}{\longrightarrow} C\end{matrix}}{\Gamma; \Delta, A_1 \oplus A_2 \overset{x+y}{\longrightarrow} C} \oplus\mathrm{L}}{\Gamma^?, \Delta, A_1 \oplus A_2 \overset{x+y}{\Longrightarrow} C} \text{ (to show)}$$

We reason as follows:

$\mathcal{D}_1' :: (A_1 \oplus A_2)^?, \Delta, A_1 \overset{x}{\Longrightarrow} C$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: (A_1 \oplus A_2)^?, \Delta, A_2 \overset{y}{\Longrightarrow} C$ by i.h. on $\mathcal{D}_2$.

If $c \nleq x + y$ then $c \nleq x$ and $c \nleq y$, and hence $(A_1 \oplus A_2)^? = \cdot$ in both derivations. Applying the $\oplus$L rule then gives the desired result.

If on the other hand we have $c \leq x + y$, and in $\mathcal{D}_1'$ we have $(A_1 \oplus A_2)^? = A_1 \oplus A_2$, then we may cut this derivation against one of $A_1 \overset{\cdot}{\Longrightarrow} A_1 \oplus A_2$ to yield a derivation of $\Delta, A_1, A_1 \overset{x}{\Longrightarrow} C$ which can be contracted into $\Delta, A_1 \overset{x}{\Longrightarrow} C$. A similar argument allows us to contract any occurrences of $A_1 \oplus A_2$ into A_2 in $\mathcal{D}_2'$, and at that point we can simply reapply the $\oplus$L rule to get the desired result.

With the above change, we have thus regained soundness, and can now state this result as follows

Theorem 7. *If $\Gamma; \Delta \overset{x}{\longrightarrow} C$, then $\Gamma^?, \Delta \overset{x}{\Longrightarrow} C$ for some $\Gamma^? \subseteq \Gamma$. If $\mathbf{c} \nleq x$ i.e. there are no occurrences of the* promote *rule in the given derivation, then $\Gamma^?$ may be chosen to be empty.*

Of course it is no longer clear that the system is internally sound, i.e. has the cut elimination property. To show this, we must go through the weakening, contraction and cut admissibility theorems and show that these properties continue to hold in this new system. Luckily, because we have fixed the & R and $\oplus$L rules, we have a powerful new tool at our disposal in the form of the following strengthening lemma:

Lemma 1 (Strengthening)**.** *If $\Gamma, A; \Delta \overset{x}{\longrightarrow} C$ then either $\Gamma; \Delta \overset{x}{\longrightarrow} C$ or $\Gamma; \Delta, A \overset{x}{\longrightarrow} C$ and $\mathbf{c} \leq x$. In either case, the height of the resulting derivation is no greater than the height of the given derivation.*

Proof. By induction on the given derivation of $\Gamma, A; \Delta \overset{x}{\longrightarrow} C$. We use the shorthand $\Gamma; \Delta, A^? \overset{x}{\longrightarrow} C$ to represent the result of appealing to the induction hypothesis in cases where it does not matter whether A was strengthened away or not. We show here two representative cases:

- *Case* $\multimap$ L*:*

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots \\ \Gamma, A, \Delta_2, A_1 \multimap A_2; \Delta_1 \overset{x}{\longrightarrow} A_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots \\ \Gamma, A, \Delta_1, A_1 \multimap A_2; \Delta_2, A_2 \overset{y}{\longrightarrow} C\end{matrix}}{\Gamma, A; \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\longrightarrow} C} \multimap \mathrm{L}$$

If applying the induction hypothesis to $\mathcal{D}_1$ and $\mathcal{D}_2$ yields either $\mathcal{D}_1' :: \Gamma, \Delta_2, A_1 \multimap A_2; \Delta_1, A \overset{x}{\longrightarrow} A_1$ or $\mathcal{D}_2' :: \Gamma, \Delta_1, A_1 \multimap A_2; \Delta_2, A_2, A \overset{y}{\longrightarrow} C$, we reason as follows:

$$\Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2, A \xrightarrow{x+y} C$$

by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2$ or $\mathcal{D}_1$ and $\mathcal{D}_2'$.

Otherwise, A must be strengthened away from each subderivation, and we reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, A_1 \multimap A_2; \Delta_1 \xrightarrow{x} A_1$ by i.h. on $\mathcal{D}_1$.

$\mathcal{D}_2' :: \Gamma, \Delta_1, A_1 \multimap A_2; \Delta_2, A_2 \xrightarrow{x} C$ by i.h. on $\mathcal{D}_2$.

$\Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2 \xrightarrow{x+y} C$ by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ R*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma, A; \Delta, C_1 \xrightarrow{x} C_2\end{matrix}}{\Gamma, A; \Delta \xrightarrow{x} C_1 \multimap C_2} \multimap \mathrm{R}$$

We reason as follows:

$\Gamma; \Delta, A^?, C_1 \xrightarrow{x} C_2$ by i.h. on $\mathcal{D}$.

$\Gamma; \Delta, A^? \xrightarrow{x} C_1 \multimap C_2$ by $\multimap$ R.

□

Although this lemma may seem somewhat tame, it will in fact help us to vastly cut down on the number of cases that we need to consider when reestablishing the contraction and cut admissibility properties.

Theorem 8 (Contraction)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma, A, A; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\,\text{ucontract} \qquad \dfrac{\Gamma, A; \Delta, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x} C}\,\text{contract} \qquad \dfrac{\Gamma; \Delta, A, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C}\,\text{pcontract}$$

Proof. First, note that by applying the strengthening lemma to

$$\Gamma, A, A; \Delta \xrightarrow{x} C$$

we get either

$$\Gamma, A; \Delta \xrightarrow{x} C$$

in which case we have shown ucontract admissible, or

$$\Gamma, A; \Delta, A \xrightarrow{x} C, \qquad \mathbf{c} \leq x$$

in which case we can appeal to the admissibility of contract to get

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma, A; \Delta, A \xrightarrow{x} C\end{matrix}}{\Gamma; \Delta, A \xrightarrow{x} C}\,\text{contract}}{\Gamma, A; \Delta \xrightarrow{x} C}\,\text{promote}$$

as $\mathbf{c} \leq x$ implies $x = x + \mathbf{c}$. Thus, the admissibility of the ucontract rule follows directly from the admissibility of the contract rule. Similarly, to prove the admissibility of the contract rule, we can apply the strengthening lemma to

$$\Gamma, A; \Delta, A \xrightarrow{x} C$$

to get either

$$\Gamma; \Delta, A \xrightarrow{x} C$$

in which case we are done, or

$$\Gamma; \Delta, A, A \xrightarrow{x} C, \qquad \mathbf{c} \leq x$$

in which case we may appeal directly to the pcontract rule. Thus, we only need to consider cases for the pcontract rule, which we prove by induction on the structure of the given derivation.. The complete proof may be seen in the appendix. ☐

The cut admissibility theorem follows in a similar way. Again, the use of the strengthening lemma enables us to skip a whole swathe of cases.

Theorem 9 (Cut admissibility)**.** *The following inference rules are admissible*

$$\frac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \Gamma; A, \Delta_2 \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C}\,\text{cut} \qquad \frac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \Gamma, A; \Delta_2 \xrightarrow{y} C}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C}\,\text{ucut}$$

Proof. By lexicographic induction on the cut formulas and the two given derivations. For the ucut rule, we reason as follows. Given

$$\frac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \begin{array}{c}\mathcal{E}\\ \vdots\\ \Gamma, A; \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C}\ \text{(to show)}$$

we first apply the strengthening lemma to $\mathcal{E}$. If this results in a derivation of

$$\Gamma; \Delta_2 \xrightarrow{y} C$$

then the result is immediate by the admissibility of weakening. If not, we have a derivation

$$\mathcal{E}' :: \ \Gamma; \Delta_2, A \xrightarrow{y} C$$

and $\mathbf{c} \leq y$. Applying the cut rule to $(A, \mathcal{D}, \mathcal{E}')$ yields a derivation of the sequent

$$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C$$

and as $\mathbf{c} \leq y$ we have $x + y = x + y + \mathbf{c}$ and we can therefore apply the promote rule repeatedly to get a derivation of the desired sequent.

The complete proof may be seen in the appendix. ☐

4 Variations

Non-contracting Rules. It is well known that for some of the left rules of the sequent calculus it is unnecessary to copy the principal formula to all the premisses of said rule. This is the case for e.g. the ⊗L and ⊸ L rules, where the following more restricted rules have the same expressive power as their unrestricted counterparts in Figure 1:

$$\frac{\Gamma; \Delta, A, B \xrightarrow{x} C}{\Gamma; \Delta, A \otimes B \xrightarrow{x} C}\otimes\text{L} \qquad \frac{\Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x} A \qquad \Gamma, \Delta_1; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C}\multimap\text{L}$$

What is more surprising is that the left rules for & also admit this kind of non-contracting presentation:

$$\frac{\Gamma, A; \Delta, B \xrightarrow{x} C}{\Gamma; \Delta, A \,\&\, B \xrightarrow{x} C}\,\&\text{L}_1 \qquad \frac{\Gamma, B; \Delta, A \xrightarrow{x} C}{\Gamma; \Delta, A \,\&\, B \xrightarrow{x} C}\,\&\text{L}_2$$

The benefit of this seems limited, however, as using these rules instead greatly complicates the proof of the admissibility of contraction.

Asymmetric splitting. One curious aspect of the system presented is the fact that although it is sound and complete at the level of provability, in general there will be many more derivations in the system with the promote rule than in the system with the contract rule. For example, there is essentially one proof of $a \overset{\mathbf{c}}{\Longrightarrow} (a \otimes a)$:

$$\dfrac{\dfrac{\dfrac{}{a \overset{\cdot}{\Longrightarrow} a}\,\text{init} \qquad \dfrac{}{a \overset{\cdot}{\Longrightarrow} a}\,\text{init}}{a, a \overset{\cdot}{\Longrightarrow} a \otimes a}\,\otimes\text{R}}{a \overset{\mathbf{c}}{\Longrightarrow} a \otimes a}\,\text{contract}$$

In the present system, however, there are two derivations, corresponding to which branch of the $\otimes$R rule gets the principal copy of a:

$$\dfrac{\dfrac{}{\cdot; a \overset{\cdot}{\longrightarrow} a}\,\text{init} \qquad \dfrac{\dfrac{}{\cdot; a \overset{\cdot}{\longrightarrow} a}\,\text{init}}{a; \cdot \overset{\mathbf{c}}{\longrightarrow} a}\,\text{promote}}{a \overset{\mathbf{c}}{\Longrightarrow} a \otimes a}\,\otimes\text{R} \qquad \dfrac{\dfrac{\dfrac{}{\cdot; a \overset{\cdot}{\longrightarrow} a}\,\text{init}}{a; \cdot \overset{\mathbf{c}}{\longrightarrow} a}\,\text{promote} \qquad \dfrac{}{\cdot; a \overset{\cdot}{\longrightarrow} a}\,\text{init}}{a \overset{\mathbf{c}}{\Longrightarrow} a \otimes a}\,\otimes\text{R}$$

One way of fixing this is to make the $\otimes$R rule, and indeed any rule that splits the context, asymmetric in how it handles the splitting.

First, observe that any instance of the promote rule can be permuted down until just above the rule that introduces the principal connective into the "may use" context. Assuming this has been done, we note that the following reduction is always permitted:

$$\dfrac{\dfrac{\vdots \atop \Gamma, \Delta_2; \Delta_1, A \overset{x}{\longrightarrow} C}{\Gamma, \Delta_2, A; \Delta_1 \overset{x+\mathbf{c}}{\longrightarrow} C}\,\text{p} \qquad {\vdots \atop \Gamma, \Delta_1; \Delta_2, A \overset{y}{\longrightarrow} D}}{\Gamma; \Delta_1, \Delta_2, A \overset{x+y+\mathbf{c}}{\longrightarrow} C \otimes D}\,\otimes\text{R} \quad \rightsquigarrow \quad \dfrac{{\vdots \atop \Gamma, \Delta_2; \Delta_1, A \overset{x}{\longrightarrow} C} \qquad \dfrac{\vdots \atop \Gamma, \Delta_1; \Delta_2, A \overset{y}{\longrightarrow} D}{\Gamma, \Delta_1, A; \Delta_2 \overset{y+\mathbf{c}}{\longrightarrow} D}\,\text{p}}{\Gamma; \Delta_1, \Delta_2, A \overset{x+y+\mathbf{c}}{\longrightarrow} C \otimes D}\,\otimes\text{R}$$

where p is used as shorthand for the promote rule. Assuming the above reduction is applied maximally, anything left over in Δ_2 in the first premiss can simply be strengthened away, as it is not promoted back into the linear context. In other words, the following *asymmetric* version of the $\otimes$R rule is sufficient:

$$\dfrac{\Gamma; \Delta_1 \overset{x}{\longrightarrow} A \qquad \Gamma, \Delta_1; \Delta_2 \overset{y}{\longrightarrow} B}{\Gamma; \Delta_1, \Delta_2 \overset{x+y}{\longrightarrow} A \otimes B}\,\otimes\text{R}$$

Adopting this change does not complicate the proof of the admissibility of cut.

5 A Strongly Focused Calculus for Strict Logic

Now that we have established both the internal and external soundness of our calculus, it is natural to consider whether the focusing methodology can be applied to this system. For the linear, affine, and intuitionistic subsystems, this is straightforward and well-known. Focusing is perhaps most elegant in the linear setting, and the addition of weakening does not make matters any more difficult, as this only affects the initial rules. Going from affine to intuitionistic logic has a notable effect on the calculus where contraction is concerned. A standard way of presenting the left implication rule for focused intuitionistic logic is as follows:

$$\dfrac{\Gamma \longrightarrow [A] \qquad \Gamma, [B] \longrightarrow C}{\Gamma, [A \multimap B] \longrightarrow C}\,\multimap\text{L}$$

Note that unlike the previous additive presentation of $\multimap$ L in intuitionistic logic, we do not explicitly make a copy of the principal formula $A \multimap B$. Thus, the rule above is closer to the rule in linear logic, with the exception that the context Γ in the conclusion is not split into two contexts, one for each premiss.

Instead of incorporating contraction in the above rule, it is made a part of the rule for selecting foci on the left, usually referred to as the *left decision* rule:

$$\frac{\Gamma, A, [A] \longrightarrow C}{\Gamma, A \longrightarrow C}\,\text{decL}$$

Essentially, in the focused setting we can restrict the use of (multiplicative) contraction to this rule, which greatly cuts down on the number of formulas that are accumulated in the sequent as we move up the proof.

Now, the above rule may seem a bit dangerous, as it is very similar to the unrestricted contraction rule that we had to work around to get a structural cut admissibility argument. The similarity is only superficial, however: the above rule is *not* an unrestricted use of contraction, precisely because the copy of A that we add to the context is under focus and must therefore be decomposed immediately. This small requirement is enough to allow a cut against the sequent $\Gamma \longrightarrow A$ to go through, by first performing a cut on A, and then a cut on $[A]$. As long as we are able to consider the second cut to be "smaller", this argument is well-founded.

For strict logic, the decL rule above is not sufficient. First of all, we must at the very least have an additional rule that picks a focus without making a copy of the principal formula:

$$\frac{\Gamma, [A] \longrightarrow C}{\Gamma, A \longrightarrow C}\,\text{decL}'$$

Unfortunately, this is not enough. In strict logic, we have that $a \longrightarrow a \otimes a$ is provable, but with the above rules, we cannot prove this. If we focus on the left, we get either the sequent $[a] \longrightarrow a \otimes a$ or $a, [a] \longrightarrow a \otimes a$, both of which which fail since $a \neq a \otimes a$. If we focus on the right, we get (without loss of generality)

$$\frac{\begin{array}{c}\vdots\\ a \longrightarrow [a]\end{array} \qquad \begin{array}{c}\vdots\\ \cdot \longrightarrow [a]\end{array}}{a \longrightarrow a \otimes a}\,\otimes\text{R}$$

and here the second premiss has no proof.

Thus, there is a fundamental tension between contraction and focusing, and this in part explains why such systems have not been presented in the past.

Despite these difficulties, we in fact have all the necessary ingredients for constructing such a calculus. The main ingredient is of course — as we have seen already — a rule that "saves" the contracted formula in a separate context and permits it to be contracted at a later point.

The calculus presented in the previous sections has one major drawback with regard to focusing, however. In the focusing methodology, connectives are assigned a *polarity* based on their inversion properties. Connectives that are invertible when they appear on the left hand side of the sequent are *positive*, and conversely the ones that are invertible on the right hand side of the sequent are *negative*. Usually, this works out nicely, and every connective gets a unique polarity. Unfortunately, two of the connectives are not invertible on *either* side of the sequent. Perhaps unsurprisingly, the culprits are the additive connectives. The & connective, for instance, has the following rules:

$$\frac{\Gamma, A_1 \,\&\, A_2; \Delta, A_i \longrightarrow C}{\Gamma; \Delta, A_1 \,\&\, A_2 \longrightarrow C}\,\&\,\text{L}_i \qquad \frac{\cdot; \Delta \longrightarrow A_1 \qquad \cdot; \Delta \longrightarrow A_2}{\Gamma; \Delta \longrightarrow A_1 \,\&\, A_2}\,\&\,\text{R}$$

If the $\& \mathrm{L}_1$ rule were invertible, then applying it to the sequent $\cdot; a \,\&\, b \longrightarrow b$ would yield the sequent $a \,\&\, b; a \longrightarrow b$. But no matter what rules we apply (bottom-up) in order to prove this sequent, there is no way of getting rid of the a in the linear context, and hence no way of proving just b. A similar argument applies for the $\& \mathrm{L}_2$ rule. To see that the $\& \mathrm{R}$ rule is non-invertible, we note that $a \,\&\, b; \cdot \longrightarrow a \,\&\, b$ is provable — simply apply the promote rule to get $\cdot; a \& b \longrightarrow a \& b$, which is straightforward to prove. If the $\& \mathrm{R}$ rule were invertible, we would get that $\cdot; \cdot \longrightarrow a$ should be provable, and it is clearly not. In this case, the problem is that the $\& \mathrm{R}$ rule discards the context Γ, thus preventing any formulas in this context from being promoted. This creates a dependency: to prove a sequent we may need to apply the promote rule before applying the $\& \mathrm{R}$ rule, and hence the $\& \mathrm{R}$ rule cannot be invertible.

Thus, neither of the rules are invertible, and a similar argument may be applied to the $\oplus \mathrm{R}_i$ and $\oplus \mathrm{L}$ rules. Off hand, it may seem surprising that this is the case, given that inversion properties usually follow from having the cut elimination and identity expansion properties. Thus, to show that $\& \mathrm{R}$ is invertible, one would usually construct the following cut:

$$\dfrac{\dfrac{\vdots}{\Gamma \longrightarrow A \,\&\, B} \qquad \dfrac{\dfrac{}{A \longrightarrow A}\,\mathrm{id}_A}{A \,\&\, B \longrightarrow A}\,\& \mathrm{L}_1}{\Gamma \longrightarrow A}\,\mathrm{cut}$$

and a similar cut would show that $\Gamma \longrightarrow B$ is likewise derivable. In the calculus presented in the previous section, the corresponding cut would be the following:

$$\dfrac{\dfrac{\vdots}{\Gamma; \Delta \longrightarrow A \,\&\, B} \qquad \dfrac{\dfrac{}{A \,\&\, B; A \longrightarrow A}\,\mathrm{id}_A}{\cdot; A \,\&\, B \longrightarrow A}\,\& \mathrm{L}_1}{\Gamma; \Delta \longrightarrow A}\,\mathrm{cut}$$

Note the difference: the resulting sequent $\Gamma; \Delta \longrightarrow A$ does not match the premiss of the $\& \mathrm{R}$ rule, since Γ may be non-empty. In a sense, the calculus is *unharmonious* (albeit for a different notion of harmony than the usual one) — the inversion properties are too weak.

In focused calculi, the counterpart to inversion is *chaining*: once a formula has been selected as a focus, it is decomposed eagerly, retaining this focus for as long as possible. Thus, one might hope that by being non-invertible on both sides of the sequent, the $\oplus$ and $\&$ connectives would instead be *chaining* on both sides. In particular, this would require the $\& \mathrm{R}$ rule to be focused on both the principal formula and its immediate subformulas:

$$\dfrac{\cdot; \Delta \longrightarrow [A] \qquad \cdot; \Delta \longrightarrow [B]}{\Gamma; \Delta \longrightarrow [A \,\&\, B]}\,\& \mathrm{R}$$

Alas, this does not work either. Attempting to prove the sequent $\cdot; a \longrightarrow (a \,\&\, a) \otimes (a \,\&\, a)$, we would (without loss of generality) end up with the following partial proof:

$$\dfrac{\dfrac{\dfrac{\vdots}{\cdot; \cdot \longrightarrow [a]} \qquad \dfrac{\vdots}{\cdot; \cdot \longrightarrow [a]}}{a; \cdot \longrightarrow [a \,\&\, a]}\,\& \mathrm{R} \qquad \dfrac{\vdots}{\cdot; a \longrightarrow [a \,\&\, a]}}{\cdot; a \longrightarrow [(a \,\&\, a) \otimes (a \,\&\, a)]}\,\otimes \mathrm{R}$$

Clearly, the above partial proof cannot be completed.

To present the calculus, we will use a *polarised* calculus, with the following formulas

$$P, Q ::= p \mid P \otimes Q \mid P \oplus Q \mid \mathbf{1} \mid \mathbf{0} \mid \downarrow N$$
$$N, M ::= n \mid P \multimap N \mid N \mathbin{\&} M \mid \top \mid \uparrow P$$

Note in particular that — as is standard — we distinguish between two polarities of the atomic formulas, and we use explicit polarity shifts to mediate between positive and negative formulas. Our system has three judgements:

$$\begin{aligned} \Gamma; \Delta, [N] &\longrightarrow \nu \\ \Gamma; \Delta &\longrightarrow [P] \\ \Sigma &\longrightarrow \langle N \rangle \end{aligned}$$

corresponding to left chaining, right chaining and inversion respectively. We will consistently use the following definitions of contexts:

$$\begin{aligned} \nu &::= n \mid \uparrow P \\ \pi &::= p \mid \downarrow N \\ \Gamma &::= \cdot \mid \Gamma, \pi \\ \Sigma &::= \Gamma \mid \Sigma, \langle P \rangle \end{aligned}$$

The mnemonic for the π and ν classes of formulas is that they are respectively *positive* and *negative*, but *non-invertible* where they occur. We will refer to such formulas as *neutral*. Observe that in particular in the judgements where foci appear, all other formulas must be neutral. Maintaining this invariant ensures that inversion phases are maximal.

The rules of the system can be seen in Figure 3. Observe that during the chaining phase, we have a dual context, just as in the calculi presented previously. Unlike the previous systems, however, any promotion of formulas in these contexts must happen before the next inversion phase begins. This ensures that the additive rules become invertible, unlike the unfocused calculus of the previous section.

When initiating a chaining phase, we populate the extra context either with the formula we focused upon (in the case of the left decision rule) or with no formulas at all (in the case of the right decision rule). Thus, this context tracks the formulas that have "disappeared" during this chaining phase, either because they were consumed when the phase began, or because a multiplicative rule split them off into a different subderivation.

To establish the cut admissibility result, we once again begin with *strengthening*:

Lemma 2 (Strengthening). *The following properties hold:*

1. *If* $\Gamma, \pi; \Delta, [N] \longrightarrow \nu$ *then either* $\Gamma; \Delta, [N] \longrightarrow \nu$ *or* $\Gamma; \Delta, \pi, [N] \longrightarrow \nu$.
2. *If* $\Gamma, \pi; \Delta \longrightarrow [P]$ *then either* $\Gamma; \Delta \longrightarrow [P]$ *or* $\Gamma; \Delta, \pi \longrightarrow [P]$.

Furthermore, the resulting output derivations have the same structure as the input derivations.

Proof. By induction over the structure of the given derivations. See the appendix for the full proof. □

In the unfocused calculus, we directly have the promote rule that always permits us to move hypotheses from the "must use" context to the "may use" context. In the focused calculus, this rule is not explicitly available, and hence we must show it admissible instead:

Left-chaining

$$\frac{}{\Gamma; [n] \longrightarrow n}\,\mathrm{init}^- \qquad \frac{\Gamma; \Delta, [N_i] \longrightarrow \nu}{\Gamma; \Delta, [N_1 \mathbin{\&} N_2] \longrightarrow \nu}\,\&\mathrm{L}_i \qquad \frac{\Gamma, \Delta_2; \Delta_1 \longrightarrow [P] \qquad \Gamma, \Delta_1; \Delta_2, [N] \longrightarrow \nu}{\Gamma; \Delta_1, \Delta_2, [P \multimap N] \longrightarrow \nu}\,{\multimap}\,\mathrm{L}$$

Right-chaining:

$$\frac{}{\Gamma; p \longrightarrow [p]}\,\mathrm{init}^+ \qquad \frac{}{\Gamma; \cdot \longrightarrow [\mathbf{1}]}\,\mathbf{1}\mathrm{R} \qquad \frac{\Gamma, \Delta_2; \Delta_1 \longrightarrow [P] \qquad \Gamma, \Delta_1; \Delta_2 \longrightarrow [Q]}{\Gamma; \Delta_1, \Delta_2 \longrightarrow [P \otimes Q]}\,\otimes\mathrm{R} \qquad \frac{\Gamma; \Delta \longrightarrow [P_i]}{\Gamma; \Delta \longrightarrow [P_1 \oplus P_2]}\,\oplus\mathrm{R}_i$$

Inversion:

$$\frac{}{\Sigma \longrightarrow \langle \top \rangle}\,\top\mathrm{R} \qquad \frac{}{\Sigma, \langle \mathbf{0} \rangle \longrightarrow \langle N \rangle}\,\mathbf{0}\mathrm{L} \qquad \frac{\Sigma \longrightarrow \langle N \rangle \qquad \Sigma \longrightarrow \langle M \rangle}{\Sigma \longrightarrow \langle N \mathbin{\&} M \rangle}\,\&\mathrm{R} \qquad \frac{\Sigma \longrightarrow \langle N \rangle}{\Sigma, \langle \mathbf{1} \rangle \longrightarrow \langle N \rangle}\,\mathbf{1}\mathrm{L}$$

$$\frac{\Sigma, \langle P \rangle, \langle Q \rangle \longrightarrow \langle N \rangle}{\Sigma, \langle P \otimes Q \rangle \longrightarrow \langle N \rangle}\,\otimes\mathrm{L} \qquad \frac{\Sigma, \langle P \rangle \longrightarrow \langle N \rangle \qquad \Sigma, \langle Q \rangle \longrightarrow \langle N \rangle}{\Sigma, \langle P \oplus Q \rangle \longrightarrow \langle N \rangle}\,\oplus\mathrm{L} \qquad \frac{\Sigma, \langle P \rangle \longrightarrow \langle N \rangle}{\Sigma \longrightarrow \langle P \multimap N \rangle}\,{\multimap}\,\mathrm{R}$$

Structural:

$$\frac{{\downarrow}N; \Gamma, [N] \longrightarrow \nu}{\Gamma, {\downarrow}N \longrightarrow \nu}\,{\downarrow}\mathrm{L} \qquad \frac{\cdot; \Gamma \longrightarrow [P]}{\Gamma \longrightarrow {\uparrow}P}\,{\uparrow}\mathrm{R} \qquad \frac{\Gamma, \Delta \longrightarrow \langle N \rangle}{\Gamma, \Gamma'; \Delta \longrightarrow [{\downarrow}N]}\,{\downarrow}\mathrm{R} \qquad \frac{\Gamma, \Delta, \langle P \rangle \longrightarrow \langle \nu \rangle}{\Gamma, \Gamma'; \Delta, [{\uparrow}P] \longrightarrow \nu}\,{\uparrow}\mathrm{L}$$

$$\frac{\Gamma, \Delta \longrightarrow \nu}{\Gamma, \langle \Delta \rangle \longrightarrow \langle \nu \rangle}\,\mathrm{rel}$$

Figure 3: Focused sequent calculus for IMALL with contraction.

Lemma 3 (Admissibility of promotion)**.** *The following inference rules are admissible*

$$\frac{\Gamma; \Delta, \pi, [N] \longrightarrow \nu}{\Gamma, \pi; \Delta, [N] \longrightarrow \nu}\,\mathrm{promote}^- \qquad \frac{\Gamma; \Delta, \pi \longrightarrow [P]}{\Gamma, \pi; \Delta \longrightarrow [P]}\,\mathrm{promote}^+$$

Proof. By induction on the structure of the given derivations. See the appendix for the full proof. ☐

Next, we need a focused version of the contraction result. We show this only for neutral positive formulas:

Theorem 10 (Neutral contraction)**.** *The following inference rules are admissible*

$$\frac{\Gamma, \pi, \pi; \Delta, [N] \longrightarrow \nu}{\Gamma, \pi; \Delta, [N] \longrightarrow \nu}\,\mathrm{ucontract}^- \qquad \frac{\Gamma, \pi; \Delta, \pi, [N] \longrightarrow \nu}{\Gamma; \Delta, \pi, [N] \longrightarrow \nu}\,\mathrm{contract}^- \qquad \frac{\Gamma; \Delta, \pi, \pi, [N] \longrightarrow \nu}{\Gamma; \Delta, \pi, [N] \longrightarrow \nu}\,\mathrm{pcontract}^-$$

$$\frac{\Gamma, \pi, \pi; \Delta \longrightarrow [P]}{\Gamma, \pi; \Delta \longrightarrow [P]}\,\mathrm{ucontract}^+ \qquad \frac{\Gamma, \pi; \Delta, \pi \longrightarrow [P]}{\Gamma; \Delta, \pi \longrightarrow [P]}\,\mathrm{contract}^+ \qquad \frac{\Gamma; \Delta, \pi, \pi \longrightarrow [P]}{\Gamma; \Delta, \pi \longrightarrow [P]}\,\mathrm{pcontract}^+$$

$$\frac{\Sigma, \pi, \pi \longrightarrow \langle N \rangle}{\Sigma, \pi \longrightarrow \langle N \rangle}\,\mathrm{icontract} \qquad \frac{\Gamma, \pi, \pi \longrightarrow \nu}{\Gamma, \pi \longrightarrow \nu}\,\mathrm{contract}$$

Proof. By repeated use of the preceding lemmas, it suffices to show that the pcontract$^-$ and pcontract$^+$ rules are admissible. Thus, for instance, given

$$\Gamma, \pi, \pi; \Delta \longrightarrow [P]$$

we apply the strengthening lemma to get either

$$\Gamma, \pi; \Delta \longrightarrow [P] \qquad \text{or} \qquad \Gamma, \pi; \Delta, \pi \longrightarrow [P]$$

In the former case, we are done, and in the latter case, we appeal to the contract$^+$ rule followed by the promote$^+$ rule. For the contract$^+$ rule, we again apply the strengthening lemma, resulting in either

$$\Gamma; \Delta, \pi \longrightarrow [P] \qquad \text{or} \qquad \Gamma; \Delta, \pi, \pi \longrightarrow [P]$$

Again, in the former case, we are done, and in the latter case we appeal to the pcontract$^+$ rule. A similar argument reduces ucontract$^-$ to contract$^-$ to pcontract$^-$.

To show these cases, we reason by induction on the given derivations. The full proof may be seen in the appendix. ☐

With these lemmas, we are now able to state and prove the admissibility of cut:

Theorem 11 (Admissibility of cut)**.** *The following rules are admissible:*

$$\frac{\Gamma; \Delta \longrightarrow [P] \qquad \Sigma, \langle P \rangle \longrightarrow \langle N \rangle}{\Gamma^?, \Delta, \Sigma \longrightarrow \langle N \rangle}\,\text{cut}^+ \qquad \frac{\Sigma \longrightarrow \langle N \rangle \qquad \Gamma; \Delta, [N] \longrightarrow \nu}{\Gamma^?, \Delta, \Sigma \longrightarrow \nu}\,\text{cut}^-$$

where $\Gamma^?$ *represents* some *subset of* Γ.

$$\frac{\Gamma \longrightarrow \langle N \rangle \qquad \Delta, {\downarrow} N \longrightarrow \nu}{\Gamma, \Delta \longrightarrow \nu}\,\text{ncut} \qquad \frac{\Gamma \longrightarrow \langle N \rangle \qquad \Delta, {\downarrow} N \longrightarrow \langle M \rangle}{\Gamma, \Delta \longrightarrow \langle M \rangle}\,\text{icut}$$

$$\frac{\Gamma_1 \longrightarrow \langle N \rangle \qquad \Gamma_2; \Delta, {\downarrow} N, [M] \longrightarrow \nu}{\Gamma_2; \Gamma_1, \Delta, [M] \longrightarrow \nu}\,\text{ncut}^- \qquad \frac{\Gamma_1 \longrightarrow \langle N \rangle \qquad \Gamma_2; \Delta, {\downarrow} N \longrightarrow [P]}{\Gamma_2; \Gamma_1, \Delta \longrightarrow [P]}\,\text{ncut}^+$$

$$\frac{\Gamma_1 \longrightarrow \langle N \rangle \qquad \Gamma_2, {\downarrow} N; \Delta, [M] \longrightarrow \nu}{\Gamma_2, \Gamma_1; \Delta, [M] \longrightarrow \nu}\,\text{ucut}^- \qquad \frac{\Gamma_1 \longrightarrow \langle N \rangle \qquad \Gamma_2, {\downarrow} N; \Delta \longrightarrow [P]}{\Gamma_2, \Gamma_1; \Delta \longrightarrow [P]}\,\text{ucut}^+$$

Proof. By induction on the given derivations. Before we start the proof proper, we will make a few simplifying observations. Consider first the cut$^+$ rule:

$$\frac{\Gamma; \Delta \longrightarrow [P] \qquad \Sigma, \langle P \rangle \longrightarrow \langle N \rangle}{\Gamma^?, \Delta, \Sigma \longrightarrow \langle N \rangle}\,\text{cut}^+$$

by repeatedly applying the strengthening lemma to the first premiss, we get a derivation of $\cdot; \Gamma^?, \Delta \longrightarrow [P]$ with the same shape as the original derivation. We are therefore justified in showing a slightly more restrictive variant of the cut$^+$ rule,

$$\frac{\cdot; \Delta \longrightarrow [P] \qquad \Sigma, \langle P \rangle \longrightarrow \langle N \rangle}{\Delta, \Sigma \longrightarrow \langle N \rangle}\,\text{cut}^+$$

in cases where this is convenient. A similar argument holds for cut$^-$. Regarding the $(-)^?$ notation, we note the following properties: $(\cdot)^? = \cdot$, and $\Gamma^? \subseteq \Gamma$, and if contraction is available, $\Gamma^?, \Gamma^?$ (where each occurrence of $(-)^?$ may represent a *different* subset of Γ) can always be contracted into $\Gamma^?$. In particular, $\Gamma^?, \Gamma$ can be contracted into Γ.

Likewise, we may use the strengthening lemma to reduce ucut$^-$ to ncut$^-$ and ucut$^+$ to ncut$^+$. Applying strengthening to the formula $\downarrow N$ in $\Gamma_2, \downarrow N; \Delta, [M] \longrightarrow \nu$ results in either a derivation of $\Gamma_2; \Delta, [M] \longrightarrow \nu$ in which case we may simply weaken the first context with Γ_1 to get the desired result, or alternatively we get a derivation of $\Gamma_2; \Delta, \downarrow N, [M] \longrightarrow \nu$. In the latter case, the ncut$^-$ rule gives us $\Gamma_2; \Delta, \Gamma_1, [M] \longrightarrow \nu$, and repeated use of the promote$^-$ rule then yields the desired sequent $\Gamma_2, \Gamma_1; \Delta, [M] \longrightarrow \nu$.

Starting with the cut$^+$ and cut$^-$ rules (which contain the principal cases of the cut admissibility argument), we will now sketch the way in which the various cuts are reduced. In the aforementioned rules, we proceed by induction on the derivation that does *not* contain the focused formula. There are essentially two cases to consider: either the cut formula $\langle P \rangle$ or $\langle N \rangle$ is the principal formula of the final rule of the derivation, in which case we have a principal cut, and reduce it to cuts on subformulas, or the cut formula is *not* principal, in which case we simply permute this rule below the cut. The remaining cuts likewise have one premiss that can only be decomposed one way: $\Delta \longrightarrow \langle N \rangle$ or $\Delta, \langle P \rangle \longrightarrow \nu$. The cuts containing these premisses are permuted up the *other* premiss until the cut formula ($\downarrow N$ or $\uparrow P$) is focused upon, at which point we return to the principal cuts. □

With the cut admissibility theorem, and the corresponding identity expansion property — that $\langle P \rangle \longrightarrow \uparrow P$ and $\downarrow N \longrightarrow \langle N \rangle$ for all P and N — it is straightforward to show that all unfocused rules can be simulated in the focused system, thus proving its completeness [Sim14, CPP08]. Soundness is trivial, as usual, since the focused rules are simply more restricted versions of their unfocused counterparts.

6 Future Work

Exponentials In this paper, we chose to use IMALL as the base system. An obvious next step would be to add exponentials to the system, to see whether the cut elimination result extends to full ILL. We conjecture that the ! modality can be implemented using the following inference rules:

$$\frac{\Gamma; \cdot \xrightarrow{x} A}{\Gamma; \cdot \xrightarrow{x} !A}\,!\mathrm{R} \qquad \frac{\Gamma, !A; \Delta, A \xrightarrow{x} C}{\Gamma, !A; \Delta \xrightarrow{x} C}\,!\mathrm{L} \qquad \frac{\Gamma, !A; \Delta \xrightarrow{x} C}{\Gamma; \Delta, !A \xrightarrow{x} C}\,!\mathrm{L}$$

Alternatively, one could add a third context representing the truly persistent assumptions.

Structural Modalities The way our system tracks the structural rules as part of the sequent suggests that it might be possible to internalise this as a modality. Thus, one might consider rules of the following form:

$$\frac{\Gamma; \Delta \xrightarrow{x} A}{\Gamma; \Delta \xrightarrow{x+s} [s]A}\,[s]\mathrm{R} \qquad \frac{\Gamma; \Delta \xrightarrow{x-s} A}{\Gamma; \Delta \xrightarrow{x} [\neg s]A}\,[\neg s]\mathrm{R}$$

Here, $[s]A$ would indicate that A is provable in the presence of the structural rules in s, whereas $[\neg s]A$ conversely indicates that A is provable in the *absence* of these rules. It is not clear, however, whether there are sensible left rules to accompany the above rules in such a way that cut elimination still holds.

7 Conclusion

We have presented a sequent calculus for Intuitionistic MALL and the affine, strict, and intuitionistic extensions arising from the addition of contraction and weakening to this system. The structural cut admissibility argument we present works uniformly for all combinations of the structural rules. Moreover, proving the soundness and completeness of this system with regard to the usual presentation can be done by a straightforward induction.

The calculus itself can be related to the "Omnibus" logic presented by Hodas in his thesis [Hod94]. This logic explicitly encodes intuitionistic, affine, strict and linear behaviour through a context with four parts, each corresponding to a logical fragment. Hypotheses in the "strict" context can be contracted into two copies — one in the intuitionistic context, and one in the linear context. This ensures that strict hypotheses must be used at least once.

It is not immediately clear whether this logic admits a straightforward proof of cut-elimination (and Hodas does not attempt to do this). One potential problem is that cutting against the contraction mentioned above requires two separate cuts on the same cut formula, much like in the usual formulation of LJ with an explicit contraction rule. Because of this, it is not immediately clear that the cuts can be justified through a simple lexicographic ordering.

Finally, we should add that Hodas' work only considers a small fragment of the possible connectives. As an example of this, one can consider the additive disjunction, for which Hodas does not include the left rule.

The main novelty of this paper is the presentation of a structural cut admissibility argument for strict logic. In contrast, the admissibility of cut for linear, affine and intuitionistic logic is comparatively well known. This is not to say that cut elimination for logics with contraction but not weakening (e.g. relevance logic) has not been attempted before. In [DR02], Dunn and Restall present a cut elimination argument that proceeds along the lines of Gentzen's original proof. Because of this, the termination metric is much more elaborate.

Finally, we have presented a strongly focused sequent calculus for strict logic, and showed how to unite focusing and contraction in the absence of weakening.

Acknowledgements

I would like to thank Frank Pfenning and others who provided useful feedback on this work. A special thanks is due to my thesis committee (Roy Dyckhoff, Dale Miller and Thore Husfeldt) for insisting I write out detailed proofs for my claims about the first calculus presented in this paper. Without them, I would probably not have discovered its unsoundness.

References

[And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.

[BP96] Andrew Barber and Gordon Plotkin. *Dual intuitionistic linear logic.* University of Edinburgh, Department of Computer Science, Laboratory for Foundations of Computer Science, 1996.

[CPP08] Kaustuv Chaudhuri, Frank Pfenning, and Greg Price. A logical characterization of forward and backward chaining in the inverse method. *J. Autom. Reasoning*, 40(2-3):133–177, 2008.

[DR02] J Michael Dunn and Greg Restall. Relevance logic. In *Handbook of philosophical logic*, pages 1–128. Springer, 2002.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Hod94] Joshua Hodas. *Logic Programming in Intuitionistic Linear Logic: Theory, Design, and Implemenation.* Dissertation, University of Pennsylvania, 1994.

[Pfe00] Frank Pfenning. Structural cut elimination - i. intuitionistic and classical logic. *Information and Computation*, 157, 2000.

[Sim14] Robert Simmons. Structural focalization. *Transactions on Compututational Logic*, 15(3):21, 2014.

A Appendix

In this appendix, we have put the complete proofs of most of the lemmas and theorems of the paper. Be warned that there may be a few inconcistencies in the presentation, as most of the text was taken directly from the author's Ph.d. thesis. I do not recommend attempting to read all these proofs in full — writing them was painful enough!

A.1 Proofs concerning the unfocused calculi

Theorem 12 (Weakening). *The following rule is admissible:*

$$\frac{\Gamma; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\text{weaken}$$

Furthermore, it is strongly *admissible, in the sense that it does not change the shape of the resulting derivation.*

Proof. We proceed by induction on the structure of $\Gamma; \Delta \xrightarrow{x} C$:

- *Case* init*:*

$$\frac{\overline{\Gamma; a \xrightarrow{x} a}\ \text{init}}{\Gamma, A; a \xrightarrow{x} a}\text{(to show)}$$

  We reason as follows:

  $\Gamma, A; a \xrightarrow{x} a$ by init.

- *Case* $\top$R*:*

$$\frac{\overline{\Gamma; \Delta \xrightarrow{x} \top}\ \top\text{R}}{\Gamma, A; \Delta \xrightarrow{x} \top}\text{(to show)}$$

  We reason as follows:

  $\Gamma, A; \Delta \xrightarrow{x} \top$ by $\top$R.

- *Case* $\mathbf{0}$L*:*

$$\frac{\overline{\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C}\ \mathbf{0}\text{L}}{\Gamma, A; \Delta, \mathbf{0} \xrightarrow{x} C}\text{(to show)}$$

  We reason as follows:

  $\Gamma, A; \Delta, \mathbf{0} \xrightarrow{x} C$ by $\mathbf{0}$L.

- *Case* $\mathbf{1}$R*:*

$$\frac{\overline{\Gamma; \cdot \xrightarrow{x} \mathbf{1}}\ \mathbf{1}\text{R}}{\Gamma, A; \cdot \xrightarrow{x} \mathbf{1}}\text{(to show)}$$

  We reason as follows:

  $\Gamma, A; \cdot \xrightarrow{x} \mathbf{1}$ by $\mathbf{1}$R.

- *Case* $\&\text{L}_i$, $i \in \{1, 2\}$*:*

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A_1 \,\&\, A_2; \Delta, A_i \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A_1 \,\&\, A_2 \xrightarrow{x} C}\,\&\text{L}_i}{\Gamma, A; \Delta, A_1 \,\&\, A_2 \xrightarrow{x} C}\text{(to show)}$$

  We reason as follows:

$\Gamma, A, A_1 \mathbin{\&} A_2; \Delta, A_i \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma, A; \Delta, A_1 \mathbin{\&} A_2 \xrightarrow{x} C$ by $\&\mathrm{L}_i$.

- *Case* &R*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma; \Delta \xrightarrow{x} C_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma; \Delta \xrightarrow{y} C_2\end{matrix}}{\Gamma; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\,\&\mathrm{R}}{\Gamma, A; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\,(\text{to show})$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, A; \Delta \xrightarrow{x} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, A; \Delta \xrightarrow{y} C_2$ by i.h. on $\mathcal{D}_2$.
$\Gamma, A; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2$ by &R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\mathbf{1}$L*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ \vdots \\ \Gamma, \mathbf{1}; \Delta \xrightarrow{x} C\end{matrix}}{\Gamma; \Delta, \mathbf{1} \xrightarrow{x} C}\,\mathbf{1}\mathrm{L}}{\Gamma, A; \Delta, \mathbf{1} \xrightarrow{x} C}\,(\text{to show})$$

We reason as follows:

$\Gamma, A, \mathbf{1}; \Delta \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma, A; \Delta, \mathbf{1} \xrightarrow{x} C$ by $\mathbf{1}$L.

- *Case* $\otimes$L*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ \vdots \\ \Gamma, B_1 \otimes B_2; \Delta, B_1, B_2 \xrightarrow{x} C\end{matrix}}{\Gamma; \Delta, B_1 \otimes B_2 \xrightarrow{x} C}\,\otimes\mathrm{L}_i}{\Gamma, A; \Delta, B_1 \otimes B_2 \xrightarrow{x} C}\,(\text{to show})$$

We reason as follows:

$\Gamma, A, B_1 \otimes B_2; \Delta, B_1, B_2 \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma, A; \Delta, B_1 \otimes B_2 \xrightarrow{x} C$ by $\otimes$L.

- *Case* $\otimes$R*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma, \Delta_2; \Delta_1 \xrightarrow{x} C_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma, \Delta_1; \Delta_2 \xrightarrow{y} C_2\end{matrix}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2}\,\otimes\mathrm{R}}{\Gamma, A; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2}\,(\text{to show})$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, A, \Delta_2; \Delta_1 \xrightarrow{x} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, A, \Delta_1; \Delta_2 \xrightarrow{y} C_2$ by i.h. on $\mathcal{D}_2$.
$\Gamma, A; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Gamma, B_1 \oplus B_2; \Delta, B_1 \xrightarrow{x} C\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \Gamma, B_1 \oplus B_2; \Delta, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C}\,\&\mathrm{R}}{\Gamma, A; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C}\,(\text{to show})$$

We reason as follows:

$\mathcal{D}'_1 :: \Gamma, A, B_1 \oplus B_2; \Delta, B_1 \xrightarrow{x} C$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}'_2 :: \Gamma, A, B_1 \oplus B_2; \Delta, B_2 \xrightarrow{y} C$ by i.h. on $\mathcal{D}_2$.
$\Gamma, A; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C$ by $\oplus$L on $\mathcal{D}'_1$ and $\mathcal{D}'_2$.

- *Case* $\oplus\mathrm{R}_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma; \Delta \xrightarrow{x} C_i\end{matrix}}{\Gamma; \Delta \xrightarrow{x} C_1 \oplus C_2}\,\oplus\mathrm{R}_i}{\Gamma, A; \Delta \xrightarrow{x} C_1 \oplus C_2}\,(\text{to show})$$

We reason as follows:

$\Gamma, A; \Delta \xrightarrow{x} C_i$ by i.h. on $\mathcal{D}$.
$\Gamma, A; \Delta \xrightarrow{x} C_1 \oplus C_2$ by $\oplus\mathrm{R}_i$.

- *Case* $\multimap$ L*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}\,\multimap\mathrm{L}}{\Gamma, A; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}\,(\text{to show})$$

We reason as follows:

$\mathcal{D}'_1 :: \Gamma, A, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}'_2 :: \Gamma, A, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C$ by i.h. on $\mathcal{D}_2$.
$\Gamma, A; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C$ by $\multimap$ L on $\mathcal{D}'_1$ and $\mathcal{D}'_2$.

- *Case* $\multimap$ R*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma; \Delta, B \xrightarrow{x} C\end{matrix}}{\Gamma; \Delta \xrightarrow{x} B \multimap C}\,\multimap\mathrm{L}}{\Gamma, A; \Delta \xrightarrow{x} B \multimap C}\,(\text{to show})$$

We reason as follows:

$\Gamma, A; \Delta, B \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma, A; \Delta \xrightarrow{x} B \multimap C$ by $\multimap$ R.

- *Case* promote*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma; \Delta, B \xrightarrow{x} C\end{matrix}}{\Gamma, B; \Delta \xrightarrow{x+\mathbf{c}} C}\,\text{promote}}{\Gamma, B, A; \Delta \xrightarrow{x+\mathbf{c}} C}\,\text{(to show)}$$

We reason as follows:

$\Gamma, A; \Delta, B \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma, B, A; \Delta \xrightarrow{x+\mathbf{c}} C$ by promote.

- *Case* demote*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma, B; \Delta \xrightarrow{x} C\end{matrix}}{\Gamma; \Delta, B \xrightarrow{x+\mathbf{w}} C}\,\text{demote}}{\Gamma, A; \Delta, B \xrightarrow{x+\mathbf{w}} C}\,\text{(to show)}$$

We reason as follows:

$\Gamma, A, B; \Delta \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma, A; \Delta, B \xrightarrow{x+\mathbf{w}} C$ by demote.

□

Theorem 13 (Contraction)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma, A, A; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\,\text{ucontract} \qquad \dfrac{\Gamma, A; \Delta, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x} C}\,\text{contract} \qquad \dfrac{\Gamma; \Delta, A, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C}\,\text{pcontract}$$

Proof. We prove the admissibility by a mutual induction the given input derivations. To increase readability, we will refer to the relevant induction hypothesis using the names given above, e.g. "by pcontract" instead of "by i.h. 3". We begin by considering the cases for the ucontract rule:

- *Case* init*:*

$$\dfrac{\dfrac{}{\Gamma, A, A; a \xrightarrow{x} a}\,\text{init}}{\Gamma, A; a \xrightarrow{x} a}\,\text{ucontract}$$

We reason as follows:

$\Gamma, A; a \xrightarrow{x} a$ by init.

- *Case* $\top$R*:*

$$\dfrac{\dfrac{}{\Gamma, A, A; \Delta \xrightarrow{x} \top}\,\top\text{R}}{\Gamma, A; \Delta \xrightarrow{x} \top}\,\text{ucontract}$$

We reason as follows:

$\Gamma, A; \Delta \xrightarrow{x} \top$ by $\top$R.

- *Case* $\mathbf{0}$L*:*

$$\dfrac{\dfrac{}{\Gamma, A, A; \Delta, \mathbf{0} \xrightarrow{x} C}\,\mathbf{0}\text{L}}{\Gamma, A; \Delta, \mathbf{0} \xrightarrow{x} C}\,\text{ucontract}$$

We reason as follows:

$\Gamma, A; \Delta, \mathbf{0} \xrightarrow{x} C$ by $\mathbf{0}$L.

- *Case* $\mathbf{1}$R*:*

$$\dfrac{\dfrac{}{\Gamma, A, A; \cdot \xrightarrow{x} \mathbf{1}}\ \mathbf{1}\mathrm{R}}{\Gamma, A; \cdot \xrightarrow{x} \mathbf{1}}\ \text{ucontract}$$

We reason as follows:

$\Gamma, A; \cdot \xrightarrow{x} \mathbf{1}$ by $\mathbf{1}$R.

- *Case* &L$_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A, A, A_1 \mathbin{\&} A_2; \Delta, A_i \xrightarrow{x} C\end{array}}{\Gamma, A, A; \Delta, A_1 \mathbin{\&} A_2 \xrightarrow{x} C}\ \&\mathrm{L}_i}{\Gamma, A; \Delta, A_1 \mathbin{\&} A_2 \xrightarrow{x} C}\ \text{ucontract}$$

We reason as follows:

$\Gamma, A, A_1 \mathbin{\&} A_2; \Delta, A_i \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma, A; \Delta, A_1 \mathbin{\&} A_2 \xrightarrow{x} C$ by &L$_i$.

- *Case* &R*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, A, A; \Delta \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, A, A; \Delta \xrightarrow{y} C_2\end{array}}{\Gamma, A, A; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\ \&\mathrm{R}}{\Gamma, A; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\ \text{ucontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, A; \Delta \xrightarrow{x} C_1$ by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, A; \Delta \xrightarrow{y} C_2$ by ucontract on $\mathcal{D}_2$.
$\Gamma, A; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2$ by &R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\mathbf{1}$L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A, A, \mathbf{1}; \Delta \xrightarrow{x} C\end{array}}{\Gamma, A, A; \Delta, \mathbf{1} \xrightarrow{x} C}\ \mathbf{1}\mathrm{L}}{\Gamma, A; \Delta, \mathbf{1} \xrightarrow{x} C}\ \text{ucontract}$$

We reason as follows:

$\Gamma, A, \mathbf{1}; \Delta \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma, A; \Delta, \mathbf{1} \xrightarrow{x} C$ by $\mathbf{1}$L.

- *Case* $\otimes$L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A, A, B_1 \otimes B_2; \Delta, B_1, B_2 \xrightarrow{x} C\end{array}}{\Gamma, A, A; \Delta, B_1 \otimes B_2 \xrightarrow{x} C}\ \otimes\mathrm{L}_i}{\Gamma, A; \Delta, B_1 \otimes B_2 \xrightarrow{x} C}\ \text{ucontract}$$

We reason as follows:

$\Gamma, A, B_1 \otimes B_2; \Delta, B_1, B_2 \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma, A; \Delta, B_1 \otimes B_2 \xrightarrow{x} C$ by $\otimes$L.

- *Case* $\otimes$R*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma, A, A, \Delta_2; \Delta_1 \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma, A, A, \Delta_1; \Delta_2 \xrightarrow{y} C_2\end{array}}{\Gamma, A, A; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2}\otimes\mathrm{R}}{\Gamma, A; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2}\text{ucontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, A, \Delta_2; \Delta_1 \xrightarrow{x} C_1$ by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, A, \Delta_1; \Delta_2 \xrightarrow{y} C_2$ by ucontract on $\mathcal{D}_2$.
$\Gamma, A; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma, A, A, B_1 \oplus B_2; \Delta, B_1 \xrightarrow{x} C\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma, A, A, B_1 \oplus B_2; \Delta, B_2 \xrightarrow{y} C\end{array}}{\Gamma, A, A; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C}\oplus\mathrm{L}}{\Gamma, A; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C}\text{ucontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, A, B_1 \oplus B_2; \Delta, B_1 \xrightarrow{x} C$ by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, A, B_1 \oplus B_2; \Delta, B_2 \xrightarrow{y} C$ by ucontract on $\mathcal{D}_2$.
$\Gamma, A; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C$ by $\oplus$L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus\mathrm{R}_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, A; \Delta \xrightarrow{x} C_i\end{array}}{\Gamma, A, A; \Delta \xrightarrow{x} C_1 \oplus C_2}\oplus\mathrm{R}_i}{\Gamma, A; \Delta \xrightarrow{x} C_1 \oplus C_2}\text{ucontract}$$

We reason as follows:

$\Gamma, A; \Delta \xrightarrow{x} C_i$ by ucontract on $\mathcal{D}$.
$\Gamma, A; \Delta \xrightarrow{x} C_1 \oplus C_2$ by $\oplus\mathrm{R}_i$.

- *Case* $\multimap$ L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma, A, A, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma, A, A, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C\end{array}}{\Gamma, A, A; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}\multimap\mathrm{L}}{\Gamma, A; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}\text{ucontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, A, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1$ by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, A, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C$ by ucontract on $\mathcal{D}_2$.
$\Gamma, A; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C$ by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ R*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, A; \Delta, B \xrightarrow{x} C\end{array}}{\Gamma, A, A; \Delta \xrightarrow{x} B \multimap C} \multimap \mathrm{R}}{\Gamma, A; \Delta \xrightarrow{x} B \multimap C} \text{ucontract}$$

We reason as follows:

$\Gamma, A; \Delta, B \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma, A; \Delta \xrightarrow{x} B \multimap C$ by $\multimap$ R.

- *Case* promote*, A non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, A; \Delta, B \xrightarrow{x} C\end{array}}{\Gamma, A, A, B; \Delta \xrightarrow{x+\mathbf{c}} C} \text{promote}}{\Gamma, B, A; \Delta \xrightarrow{x+\mathbf{c}} C} \text{ucontract}$$

We reason as follows:

$\Gamma, A; \Delta, B \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma, B, A; \Delta \xrightarrow{x+\mathbf{c}} C$ by promote.

- *Case* promote*, A principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A; \Delta, A \xrightarrow{x} C\end{array}}{\Gamma, A, A; \Delta \xrightarrow{x+\mathbf{c}} C} \text{promote}}{\Gamma, A; \Delta \xrightarrow{x+\mathbf{c}} C} \text{ucontract}$$

We reason as follows:

$\Gamma; \Delta, A \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma, A; \Delta \xrightarrow{x+\mathbf{c}} C$ by promote.

- *Case* demote*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, A, B; \Delta \xrightarrow{x} C\end{array}}{\Gamma, A, A; \Delta, B \xrightarrow{x+\mathbf{w}} C} \text{demote}}{\Gamma, A; \Delta, B \xrightarrow{x+\mathbf{w}} C} \text{ucontract}$$

We reason as follows:

$\Gamma, A, B; \Delta \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma, A; \Delta, B \xrightarrow{x+\mathbf{w}} C$ by demote.

Next, we consider the cases for the contract rule:

- *Case* init*:*

$$\dfrac{\dfrac{}{\Gamma, A; a \xrightarrow{x} a} \text{init}}{\Gamma; a \xrightarrow{x} a} \text{contract}$$

We reason as follows:

$\Gamma; a \xrightarrow{x} a$ by init.

- *Case* $\top$R*:*

$$\dfrac{\dfrac{}{\Gamma, A; \Delta, A \xrightarrow{x} \top}\ \top\mathrm{R}}{\Gamma; \Delta, A \xrightarrow{x} \top}\ \text{contract}$$

We reason as follows:

$\Gamma; \Delta, A \xrightarrow{x} \top$ by $\top$R.

- *Case* $\mathbf{0}$L*, A non-principal:*

$$\dfrac{\dfrac{}{\Gamma, A; \Delta, A, \mathbf{0} \xrightarrow{x} C}\ \mathbf{0}\mathrm{L}}{\Gamma; \Delta, A, \mathbf{0} \xrightarrow{x} C}\ \text{contract}$$

We reason as follows:

$\Gamma; \Delta, A, \mathbf{0} \xrightarrow{x} C$ by $\mathbf{0}$L.

- *Case* $\mathbf{0}$L*, $A = \mathbf{0}$ principal:*

$$\dfrac{\dfrac{}{\Gamma, \mathbf{0}; \Delta, \mathbf{0} \xrightarrow{x} C}\ \mathbf{0}\mathrm{L}}{\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C}\ \text{contract}$$

We reason as follows:

$\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C$ by $\mathbf{0}$L.

- *Case* $\mathbf{1}$R*:* Impossible.
- *Case* $\&\mathrm{L}_i$*, $i \in \{1, 2\}$, A non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A, A_1 \,\&\, A_2; \Delta, A, A_i \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A, A_1 \,\&\, A_2 \xrightarrow{x} C}\ \&\mathrm{L}_i}{\Gamma; \Delta, A, A_1 \,\&\, A_2 \xrightarrow{x} C}\ \text{contract}$$

We reason as follows:

$\Gamma, A_1 \,\&\, A_2; \Delta, A, A_i \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma; \Delta, A, A_1 \,\&\, A_2 \xrightarrow{x} C$ by $\&\mathrm{L}_i$.

- *Case* $\&\mathrm{L}_i$*, $i \in \{1, 2\}$, $A = A_1 \,\&\, A_2$ principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A_1 \,\&\, A_2, A_1 \,\&\, A_2; \Delta, A_i \xrightarrow{x} C\end{array}}{\Gamma, A_1 \,\&\, A_2; \Delta, A_1 \,\&\, A_2 \xrightarrow{x} C}\ \&\mathrm{L}_i}{\Gamma; \Delta, A_1 \,\&\, A_2 \xrightarrow{x} C}\ \text{contract}$$

We reason as follows:

$\Gamma, A_1 \,\&\, A_2; \Delta, A_i \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma; \Delta, A_1 \,\&\, A_2 \xrightarrow{x} C$ by $\&\mathrm{L}_i$.

- *Case* &R*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma, A; \Delta, A \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma, A; \Delta, A \xrightarrow{y} C_2\end{array}}{\Gamma, A; \Delta, A \xrightarrow{x+y} C_1 \,\&\, C_2}\,\&\mathrm{R}}{\Gamma; \Delta, A \xrightarrow{x+y} C_1 \,\&\, C_2}\,\text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma; \Delta, A \xrightarrow{x} C_1$ by contract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma; \Delta, A \xrightarrow{y} C_2$ by contract on $\mathcal{D}_2$.
$\Gamma; \Delta, A \xrightarrow{x+y} C_1 \,\&\, C_2$ by &R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* **1**L*, A non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, \mathbf{1}; \Delta, A \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A, \mathbf{1} \xrightarrow{x} C}\,\mathbf{1}\mathrm{L}}{\Gamma; \Delta, A, \mathbf{1} \xrightarrow{x} C}\,\text{contract}$$

We reason as follows:

$\Gamma, \mathbf{1}; \Delta, A \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma; \Delta, A, \mathbf{1} \xrightarrow{x} C$ by $\mathbf{1}$L.

- *Case* **1**L*, $A = \mathbf{1}$ principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, \mathbf{1}, \mathbf{1}; \Delta \xrightarrow{x} C\end{array}}{\Gamma, \mathbf{1}; \Delta, \mathbf{1} \xrightarrow{x} C}\,\mathbf{1}\mathrm{L}}{\Gamma; \Delta, \mathbf{1} \xrightarrow{x} C}\,\text{contract}$$

We reason as follows:

$\Gamma, \mathbf{1}; \Delta \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma; \Delta, \mathbf{1} \xrightarrow{x} C$ by $\mathbf{1}$L.

- *Case* $\otimes$L*, A non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, B_1 \otimes B_2; \Delta, A, B_1, B_2 \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A, B_1 \otimes B_2 \xrightarrow{x} C}\,\otimes\mathrm{L}}{\Gamma; \Delta, A, B_1 \otimes B_2 \xrightarrow{x} C}\,\text{contract}$$

We reason as follows:

$\Gamma, B_1 \otimes B_2; \Delta, A, B_1, B_2 \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma; \Delta, A, B_1 \otimes B_2 \xrightarrow{x} C$ by $\otimes$L.

- *Case* $\otimes$L, $A = B_1 \otimes B_2$ *principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, B_1 \otimes B_2, B_1 \otimes B_2; \Delta, B_1, B_2 \xrightarrow{x} C\end{array}}{\Gamma, B_1 \otimes B_2; \Delta, B_1 \otimes B_2 \xrightarrow{x} C}\otimes\text{L}}{\Gamma; \Delta, B_1 \otimes B_2 \xrightarrow{x} C}\text{contract}$$

We reason as follows:

$\Gamma, B_1 \otimes B_2; \Delta, B_1, B_2 \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma; \Delta, B_1 \otimes B_2 \xrightarrow{x} C$ by $\otimes$L.

- *Case* $\otimes$R, A *in linear context of first premise:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, A, \Delta_2; \Delta_1, A \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, A, \Delta_1, A; \Delta_2 \xrightarrow{y} C_2\end{array}}{\Gamma, A; \Delta_1, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2}\otimes\text{R}}{\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2}\text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2; \Delta_1, A \xrightarrow{x} C_1$ by contract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, \Delta_1, A; \Delta_2 \xrightarrow{y} C_2$ by ucontract on $\mathcal{D}_2$.
$\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\otimes$R, A *in linear context of second premise:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, A, \Delta_2, A; \Delta_1 \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, A, \Delta_1; \Delta_2, A \xrightarrow{y} C_2\end{array}}{\Gamma, A; \Delta_1, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2}\otimes\text{R}}{\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2}\text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, A; \Delta_1 \xrightarrow{x} C_1$ by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, \Delta_1; \Delta_2, A \xrightarrow{y} C_2$ by contract on $\mathcal{D}_2$.
$\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L, A *non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, A, B_1 \oplus B_2; \Delta, A, B_1 \xrightarrow{x} C\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, A, B_1 \oplus B_2; \Delta, A, B_2 \xrightarrow{y} C\end{array}}{\Gamma, A; \Delta, A, B_1 \oplus B_2 \xrightarrow{x+y} C}\oplus\text{L}}{\Gamma; \Delta, A, B_1 \oplus B_2 \xrightarrow{x+y} C}\text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, B_1 \oplus B_2; \Delta, A, B_1 \xrightarrow{x} C$ by contract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, B_1 \oplus B_2; \Delta, A, B_2 \xrightarrow{y} C$ by contract on $\mathcal{D}_2$.
$\Gamma; \Delta, A, B_1 \oplus B_2 \xrightarrow{x+y} C$ by $\oplus$L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L, $A = B_1 \oplus B_2$ *principal:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Gamma, B_1 \oplus B_2, B_1 \oplus B_2; \Delta, B_1 \xrightarrow{x} C\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \Gamma, B_1 \oplus B_2, B_1 \oplus B_2; \Delta, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma, B_1 \oplus B_2; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C}\,\oplus\mathrm{L}}{\Gamma; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C}\,\text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, B_1 \oplus B_2; \Delta, B_1 \xrightarrow{x} C$	by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, B_1 \oplus B_2; \Delta, B_2 \xrightarrow{y} C$	by ucontract on $\mathcal{D}_2$.
$\Gamma; \Delta, B_1 \oplus B_2 \xrightarrow{x+y} C$	by $\oplus$L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus\mathrm{R}_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma, A; \Delta, A \xrightarrow{x} C_i\end{matrix}}{\Gamma, A; \Delta, A \xrightarrow{x} C_1 \oplus C_2}\,\oplus\mathrm{R}_i}{\Gamma; \Delta, A \xrightarrow{x} C_1 \oplus C_2}\,\text{contract}$$

We reason as follows:

$\Gamma; \Delta, A \xrightarrow{x} C_i$	by contract on $\mathcal{D}$.
$\Gamma; \Delta, A \xrightarrow{x} C_1 \oplus C_2$	by $\oplus\mathrm{R}_i$.

- *Case* $\multimap$ L, A *in linear context of first premise:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Gamma, A, \Delta_2, B_1 \multimap B_2; \Delta_1, A \xrightarrow{x} B_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \Gamma, A, \Delta_1, A, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma, A; \Delta_1, A, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}\,\multimap\mathrm{L}}{\Gamma; \Delta_1, A, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}\,\text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1, A \xrightarrow{x} B_1$	by contract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, \Delta_1, A, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C$	by ucontract on $\mathcal{D}_2$.
$\Gamma; \Delta_1, A, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C$	by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ L, A *in linear context of second premise:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \Gamma, A, \Delta_2, A, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \Gamma, A, \Delta_1, B_1 \multimap B_2; \Delta_2, A, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma, A; \Delta_1, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y} C}\,\multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y} C}\,\text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, A, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1$	by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, A, B_2 \xrightarrow{y} C$	by contract on $\mathcal{D}_2$.
$\Gamma; \Delta_1, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y} C$	by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ L*,* $A = B_1 \multimap B_2$ *principal:*

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, B_1 \multimap B_2, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1} \quad \dfrac{\mathcal{D}_2}{\Gamma, B_1 \multimap B_2, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C}}{\Gamma, B_1 \multimap B_2; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C} \multimap \mathrm{L}}{\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C} \text{contract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1$ by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C$ by ucontract on $\mathcal{D}_2$.
$\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C$ by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ R*:*

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma, A; \Delta, A, B \xrightarrow{x} C}}{\Gamma, A; \Delta, A \xrightarrow{x} B \multimap C} \multimap \mathrm{R}}{\Gamma; \Delta, A \xrightarrow{x} B \multimap C} \text{contract}$$

We reason as follows:

$\Gamma; \Delta, A, B \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma; \Delta, A \xrightarrow{x} B \multimap C$ by $\multimap$ R.

- *Case* promote*,* A *non-principal:*

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma, A; \Delta, A, B \xrightarrow{x} C}}{\Gamma, A, B; \Delta, A \xrightarrow{x+\mathfrak{c}} C} \text{promote}}{\Gamma, B; \Delta, A \xrightarrow{x+\mathfrak{c}} C} \text{contract}$$

We reason as follows:

$\Gamma; \Delta, A, B \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma, B; \Delta, A \xrightarrow{x+\mathfrak{c}} C$ by promote.

- *Case* promote*,* A *principal:*

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma; \Delta, A, A \xrightarrow{x} C}}{\Gamma, A; \Delta, A \xrightarrow{x+\mathfrak{c}} C} \text{promote}}{\Gamma; \Delta, A \xrightarrow{x+\mathfrak{c}} C} \text{contract}$$

We reason as follows:

$\Gamma; \Delta, A \xrightarrow{x+\mathfrak{c}} C$ by pcontract on $\mathcal{D}$.

- *Case* demote, A *non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, B; \Delta, A \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A, B \xrightarrow{x+\mathbf{w}} C}\text{demote}}{\Gamma; \Delta, A, B \xrightarrow{x+\mathbf{w}} C}\text{contract}$$

We reason as follows:

$\Gamma, B; \Delta, A \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma; \Delta, A, B \xrightarrow{x+\mathbf{w}} C$ by demote.

- *Case* demote, A *principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma, A, A; \Delta \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A \xrightarrow{x+\mathbf{w}} C}\text{demote}}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{w}} C}\text{contract}$$

We reason as follows:

$\Gamma, A; \Delta \xrightarrow{x} C$ by ucontract on $\mathcal{D}$.
$\Gamma; \Delta, A \xrightarrow{x+\mathbf{w}} C$ by demote.

Finally, we look at the cases for the pcontract rule:

- *Case* init*:* Impossible.
- *Case* $\top$R*:*

$$\dfrac{\dfrac{}{\Gamma; \Delta, A, A \xrightarrow{x+\mathbf{c}} \top}\top\text{R}}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} \top}\text{pcontract}$$

We reason as follows:

$\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} \top$ by $\top$R.

- *Case* $\mathbf{0}$L, A *non-principal:*

$$\dfrac{\dfrac{}{\Gamma; \Delta, A, A, \mathbf{0} \xrightarrow{x+\mathbf{c}} C}\mathbf{0}\text{L}}{\Gamma; \Delta, A, \mathbf{0} \xrightarrow{x+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$\Gamma; \Delta, A, \mathbf{0} \xrightarrow{x+\mathbf{c}} C$ by $\mathbf{0}$L.

- *Case* $\mathbf{0}$L, $A = \mathbf{0}$ *principal:*

$$\dfrac{\dfrac{}{\Gamma; \Delta, \mathbf{0}, \mathbf{0} \xrightarrow{x+\mathbf{c}} C}\mathbf{0}\text{L}}{\Gamma; \Delta, \mathbf{0} \xrightarrow{x+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$\Gamma; \Delta, \mathbf{0} \xrightarrow{x+\mathbf{c}} C$ by $\mathbf{0}$L.

- *Case* $\mathbf{1}$R*:* Impossible.

- *Case* $\&\mathrm{L}_i$, $i \in \{1,2\}$, A *non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A_1 \mathbin{\&} A_2; \Delta, A, A, A_i \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A, A, A_1 \mathbin{\&} A_2 \xrightarrow{x} C}\ \&\mathrm{L}_i}{\Gamma; \Delta, A, A_1 \mathbin{\&} A_2 \xrightarrow{x+\mathfrak{c}} C}\ \text{pcontract}$$

We reason as follows:

$\Gamma, A_1 \mathbin{\&} A_2; \Delta, A, A_i \xrightarrow{x+\mathfrak{c}} C$ by pcontract on $\mathcal{D}$.
$\Gamma; \Delta, A, A_1 \mathbin{\&} A_2 \xrightarrow{x+\mathfrak{c}} C$ by $\&\mathrm{L}_i$.

- *Case* $\&\mathrm{L}_i$, $i \in \{1,2\}$, $A = A_1 \mathbin{\&} A_2$ *principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A_1 \mathbin{\&} A_2; \Delta, A_1 \mathbin{\&} A_2, A_i \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A_1 \mathbin{\&} A_2, A_1 \mathbin{\&} A_2 \xrightarrow{x} C}\ \&\mathrm{L}_i}{\Gamma; \Delta, A_1 \mathbin{\&} A_2 \xrightarrow{x+\mathfrak{c}} C}\ \text{pcontract}$$

We reason as follows:

$\Gamma; \Delta, A_1 \mathbin{\&} A_2, A_i \xrightarrow{x} C$ by contract on $\mathcal{D}$.
$\Gamma, A_1 \mathbin{\&} A_2; \Delta, A_i \xrightarrow{x+\mathfrak{c}} C$ by promote.
$\Gamma; \Delta, A_1 \mathbin{\&} A_2 \xrightarrow{x+\mathfrak{c}} C$ by $\&\mathrm{L}_i$.

- *Case* $\&\mathrm{R}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma; \Delta, A, A \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma; \Delta, A, A \xrightarrow{y} C_2\end{array}}{\Gamma; \Delta, A, A \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\ \&\mathrm{R}}{\Gamma; \Delta, A \xrightarrow{x+y+\mathfrak{c}} C_1 \mathbin{\&} C_2}\ \text{pcontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma; \Delta, A \xrightarrow{x+\mathfrak{c}} C_1$ by pcontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma; \Delta, A \xrightarrow{y+\mathfrak{c}} C_2$ by pcontract on $\mathcal{D}_2$.
$\Gamma; \Delta, A \xrightarrow{x+y+\mathfrak{c}} C_1 \mathbin{\&} C_2$ by $\&\mathrm{R}$ on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\mathbf{1}\mathrm{L}$, A *non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, \mathbf{1}; \Delta, A, A \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A, A, \mathbf{1} \xrightarrow{x} C}\ \mathbf{1}\mathrm{L}}{\Gamma; \Delta, A, \mathbf{1} \xrightarrow{x+\mathfrak{c}} C}\ \text{pcontract}$$

We reason as follows:

$\Gamma, \mathbf{1}; \Delta, A \xrightarrow{x+\mathfrak{c}} C$ by pcontract on $\mathcal{D}$.
$\Gamma; \Delta, A, \mathbf{1} \xrightarrow{x+\mathfrak{c}} C$ by $\mathbf{1}\mathrm{L}$.

- *Case* **1**L*,* $A = \mathbf{1}$ *principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma,\mathbf{1};\Delta,\mathbf{1}\xrightarrow{x}C\end{array}}{\Gamma;\Delta,\mathbf{1},\mathbf{1}\xrightarrow{x}C}\,\mathbf{1}\mathrm{L}}{\Gamma;\Delta,\mathbf{1}\xrightarrow{x+\mathfrak{c}}C}\,\text{pcontract}$$

We reason as follows:

$$\begin{array}{ll}\Gamma;\Delta,\mathbf{1}\xrightarrow{x}C & \text{by contract on } \mathcal{D}.\\ \Gamma,\mathbf{1};\Delta\xrightarrow{x+\mathfrak{c}}C & \text{by promote.}\\ \Gamma;\Delta,\mathbf{1}\xrightarrow{x+\mathfrak{c}}C & \text{by } \mathbf{1}\mathrm{L}.\end{array}$$

- *Case* $\otimes$L*,* A *non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma,B_1\otimes B_2;\Delta,A,A,B_1,B_2\xrightarrow{x}C\end{array}}{\Gamma;\Delta,A,A,B_1\otimes B_2\xrightarrow{x}C}\,\otimes\mathrm{L}}{\Gamma;\Delta,A,B_1\otimes B_2\xrightarrow{x+\mathfrak{c}}C}\,\text{pcontract}$$

We reason as follows:

$$\begin{array}{ll}\Gamma,B_1\otimes B_2;\Delta,A,B_1,B_2\xrightarrow{x+\mathfrak{c}}C & \text{by pcontract on } \mathcal{D}.\\ \Gamma;\Delta,A,B_1\otimes B_2\xrightarrow{x+\mathfrak{c}}C & \text{by } \otimes\mathrm{L}.\end{array}$$

- *Case* $\otimes$L*,* $A = B_1 \otimes B_2$ *principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma,B_1\otimes B_2;\Delta,B_1\otimes B_2,B_1,B_2\xrightarrow{x}C\end{array}}{\Gamma;\Delta,B_1\otimes B_2,B_1\otimes B_2\xrightarrow{x}C}\,\otimes\mathrm{L}}{\Gamma;\Delta,B_1\otimes B_2\xrightarrow{x+\mathfrak{c}}C}\,\text{pcontract}$$

We reason as follows:

$$\begin{array}{ll}\Gamma;\Delta,B_1\otimes B_2,B_1,B_2\xrightarrow{x}C & \text{by contract on } \mathcal{D}.\\ \Gamma,B_1\otimes B_2;\Delta,B_1,B_2\xrightarrow{x+\mathfrak{c}}C & \text{by promote.}\\ \Gamma;\Delta,B_1\otimes B_2\xrightarrow{x+\mathfrak{c}}C & \text{by } \otimes\mathrm{L}.\end{array}$$

- *Case* $\otimes$R*,* A, A *in linear context of first premise:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma,\Delta_2;\Delta_1,A,A\xrightarrow{x}C_1\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma,\Delta_1,A,A;\Delta_2\xrightarrow{y}C_2\end{array}}{\Gamma;\Delta_1,A,A,\Delta_2\xrightarrow{x+y}C_1\otimes C_2}\,\otimes\mathrm{R}}{\Gamma;\Delta_1,A,\Delta_2\xrightarrow{x+y+\mathfrak{c}}C_1\otimes C_2}\,\text{pcontract}$$

We reason as follows:

$$\begin{array}{ll}\mathcal{D}_1' :: \Gamma,\Delta_2;\Delta_1,A\xrightarrow{x+\mathfrak{c}}C_1 & \text{by pcontract on } \mathcal{D}_1.\\ \mathcal{D}_2' :: \Gamma,\Delta_1,A;\Delta_2\xrightarrow{y}C_2 & \text{by ucontract on } \mathcal{D}_2.\\ \Gamma;\Delta_1,\Delta_2,A\xrightarrow{x+y+\mathfrak{c}}C_1\otimes C_2 & \text{by } \otimes\mathrm{R} \text{ on } \mathcal{D}_1' \text{ and } \mathcal{D}_2'.\end{array}$$

- *Case* $\otimes$R*, A and A in separate linear contexts:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma, \Delta_2, A; \Delta_1, A \xrightarrow{x} C_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma, \Delta_1, A; \Delta_2, A \xrightarrow{y} C_2\end{matrix}}{\Gamma; \Delta_1, A, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2}\otimes\text{R}}{\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y+\mathbf{c}} C_1 \otimes C_2}\text{pcontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2; \Delta_1, A \xrightarrow{x} C_1$	by contract on $\mathcal{D}_1$.
$\Gamma, \Delta_1; \Delta_2, A \xrightarrow{y} C_2$	by contract on $\mathcal{D}_2$.
$\mathcal{D}_2' :: \Gamma, \Delta_1, A; \Delta_2 \xrightarrow{y+\mathbf{c}} C_2$	by promote.
$\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y+\mathbf{c}} C_1 \otimes C_2$	by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\otimes$R*, A, A in linear context of second premise:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma, \Delta_2, A, A; \Delta_1 \xrightarrow{x} C_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma, \Delta_1; \Delta_2, A, A \xrightarrow{y} C_2\end{matrix}}{\Gamma; \Delta_1, \Delta_2, A, A \xrightarrow{x+y} C_1 \otimes C_2}\otimes\text{R}}{\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y+\mathbf{c}} C_1 \otimes C_2}\text{pcontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, A; \Delta_1 \xrightarrow{x} C_1$	by ucontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, \Delta_1; \Delta_2, A \xrightarrow{y+\mathbf{c}} C_2$	by pcontract on $\mathcal{D}_2$.
$\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y+\mathbf{c}} C_1 \otimes C_2$	by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L*, A non-principal:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma, B_1 \oplus B_2; \Delta, A, A, B_1 \xrightarrow{x} C\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma, B_1 \oplus B_2; \Delta, A, A, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta, A, A, B_1 \oplus B_2 \xrightarrow{x+y} C}\oplus\text{L}}{\Gamma; \Delta, A, B_1 \oplus B_2 \xrightarrow{x+y+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, B_1 \oplus B_2; \Delta, A, B_1 \xrightarrow{x+\mathbf{c}} C$	by pcontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, B_1 \oplus B_2; \Delta, A, B_2 \xrightarrow{y+\mathbf{c}} C$	by pcontract on $\mathcal{D}_2$.
$\Gamma; \Delta, A, B_1 \oplus B_2 \xrightarrow{x+y+\mathbf{c}} C$	by $\oplus$L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L*, $A = B_1 \oplus B_2$ principal:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma, B_1 \oplus B_2; \Delta, B_1 \oplus B_2, B_1 \xrightarrow{x} C\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ \Gamma, B_1 \oplus B_2; \Delta, B_1 \oplus B_2, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta, B_1 \oplus B_2, B_1 \oplus B_2 \xrightarrow{x+y} C}\oplus\text{L}}{\Gamma; \Delta, B_1 \oplus B_2 \xrightarrow{x+y+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$$\begin{array}{lr}
\Gamma; \Delta, B_1 \oplus B_2, B_1 \xrightarrow{x} C & \text{by contract on } \mathcal{D}_1. \\
\mathcal{D}_1' :: \Gamma, B_1 \oplus B_2; \Delta, B_1 \xrightarrow{x+\mathbf{c}} C & \text{by promote.} \\
\Gamma; \Delta, B_1 \oplus B_2, B_2 \xrightarrow{y} C & \text{by contract on } \mathcal{D}_2. \\
\mathcal{D}_2' :: \Gamma, B_1 \oplus B_2; \Delta, B_2 \xrightarrow{y+\mathbf{c}} C & \text{by promote.} \\
\Gamma; \Delta, B_1 \oplus B_2 \xrightarrow{x+y+\mathbf{c}} C & \text{by } \oplus\text{L on } \mathcal{D}_1' \text{ and } \mathcal{D}_2'.
\end{array}$$

- *Case* $\oplus\mathrm{R}_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta, A, A \xrightarrow{x} C_i\end{array}}{\Gamma; \Delta, A, A \xrightarrow{x} C_1 \oplus C_2}\oplus\mathrm{R}_i}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C_1 \oplus C_2}\text{pcontract}$$

We reason as follows:

$$\begin{array}{lr}
\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C_i & \text{by pcontract on } \mathcal{D}. \\
\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C_1 \oplus C_2 & \text{by } \oplus\mathrm{R}_i.
\end{array}$$

- *Case* $\multimap$ L*,* A, A *in linear context of first premise:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1, A, A \xrightarrow{x} B_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, \Delta_1, A, A, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_1, A, A, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}\multimap\mathrm{L}}{\Gamma; \Delta_1, A, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$$\begin{array}{lr}
\mathcal{D}_1' :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1, A \xrightarrow{x+\mathbf{c}} B_1 & \text{by pcontract on } \mathcal{D}_1. \\
\mathcal{D}_2' :: \Gamma, \Delta_1, A, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C & \text{by ucontract on } \mathcal{D}_2. \\
\Gamma; \Delta_1, A, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C & \text{by } \multimap\text{L on } \mathcal{D}_1' \text{ and } \mathcal{D}_2'.
\end{array}$$

- *Case* $\multimap$ L*,* A, A *in linear context of second premise:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, \Delta_2, A, A, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, A, A, B_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_1, \Delta_2, A, A, B_1 \multimap B_2 \xrightarrow{x+y} C}\multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$$\begin{array}{lr}
\mathcal{D}_1' :: \Gamma, \Delta_2, A, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1 & \text{by ucontract on } \mathcal{D}_1. \\
\mathcal{D}_2' :: \Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, A, B_2 \xrightarrow{y+\mathbf{c}} C & \text{by pcontract on } \mathcal{D}_2. \\
\Gamma; \Delta_1, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C & \text{by } \multimap\text{L on } \mathcal{D}_1' \text{ and } \mathcal{D}_2'.
\end{array}$$

- *Case* $\multimap$ L*,* A *and* A *in separate linear contexts:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, \Delta_2, A, B_1 \multimap B_2; \Delta_1, A \xrightarrow{x} B_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, \Delta_1, A, B_1 \multimap B_2; \Delta_2, A, B_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_1, A, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y} C}\multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1, A \xrightarrow{x} B_1$ by contract on $\mathcal{D}_1$.
$\Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, A, B_2 \xrightarrow{y} C$ by contract on $\mathcal{D}_2$.
$\mathcal{D}_2' :: \Gamma, \Delta_1, A, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y+\mathbf{c}} C$ by promote.
$\Gamma; \Delta_1, \Delta_2, A, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C$ by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ L*,* $A = B_1 \multimap B_2$ *principal and in linear context of first premise:*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdots}}{\Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1, B_1 \multimap B_2 \xrightarrow{x} B_1} \quad \dfrac{\overset{\mathcal{D}_2}{\vdots}}{\Gamma, \Delta_1, B_1 \multimap B_2, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C}}{\dfrac{\Gamma; \Delta_1, B_1 \multimap B_2, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} C}{\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C}\ \text{pcontract}}\ \multimap\text{L}$$

We reason as follows:

$\Gamma, \Delta_2; \Delta_1, B_1 \multimap B_2 \xrightarrow{x} B_1$ by contract on $\mathcal{D}_1$.
$\mathcal{D}_1' :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x+\mathbf{c}} B_1$ by promote.
$\mathcal{D}_2' :: \Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} C$ by ucontract on $\mathcal{D}_2$.
$\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C$ by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ L*,* $A = B_1 \multimap B_2$ *principal and in linear context of second premise:*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdots}}{\Gamma, \Delta_2, B_1 \multimap B_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1} \quad \dfrac{\overset{\mathcal{D}_2}{\vdots}}{\Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, B_1 \multimap B_2, B_2 \xrightarrow{y} C}}{\dfrac{\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2, B_1 \multimap B_2 \xrightarrow{x+y} C}{\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C}\ \text{pcontract}}\ \multimap\text{L}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1$ by ucontract on $\mathcal{D}_1$.
$\Gamma, \Delta_1; \Delta_2, B_1 \multimap B_2, B_2 \xrightarrow{y} C$ by contract on $\mathcal{D}_2$.
$\mathcal{D}_2' :: \Gamma, \Delta_1, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y+\mathbf{c}} C$ by promote.
$\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y+\mathbf{c}} C$ by $\multimap$ L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap$ R*:*

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\vdots}}{\Gamma; \Delta, A, A, B \xrightarrow{x} C}}{\dfrac{\Gamma; \Delta, A, A \xrightarrow{x} B \multimap C}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} B \multimap C}\ \text{pcontract}}\ \multimap\text{R}$$

We reason as follows:

$\Gamma; \Delta, A, B \xrightarrow{x+\mathbf{c}} C$ by pcontract on $\mathcal{D}$.
$\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} B \multimap C$ by $\multimap$ R.

- *Case* promote*:*

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\vdots}}{\Gamma; \Delta, A, A, B \xrightarrow{x} C}}{\dfrac{\Gamma, B; \Delta, A, A \xrightarrow{x+\mathbf{c}} C}{\Gamma, B; \Delta, A \xrightarrow{x+\mathbf{c}} C}\ \text{pcontract}}\ \text{promote}$$

We reason as follows:

$\Gamma; \Delta, A, B \xrightarrow{x+\mathbf{c}} C$ by pcontract on $\mathcal{D}$.
$\Gamma, B; \Delta, A \xrightarrow{x+\mathbf{c}} C$ by promote.

- *Case* demote*, A non-principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, B; \Delta, A, A \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A, A, B \xrightarrow{x+\mathbf{w}} C}\text{demote}}{\Gamma; \Delta, A, B \xrightarrow{x+\mathbf{w}+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$\Gamma, B; \Delta, A \xrightarrow{x+\mathbf{c}} C$ by pcontract on $\mathcal{D}$.
$\Gamma; \Delta, A, B \xrightarrow{x+\mathbf{w}+\mathbf{c}} C$ by demote.

- *Case* demote*, A principal:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A; \Delta, A \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A, A \xrightarrow{x+\mathbf{w}} C}\text{demote}}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{w}+\mathbf{c}} C}\text{pcontract}$$

We reason as follows:

$\Gamma; \Delta, A \xrightarrow{x+\mathbf{w}+\mathbf{c}} C$ by contract on $\mathcal{D}$.

This completes the proof. □

Theorem 14 (Cut admissibility)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \Gamma; A, \Delta_2 \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C}\text{cut} \qquad \dfrac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \Gamma, A; \Delta_2 \xrightarrow{y} C}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C}\text{ucut}$$

Proof. We proceed by lexicographic induction on triples $(A, \mathcal{D}, \mathcal{E})$, where A is the cut formula, and $\mathcal{D}$ and $\mathcal{E}$ are the two premises to the cut rule at hand.

The cases of the cut admissibility argument naturally fall into various classes depending on how we handle them in the proof. First, we handle all cases for the cut rule where the first premise ends in a left rule:

- *Case* **0**L*:*

$$\dfrac{\dfrac{}{\Gamma; \Delta_1, \mathbf{0} \xrightarrow{x} A}\mathbf{0}\text{L} \qquad \begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_1, \mathbf{0}, \Delta_2 \xrightarrow{x+y} C}\text{cut (to show)}$$

We reason as follows:

$\Gamma; \Delta_1, \mathbf{0}, \Delta_2 \xrightarrow{x+y} C$ by **0**L.

- *Case* $\&\mathrm{L}_i$, $i \in \{1,2\}$:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma, B_1 \mathbin{\&} B_2; \Delta_1, B_i \xrightarrow{x} A\end{matrix}}{\Gamma; \Delta_1, B_1 \mathbin{\&} B_2 \xrightarrow{x} A}\,\&\mathrm{L}_i \qquad \begin{matrix}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta_1, B_1 \mathbin{\&} B_2, \Delta_2 \xrightarrow{x+y} C}\,\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{lll}
\mathcal{E}' :: & \Gamma, B_1 \mathbin{\&} B_2; A, \Delta_2 \xrightarrow{x} C & \text{by weakening on } \mathcal{E}. \\
& \Gamma, B_1 \mathbin{\&} B_2; \Delta_1, B_i, \Delta_2 \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}, \mathcal{E}'). \\
& \Gamma; \Delta_1, B_1 \mathbin{\&} B_2, \Delta_2 \xrightarrow{x+y} C & \text{by } \&\mathrm{L}_i.
\end{array}
$$

- *Case* $\otimes\mathrm{L}$:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma, B_1 \otimes B_2; \Delta_1, B_1, B_2 \xrightarrow{x} A\end{matrix}}{\Gamma; \Delta_1, B_1 \otimes B_2 \xrightarrow{x} A}\,\otimes\mathrm{L} \qquad \begin{matrix}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta_1, B_1 \otimes B_2, \Delta_2 \xrightarrow{x+y} C}\,\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{lll}
\mathcal{E}' :: & \Gamma, B_1 \otimes B_2; A, \Delta_2 \xrightarrow{x} C & \text{by weakening on } \mathcal{E}. \\
& \Gamma, B_1 \otimes B_2; \Delta_1, B_1, B_2, \Delta_2 \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}, \mathcal{E}'). \\
& \Gamma; \Delta_1, B_1 \otimes B_2, \Delta_2 \xrightarrow{x+y} C & \text{by } \otimes\mathrm{L}.
\end{array}
$$

- *Case* $\oplus\mathrm{L}$:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots \\ \Gamma, B_1 \oplus B_2; \Delta_1, B_1 \xrightarrow{x} A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \vdots \\ \Gamma, B_1 \oplus B_2; \Delta_1, B_2 \xrightarrow{x} A\end{matrix}}{\Gamma; \Delta_1, B_1 \oplus B_2 \xrightarrow{x} A}\,\oplus\mathrm{L} \qquad \begin{matrix}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta_1, B_1 \oplus B_2, \Delta_2 \xrightarrow{x+y} C}\,\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{lll}
\mathcal{E}' :: & \Gamma, B_1 \oplus B_2; A, \Delta_2 \xrightarrow{x} C & \text{by weakening on } \mathcal{E}. \\
\mathcal{D}_1' :: & \Gamma, B_1 \oplus B_2; \Delta_1, B_1, \Delta_2 \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}_1, \mathcal{E}'). \\
\mathcal{D}_2' :: & \Gamma, B_1 \oplus B_2; \Delta_1, B_2, \Delta_2 \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}_2, \mathcal{E}'). \\
& \Gamma; \Delta_1, B_1 \oplus B_2, \Delta_2 \xrightarrow{x+y} C & \text{by } \oplus\mathrm{L} \text{ on } \mathcal{D}_1' \text{ and } \mathcal{D}_2'.
\end{array}
$$

- *Case* $\multimap\mathrm{L}$:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots \\ \Gamma, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} B_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \vdots \\ \Gamma, B_1 \multimap B_2; \Delta_2, B_2 \xrightarrow{y} A\end{matrix}}{\Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2 \xrightarrow{x+y} A}\,\multimap\mathrm{L} \qquad \begin{matrix}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_3 \xrightarrow{z} C\end{matrix}}{\Gamma; \Delta_1, \Delta_2, , B_1 \multimap B_2, \Delta_3 \xrightarrow{x+y+z} C}\,\text{cut (to show)}
$$

We reason as follows:

$$\begin{array}{lll}
\mathcal{E}' :: & \Gamma, B_1 \multimap B_2; A, \Delta_3 \xrightarrow{z} C & \text{by weakening on } \mathcal{E}. \\
\mathcal{D}_2' :: & \Gamma, B_1 \multimap B_2; \Delta_2, B_2, \Delta_3 \xrightarrow{y+z} C & \text{by cut on } (A, \mathcal{D}_2, \mathcal{E}'). \\
& \Gamma; \Delta_1, \Delta_2, B_1 \multimap B_2, \Delta_3 \xrightarrow{x+y+z} C & \text{by } \multimap \text{L on } \mathcal{D}_1 \text{ and } \mathcal{D}_2'.
\end{array}$$

- *Case* promote*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1, B \xrightarrow{x} A\end{array}}{\Gamma, B; \Delta_1 \xrightarrow{x+\mathbf{c}} A}\,\text{promote} \qquad \begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma, B; A, \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma, B; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{c}} C}\,\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}
\mathcal{D}' :: & \Gamma, B; \Delta_1, B \xrightarrow{x} A & \text{by weakening on } \mathcal{D}. \\
& \Gamma, B; \Delta_1, B, \Delta_2 \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}). \\
& \Gamma, B, B; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{c}} C & \text{by promote.} \\
& \Gamma, B; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{c}} C & \text{by ucontract.}
\end{array}$$

- *Case* demote*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, B; \Delta_1 \xrightarrow{x} A\end{array}}{\Gamma; \Delta_1, B \xrightarrow{x+\mathbf{w}} A}\,\text{demote} \qquad \begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_1, B, \Delta_2 \xrightarrow{x+y+\mathbf{w}} C}\,\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}
\mathcal{E}' :: & \Gamma, B; A, \Delta_2 \xrightarrow{x} C & \text{by weakening on } \mathcal{E}. \\
& \Gamma, B; \Delta_1, \Delta_2 \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}, \mathcal{E}'). \\
& \Gamma; \Delta_1, B, \Delta_2 \xrightarrow{x+y+\mathbf{w}} C & \text{by demote.}
\end{array}$$

Next, the cases where the second premise ends in a right rule:

- *Case* $\top$R*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \qquad \dfrac{}{\Gamma; A, \Delta_2 \xrightarrow{y} \top}\,\top\text{R}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} \top}\,\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{ll}
\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} \top & \text{by } \top\text{R.}
\end{array}$$

- *Case* ***1***R*:* Impossible
- *Case* &R*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{z} C_2\end{array}}{\Gamma; A, \Delta_2 \xrightarrow{y+z} C_1 \,\&\, C_2}\,\&\text{R}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C_1 \,\&\, C_2}\,\text{cut (to show)}$$

We reason as follows:

$\mathcal{E}_1' :: \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1$ by cut on $(A, \mathcal{D}, \mathcal{E}_1)$.
$\mathcal{E}_2' :: \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+z} C_2$ by cut on $(A, \mathcal{D}, \mathcal{E}_2)$.
$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C_1 \,\&\, C_2$ by &R on $\mathcal{E}_1'$ and $\mathcal{E}_2'$.

- *Case* $\otimes$R, *A in linear context of first premise:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots \\ \Gamma, \Delta_3; A, \Delta_2 \xrightarrow{y} C_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots \\ \Gamma, A, \Delta_2; \Delta_3 \xrightarrow{z} C_2\end{array}}{\Gamma; A, \Delta_2, \Delta_3 \xrightarrow{y+z} C_1 \otimes C_2}\otimes\mathrm{R}}{\Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C_1 \otimes C_2}\text{cut (to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, \Delta_3; \Delta_1 \xrightarrow{x} A$ by weakening on $\mathcal{D}$.
$\mathcal{E}_1' :: \Gamma, \Delta_3; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1$ by cut on $(A, \mathcal{D}', \mathcal{E}_1)$.
$\mathcal{E}_2' :: \Gamma, \Delta_1, \Delta_2; \Delta_3 \xrightarrow{z} C_2$ by ucut on $(A, \mathcal{D}, \mathcal{E}_2)$.
$\Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{E}_1'$ and $\mathcal{E}_2'$.

- *Case* $\otimes$R, *A in linear context of second premise:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots \\ \Gamma, A, \Delta_3; \Delta_2 \xrightarrow{y} C_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots \\ \Gamma, \Delta_2; A, \Delta_3 \xrightarrow{z} C_2\end{array}}{\Gamma; A, \Delta_2, \Delta_3 \xrightarrow{y+z} C_1 \otimes C_2}\otimes\mathrm{R}}{\Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C_1 \otimes C_2}\text{cut (to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, \Delta_2; \Delta_1 \xrightarrow{x} A$ by weakening on $\mathcal{D}$.
$\mathcal{E}_2' :: \Gamma, \Delta_2; \Delta_1, \Delta_3 \xrightarrow{x+y} C_1$ by cut on $(A, \mathcal{D}', \mathcal{E}_2)$.
$\mathcal{E}_1' :: \Gamma, \Delta_1, \Delta_3; \Delta_2 \xrightarrow{z} C_2$ by ucut on $(A, \mathcal{D}, \mathcal{E}_1)$.
$\Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{E}_1'$ and $\mathcal{E}_2'$.

- *Case* $\oplus\mathrm{R}_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C_i\end{array}}{\Gamma; A, \Delta_2 \xrightarrow{y} C_1 \oplus C_2}\oplus\mathrm{R}_i}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \oplus C_2}\text{cut (to show)}$$

We reason as follows:

$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_i$ by cut on $(A, \mathcal{D}, \mathcal{E})$.
$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \oplus C_2$ by $\oplus\mathrm{R}_i$.

- *Case* $\multimap$ R*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2, B \xrightarrow{y} C\end{array}}{\Gamma; A, \Delta_2 \xrightarrow{y} B \multimap C}\multimap\mathrm{R}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} B \multimap C}\text{cut (to show)}$$

We reason as follows:

$\Gamma; \Delta_1, \Delta_2, B \xrightarrow{x+y} C$ by cut on $(A, \mathcal{D}, \mathcal{E})$.
$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} B \multimap C$ by $\multimap$R.

For the last set of commutative cases for the cut rule, we consider the cases where the second premise ends in a left rule for which the cut formula is *not* the principal formula:

- *Case* $\mathbf{0}$L*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma; \Delta_1 \xrightarrow{x} A\end{matrix} \qquad \dfrac{}{\Gamma; A, \Delta_2, \mathbf{0} \xrightarrow{y} C}\,\mathbf{0}\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, \mathbf{0} \xrightarrow{x+y} C}\,\text{cut (to show)}$$

We reason as follows:

$\Gamma; \Delta_1, \Delta_2, \mathbf{0} \xrightarrow{x+y} C$ by $\mathbf{0}$L.

- *Case* &L_i, $i \in \{1, 2\}$*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma; \Delta_1 \xrightarrow{x} A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots\\ \Gamma, B_1 \,\&\, B_2; A, \Delta_2, B_i \xrightarrow{y} C\end{matrix}}{\Gamma; A, \Delta_2, B_1 \,\&\, B_2 \xrightarrow{y} C}\,\&\mathrm{L}_i}{\Gamma; \Delta_1, \Delta_2, B_1 \,\&\, B_2 \xrightarrow{x+y} C}\,\text{cut (to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, B_1 \,\&\, B_2; \Delta_1 \xrightarrow{x} A$ by weakening on $\mathcal{D}$.
$\Gamma, B_1 \,\&\, B_2; \Delta_1, \Delta_2, B_i \xrightarrow{x+y} C$ by cut on $(A, \mathcal{D}', \mathcal{E})$.
$\Gamma; \Delta_1, \Delta_2, B_1 \,\&\, B_2 \xrightarrow{x+y} C$ by &L_i.

- *Case* $\mathbf{1}$L*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma; \Delta_1 \xrightarrow{x} A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots\\ \Gamma, \mathbf{1}; A, \Delta_2 \xrightarrow{y} C\end{matrix}}{\Gamma; A, \Delta_2, \mathbf{1} \xrightarrow{y} C}\,\mathbf{1}\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, \mathbf{1} \xrightarrow{x+y} C}\,\text{cut (to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, \mathbf{1}; \Delta_1 \xrightarrow{x} A$ by weakening on $\mathcal{D}$.
$\Gamma, \mathbf{1}; \Delta_1, \Delta_2 \xrightarrow{x+y} C$ by cut on $(A, \mathcal{D}', \mathcal{E})$.
$\Gamma; \Delta_1, \Delta_2, \mathbf{1} \xrightarrow{x+y} C$ by $\mathbf{1}$L.

- *Case* $\otimes$L*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma; \Delta_1 \xrightarrow{x} A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots\\ \Gamma, B_1 \otimes B_2; A, \Delta_2, B_1, B_2 \xrightarrow{y} C\end{matrix}}{\Gamma; A, \Delta_2, B_1 \otimes B_2 \xrightarrow{y} C}\,\otimes\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, B_1 \otimes B_2 \xrightarrow{x+y} C}\,\text{cut (to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, B_1 \otimes B_2; \Delta_1 \xrightarrow{x} A$ by weakening on $\mathcal{D}$.
$\Gamma, B_1 \otimes B_2; \Delta_1, \Delta_2, B_1, B_2 \xrightarrow{x+y} C$ by cut on $(A, \mathcal{D}', \mathcal{E})$.
$\Gamma; \Delta_1, \Delta_2, B_1 \otimes B_2 \xrightarrow{x+y} C$ by $\otimes$L.

- *Case* $\oplus$L*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots \\ \Gamma, B_1 \oplus B_2; A, \Delta_2, B_1 \xrightarrow{y} C\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots \\ \Gamma, B_1 \oplus B_2; A, \Delta_2, B_2 \xrightarrow{z} C\end{array}}{\Gamma; A, \Delta_2, B_1 \oplus B_2 \xrightarrow{y+z} C}\oplus\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}\mathcal{D}' & :: \Gamma, B_1 \oplus B_2; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}.\\ \mathcal{E}_1' & :: \Gamma, B_1 \oplus B_2; \Delta_1, \Delta_2, B_1 \xrightarrow{x+z} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_1).\\ \mathcal{E}_2' & :: \Gamma, B_1 \oplus B_2; \Delta_1, \Delta_2, B_2 \xrightarrow{y+z} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_2).\\ & \Gamma; \Delta_1, \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z} C & \text{by } \oplus\mathrm{L} \text{ on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'.\end{array}$$

- *Case* $\multimap$ L*, A in linear context of first premise:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots \\ \Gamma, \Delta_3, B_1 \multimap B_2; A, \Delta_2 \xrightarrow{y} B_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots \\ \Gamma, A, \Delta_2, B_1 \multimap B_2; \Delta_3, B_2 \xrightarrow{z} C\end{array}}{\Gamma; A, \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{y+z} C}\multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{x+y+z} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}\mathcal{D}' & :: \Gamma, \Delta_3, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}.\\ \mathcal{E}_1' & :: \Gamma, \Delta_3, B_1 \multimap B_2; \Delta_1, \Delta_2 \xrightarrow{x+y} B_1 & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_1).\\ \mathcal{E}_2' & :: \Gamma, \Delta_1, \Delta_2, B_1 \multimap B_2; \Delta_3, B_2 \xrightarrow{z} C & \text{by ucut on } (A, \mathcal{D}', \mathcal{E}_2).\\ & \Gamma; \Delta_1, \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{x+y+z} C & \text{by } \multimap\mathrm{L} \text{ on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'.\end{array}$$

- *Case* $\multimap$ L*, A in linear context of second premise:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots \\ \Gamma, \Delta_3, B_1 \multimap B_2; \Delta_2 \xrightarrow{y} B_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots \\ \Gamma, \Delta_2, B_1 \multimap B_2; A, \Delta_3, B_2 \xrightarrow{z} C\end{array}}{\Gamma; A, \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{y+z} C}\multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{x+y+z} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}\mathcal{D}' & :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}.\\ \mathcal{E}_1' & :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1, \Delta_3, B_2 \xrightarrow{x+z} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_1).\\ \mathcal{E}_2' & :: \Gamma, \Delta_1, \Delta_3, B_1 \multimap B_2; \Delta_2 \xrightarrow{y} B_1 & \text{by ucut on } (A, \mathcal{D}', \mathcal{E}_2).\\ & \Gamma; \Delta_1, \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{x+y+z} C & \text{by } \multimap\mathrm{L} \text{ on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'.\end{array}$$

- *Case* promote*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, B;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2, B \xrightarrow{y} C\end{array}}{\Gamma, B; A, \Delta_2 \xrightarrow{y+\mathbf{c}} C}\text{promote}}{\Gamma, B; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{c}} C}\text{cut (to show)}$$

We reason as follows:

$$
\begin{array}{lll}
\mathcal{E}' :: & \Gamma, B; A, \Delta_2, B \xrightarrow{y} C & \text{by weakening on } \mathcal{E}. \\
& \Gamma, B; \Delta_1, \Delta_2, B \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}, \mathcal{E}'). \\
& \Gamma, B, B; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{c}} C & \text{by promote.} \\
& \Gamma, B; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{c}} C & \text{by ucontract.}
\end{array}
$$

- *Case* demote*:*

$$
\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma, B; A, \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma; A, \Delta_2, B \xrightarrow{y+\mathbf{w}} C}\,\text{demote}}{\Gamma; \Delta_1, \Delta_2, B \xrightarrow{x+y+\mathbf{w}} C}\,\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{lll}
\mathcal{D}' :: & \Gamma, B; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}. \\
& \Gamma, B; \Delta_1, \Delta_2 \xrightarrow{x+y} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}). \\
& \Gamma; \Delta_1, \Delta_2, B \xrightarrow{x+y+\mathbf{w}} C & \text{by promote.}
\end{array}
$$

We have now covered all cases where the cut formula A does not appear as the principal formula in either premise of the cut rule. This leaves the cases where A is principal in both premises:

- *Case* $A = a$*:*

$$
\dfrac{\dfrac{}{\Gamma; a \xrightarrow{x} a}\,\text{init} \quad \dfrac{}{\Gamma; a \xrightarrow{y} a}\,\text{init}}{\Gamma; a \xrightarrow{x+y} a}\,\text{cut (to show)}
$$

We reason as follows:

$$
\Gamma; a \xrightarrow{x+y} a \qquad \text{by init.}
$$

- *Case* $A = \mathbf{1}$*:*

$$
\dfrac{\dfrac{}{\mathcal{D} :: \Gamma; \cdot \xrightarrow{x} \mathbf{1}}\,\mathbf{1}\text{R} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma, \mathbf{1}; \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_2, \mathbf{1} \xrightarrow{y} C}\,\mathbf{1}\text{L}}{\Gamma; \Delta_2 \xrightarrow{x+y} C}\,\text{cut (to show)}
$$

Here and in the following cases, we will use $\mathcal{D}$ to refer to the first subderiation of the cut rule. As a reminder of this, the conclusion of the first subderivation is written as $\mathcal{D} :: \Gamma; \Delta_1 \xrightarrow{x} A$ in the general case.

We reason as follows:

$$
\Gamma; \Delta_2 \xrightarrow{x+y} C \qquad \text{by ucut on } (\mathbf{1}, \mathcal{D}, \mathcal{E}).
$$

- *Case* $A = \top$*:* Impossible, as there is no left rule for $\top$.
- *Case* $A = \mathbf{0}$*:* Impossible, as there is no right rule for $\mathbf{0}$.
- *Case* $A = A_1 \mathbin{\&} A_2$, $i \in \{1, 2\}$*:*

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{y} A_2\end{array}}{\mathcal{D} :: \Gamma; \Delta_1 \xrightarrow{x+y} A_1 \mathbin{\&} A_2}\,\&\text{R} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma, A_1 \mathbin{\&} A_2; \Delta_2, A_i \xrightarrow{z} C\end{array}}{\Gamma; \Delta_2, A_1 \mathbin{\&} A_2 \xrightarrow{z} C}\,\&\text{L}_i}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C}\,\text{cut (to show)}
$$

We reason as follows:

$$\begin{array}{lll}
\mathcal{E}' & :: \Gamma, \Delta_1; \Delta_2, A_i \xrightarrow{x+y+z} C & \text{by ucut on } (A_1 \,\&\, A_2, \mathcal{D}, \mathcal{E}). \\
\mathcal{D}'_i & :: \Gamma, \Delta_1; \Delta_1 \xrightarrow{x} A_i & \text{by weakening on } \mathcal{D}_i. \\
 & \Gamma, \Delta_1; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by cut on } (A_i, \mathcal{D}'_i, \mathcal{E}'). \\
 & \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by contract (repeated).}
\end{array}$$

- *Case* $A = A_1 \otimes A_2$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma, \Delta_2; \Delta_1 \xrightarrow{x} A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma, \Delta_1; \Delta_2 \xrightarrow{y} A_2\end{array}}{\mathcal{D} :: \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} A_1 \otimes A_2}\otimes\mathrm{R} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots\\ \Gamma, A_1 \otimes A_2; \Delta_3, A_1, A_2 \xrightarrow{z} C\end{array}}{\Gamma; \Delta_3, A_1 \otimes A_2 \xrightarrow{z} C}\otimes\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}
\mathcal{E}' & :: \Gamma, \Delta_1, \Delta_2; \Delta_3, A_1, A_2 \xrightarrow{x+y+z} C & \text{by ucut on } (A_1 \otimes A_2, \mathcal{D}, \mathcal{E}). \\
\mathcal{D}'_1 & :: \Gamma, \Delta_1, \Delta_2; \Delta_1 \xrightarrow{x} A_1 & \text{by weakening on } \mathcal{D}_1. \\
\mathcal{F} & :: \Gamma, \Delta_1, \Delta_2; \Delta_1, \Delta_3, A_2 \xrightarrow{x+y+z} C & \text{by cut on } (A, \mathcal{D}'_1, \mathcal{E}'). \\
\mathcal{D}'_2 & :: \Gamma, \Delta_1, \Delta_2; \Delta_2 \xrightarrow{x} A_2 & \text{by weakening on } \mathcal{D}_2. \\
 & \Gamma, \Delta_1, \Delta_2; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C & \text{by cut on } (A, \mathcal{D}'_2, \mathcal{F}). \\
 & \Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C & \text{by contract (repeated).}
\end{array}$$

- *Case* $A = A_1 \oplus A_2$*,* $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \vdots\\ \Gamma; \Delta_1 \xrightarrow{x} A_i\end{array}}{\mathcal{D} :: \Gamma; \Delta_1 \xrightarrow{x} A_1 \oplus A_2}\oplus\mathrm{R}_i \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots\\ \Gamma, A_1 \oplus A_2; \Delta_2, A_1 \xrightarrow{y} C\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots\\ \Gamma, A_1 \oplus A_2; \Delta_2, A_2 \xrightarrow{z} C\end{array}}{\Gamma; \Delta_2, A_1 \oplus A_2 \xrightarrow{y+z} C}\oplus\mathrm{L}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}
\mathcal{E}'_i & :: \Gamma, \Delta_1; \Delta_2, A_i \xrightarrow{x+y+z} C & \text{by ucut on } (A_1 \oplus A_2, \mathcal{D}, \mathcal{E}_i). \\
\mathcal{D}'' & :: \Gamma, \Delta_1; \Delta_1 \xrightarrow{x} A_i & \text{by weakening on } \mathcal{D}. \\
 & \Gamma, \Delta_1; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by cut on } (A_i, \mathcal{D}'', \mathcal{E}_i). \\
 & \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by contract (repeated).}
\end{array}$$

- *Case* $A = A_1 \multimap A_2$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \vdots\\ \Gamma; \Delta_1, A_1 \xrightarrow{x} A_2\end{array}}{\mathcal{D} :: \Gamma; \Delta_1 \xrightarrow{x} A_1 \multimap A_2}\multimap\mathrm{R} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots\\ \Gamma, \Delta_3, A_1 \multimap A_2; \Delta_2 \xrightarrow{y} A_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots\\ \Gamma, \Delta_2, A_1 \multimap A_2; \Delta_3, A_2 \xrightarrow{z} C\end{array}}{\Gamma; \Delta_2, \Delta_3, A_1 \multimap A_2 \xrightarrow{y+z} C}\multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll}
\mathcal{E}'_1 & :: \Gamma, \Delta_1, \Delta_3; \Delta_2 \xrightarrow{x+y} A_1 & \text{by ucut on } (A_1 \multimap A_2, \mathcal{D}, \mathcal{E}_1). \\
\mathcal{D}'' & :: \Gamma, \Delta_1, \Delta_3; \Delta_1, A_1 \xrightarrow{x} A_2 & \text{by weakening on } \mathcal{D}'. \\
 & \Gamma, \Delta_1, \Delta_3; \Delta_1, \Delta_2 \xrightarrow{x+y+z} A_2 & \text{by cut on } (A_1, \mathcal{E}'_1, \mathcal{D}''). \\
\mathcal{F} & :: \Gamma, \Delta_1, \Delta_2, \Delta_3; \Delta_1, \Delta_2 \xrightarrow{x+y+z} A_2 & \text{by weakening.}
\end{array}$$

$$\begin{array}{lll} \mathcal{E}_2' :: & \Gamma, \Delta_1, \Delta_2, \Delta_3; \Delta_3, A_2 \xrightarrow{x+z} C & \text{by ucut on } (A_1 \multimap A_2, \mathcal{D}, \mathcal{E}_2). \\ & \Gamma, \Delta_1, \Delta_2, \Delta_3; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C & \text{by cut on } (A_2, \mathcal{F}, \mathcal{E}_2'). \\ & \Gamma; \Delta_1, \Delta_2, \Delta_3 \xrightarrow{x+y+z} C & \text{by contract (repeated).} \end{array}$$

- *Case* demote*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots \\ \Gamma, A; \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta_2, A \xrightarrow{y+\mathbf{w}} C}\text{demote}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{w}} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{ll} \Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C & \text{by ucut on } (A, \mathcal{D}, \mathcal{E}). \\ \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+\mathbf{w}} C & \text{by demote (repeated).} \end{array}$$

This completes the cases for the cut rule. To complete the proof, we now need to consider the cases for the ucut rule. Again, these can be dealt with in various ways. We start by handling the cases where the second premise ends in a right rule:

- *Case* $\top$R*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{}{\Gamma, A; \Delta_2 \xrightarrow{y} \top}\top\text{R}}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} \top}\text{ucut (to show)}$$

We reason as follows:

$$\begin{array}{ll} \Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} \top & \text{by } \top\text{R}. \end{array}$$

- *Case* $\mathbf{1}$R*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{}{\Gamma, A; \cdot \xrightarrow{y} \mathbf{1}}\mathbf{1}\text{R}}{\Gamma, \Delta_1; \cdot \xrightarrow{x+y} \mathbf{1}}\text{ucut (to show)}$$

We reason as follows:

$$\begin{array}{ll} \Gamma, \Delta_1; \cdot \xrightarrow{x+y} \mathbf{1} & \text{by } \mathbf{1}\text{R}. \end{array}$$

- *Case* &R*:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots \\ \Gamma, A; \Delta_2 \xrightarrow{y} C_1\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots \\ \Gamma, A; \Delta_2 \xrightarrow{z} C_2\end{array}}{\Gamma, A; \Delta_2 \xrightarrow{y+z} C_1 \mathbin{\&} C_2}\&\text{R}}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y+z} C_1 \mathbin{\&} C_2}\text{ucut (to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{E}_1' :: & \Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C_1 & \text{by ucut on } (A, \mathcal{D}, \mathcal{E}_1). \\ \mathcal{E}_2' :: & \Gamma, \Delta_1; \Delta_2 \xrightarrow{x+z} C_2 & \text{by ucut on } (A, \mathcal{D}, \mathcal{E}_2). \\ & \Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y+z} C_1 \mathbin{\&} C_2 & \text{by \&R on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'. \end{array}$$

- *Case* $\otimes$R*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma;\Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}_1\\ \vdots\\ \Gamma,\Delta_3,A;\Delta_2 \xrightarrow{y} C_1\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ \vdots\\ \Gamma,\Delta_2,A;\Delta_3 \xrightarrow{z} C_2\end{matrix}}{\Gamma,A;\Delta_2,\Delta_3 \xrightarrow{y+z} C_1 \otimes C_2}\otimes\mathrm{R}}{\Gamma,\Delta_1;\Delta_2,\Delta_3 \xrightarrow{x+y+z} C_1 \otimes C_2}\text{ucut (to show)}$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma,\Delta_3;\Delta_1 \xrightarrow{x} A$	by weakening on $\mathcal{D}$.
$\mathcal{E}_1' :: \Gamma,\Delta_3,\Delta_1;\Delta_2 \xrightarrow{x+y} C_1$	by ucut on $(A,\mathcal{D}_1',\mathcal{E}_1)$.
$\mathcal{D}_2' :: \Gamma,\Delta_2;\Delta_1 \xrightarrow{x} A$	by weakening on $\mathcal{D}$.
$\mathcal{E}_2' :: \Gamma,\Delta_2,\Delta_1;\Delta_3 \xrightarrow{x+y} C_2$	by ucut on $(A,\mathcal{D}_2',\mathcal{E}_2)$.
$\Gamma,\Delta_1;\Delta_2,\Delta_3 \xrightarrow{x+y+z} C_1 \otimes C_2$	by $\otimes$R on $\mathcal{E}_1'$ and $\mathcal{E}_2'$.

- *Case* $\oplus\mathrm{R}_i$, $i \in \{1,2\}$*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma;\Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots\\ \Gamma,A;\Delta_2 \xrightarrow{y} C_i\end{matrix}}{\Gamma,A;\Delta_2 \xrightarrow{y} C_1 \oplus C_2}\oplus\mathrm{R}_i}{\Gamma,\Delta_1;\Delta_2 \xrightarrow{x+y} C_1 \oplus C_2}\text{ucut (to show)}$$

We reason as follows:

$\Gamma,\Delta_1;\Delta_2 \xrightarrow{x+y} C_i$	by ucut on $(A,\mathcal{D},\mathcal{E})$.
$\Gamma,\Delta_1;\Delta_2 \xrightarrow{x+y} C_1 \oplus C_2$	by $\oplus\mathrm{R}_i$.

- *Case* $\multimap$ R*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma;\Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots\\ \Gamma,A;\Delta_2,B \xrightarrow{y} C\end{matrix}}{\Gamma,A;\Delta_2 \xrightarrow{y} B \multimap C}\multimap\mathrm{R}}{\Gamma,\Delta_1;\Delta_2 \xrightarrow{x+y} B \multimap C}\text{ucut (to show)}$$

We reason as follows:

$\Gamma,\Delta_1;\Delta_2,B \xrightarrow{x+y} C$	by ucut on $(A,\mathcal{D},\mathcal{E})$.
$\Gamma,\Delta_1;\Delta_2 \xrightarrow{x+y} B \multimap C$	by $\multimap$ R.

Next, the cases where the second derivation ends in a left rule:

- *Case* **0**L*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma;\Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{}{\Gamma,A;\Delta_2,\mathbf{0} \xrightarrow{y} C}\mathbf{0}\mathrm{L}}{\Gamma,\Delta_1;\Delta_2,\mathbf{0} \xrightarrow{x+y} C}\text{cut (to show)}$$

We reason as follows:

$\Gamma,\Delta_1;\Delta_2,\mathbf{0} \xrightarrow{x+y} C$	by **0**L.

- *Case* $\&\mathrm{L}_i$, $i \in \{1,2\}$:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots\\ \Gamma, B_1 \& B_2, A; \Delta_2, B_i \xrightarrow{y} C\end{array}}{\Gamma, A;\Delta_2, B_1 \& B_2 \xrightarrow{y} C}\,\&\mathrm{L}_i}{\Gamma,\Delta_1;\Delta_2,B_1 \& B_2 \xrightarrow{x+y} C}\,\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Gamma, B_1 \& B_2; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}.\\ & \Gamma, B_1 \& B_2, \Delta_1; \Delta_2, B_i \xrightarrow{x+y} C & \text{by ucut on } (A, \mathcal{D}', \mathcal{E}).\\ & \Gamma, \Delta_1; \Delta_2, B_1 \& B_2 \xrightarrow{x+y} C & \text{by } \&\mathrm{L}_i. \end{array}$$

- *Case* $\mathbf{1}$L:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots\\ \Gamma, \mathbf{1}, A; \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma, A;\Delta_2, \mathbf{1} \xrightarrow{y} C}\,\mathbf{1}\mathrm{L}}{\Gamma,\Delta_1;\Delta_2,\mathbf{1} \xrightarrow{x+y} C}\,\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Gamma, \mathbf{1}; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}.\\ & \Gamma, \mathbf{1}, \Delta_1; \Delta_2 \xrightarrow{x+y} C & \text{by ucut on } (A, \mathcal{D}', \mathcal{E}).\\ & \Gamma, \Delta_1; \Delta_2, \mathbf{1} \xrightarrow{x+y} C & \text{by } \mathbf{1}\mathrm{L}. \end{array}$$

- *Case* $\otimes$L:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \vdots\\ \Gamma, B_1 \otimes B_2, A; \Delta_2, B_1, B_2 \xrightarrow{y} C\end{array}}{\Gamma, A;\Delta_2, B_1 \otimes B_2 \xrightarrow{y} C}\,\otimes\mathrm{L}}{\Gamma,\Delta_1;\Delta_2,B_1 \otimes B_2 \xrightarrow{x+y} C}\,\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Gamma, B_1 \otimes B_2; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}.\\ & \Gamma, B_1 \otimes B_2, \Delta_1; \Delta_2, B_1, B_2 \xrightarrow{x+y} C & \text{by ucut on } (A, \mathcal{D}', \mathcal{E}).\\ & \Gamma, \Delta_1; \Delta_2, B_1 \otimes B_2 \xrightarrow{x+y} C & \text{by } \otimes\mathrm{L}. \end{array}$$

- *Case* $\oplus$L:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma;\Delta_1 \xrightarrow{x} A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdots\\ \Gamma, B_1 \oplus B_2, A; \Delta_2, B_1 \xrightarrow{y} C\end{array} \quad \begin{array}{c}\mathcal{E}_2\\ \vdots\\ \Gamma, B_1 \oplus B_2, A; \Delta_2, B_2 \xrightarrow{z} C\end{array}}{\Gamma, A;\Delta_2, B_1 \oplus B_2 \xrightarrow{y+z} C}\,\oplus\mathrm{L}}{\Gamma,\Delta_1;\Delta_2,B_1 \oplus B_2 \xrightarrow{x+y+z} C}\,\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Gamma, B_1 \oplus B_2; \Delta_1 \xrightarrow{x} A & \text{by weakening on } \mathcal{D}.\\ \mathcal{E}_1' :: & \Gamma, B_1 \oplus B_2, \Delta_1; \Delta_2, B_1 \xrightarrow{x+z} C & \text{by ucut on } (A, \mathcal{D}', \mathcal{E}_1).\\ \mathcal{E}_2' :: & \Gamma, B_1 \oplus B_2, \Delta_1; \Delta_2, B_2 \xrightarrow{y+z} C & \text{by ucut on } (A, \mathcal{D}', \mathcal{E}_2).\\ & \Gamma, \Delta_1; \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z} C & \text{by } \oplus\mathrm{L} \text{ on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'. \end{array}$$

- *Case* $\multimap$ L*:*

$$
\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}_1\\ \vdots \\ \Gamma, \Delta_3, B_1 \multimap B_2, A; \Delta_2 \xrightarrow{y} B_1\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ \vdots \\ \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_3, B_2 \xrightarrow{z} C\end{matrix}}{\Gamma, A; \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{y+z} C}\multimap\text{L}}{\Gamma, \Delta_1; \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{x+y+z} C}\text{cut (to show)}
$$

We reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_3, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} A$	by weakening on $\mathcal{D}$.
$\mathcal{E}_1' :: \Gamma, \Delta_3, B_1 \multimap B_2, \Delta_1; \Delta_2 \xrightarrow{x+y} B_1$	by ucut on $(A, \mathcal{D}_1', \mathcal{E}_1)$.
$\mathcal{D}_2' :: \Gamma, \Delta_2, B_1 \multimap B_2; \Delta_1 \xrightarrow{x} A$	by weakening on $\mathcal{D}$.
$\mathcal{E}_1' :: \Gamma, \Delta_2, B_1 \multimap B_2, \Delta_1; \Delta_3, B_2 \xrightarrow{x+z} C$	by ucut on $(A, \mathcal{D}_2', \mathcal{E}_1)$.
$\Gamma, \Delta_1; \Delta_2, \Delta_3, B_1 \multimap B_2 \xrightarrow{x+y+z} C$	by $\multimap$ L on $\mathcal{E}_1'$ and $\mathcal{E}_2'$.

- *Case* promote*, A not principal:*

$$
\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma, B; \Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots \\ \Gamma, A; \Delta_2, B \xrightarrow{y} C\end{matrix}}{\Gamma, B, A; \Delta_2 \xrightarrow{y+\mathbf{c}} C}\text{promote}}{\Gamma, B, \Delta_1; \Delta_2 \xrightarrow{x+y+\mathbf{c}} C}\text{cut (to show)}
$$

We reason as follows:

$\mathcal{E}' :: \Gamma, B, A; \Delta_2, B \xrightarrow{y} C$	by weakening on $\mathcal{E}$.
$\Gamma, B, \Delta_1; \Delta_2, B \xrightarrow{x+y} C$	by ucut on $(A, \mathcal{D}, \mathcal{E})$.
$\Gamma, B, B, \Delta_1; \Delta_2 \xrightarrow{x+y+\mathbf{c}} C$	by promote.
$\Gamma, B, \Delta_1; \Delta_2 \xrightarrow{x+y+\mathbf{c}} C$	by ucontract.

- *Case* promote*, A principal:*

$$
\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots \\ \Gamma; A, \Delta_2 \xrightarrow{y} C\end{matrix}}{\Gamma, A; \Delta_2 \xrightarrow{y+\mathbf{c}} C}\text{promote}}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y+\mathbf{c}} C}\text{cut (to show)}
$$

We reason as follows:

$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C$	by cut on $(A, \mathcal{D}, \mathcal{E})$.
$\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y+\mathbf{c}} C$	by promote (repeated).

- *Case* demote*:*

$$
\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}\\ \vdots \\ \Gamma, B, A; \Delta_2 \xrightarrow{y} C\end{matrix}}{\Gamma, A; \Delta_2, B \xrightarrow{y+\mathbf{w}} C}\text{demote}}{\Gamma, \Delta_1; \Delta_2, B \xrightarrow{x+y+\mathbf{w}} C}\text{cut (to show)}
$$

We reason as follows:

$\mathcal{D}' :: \Gamma, B; \Delta_1 \xrightarrow{x} A$ by weakening on $\mathcal{D}$.
$\Gamma, B, \Delta_1; \Delta_2 \xrightarrow{x+y} C$ by ucut on $(A, \mathcal{D}', \mathcal{E})$.
$\Gamma, \Delta_1; \Delta_2, B \xrightarrow{x+y+\mathbf{w}} C$ by demote.

This completes the proof. □

Theorem 15 (Completeness)**.** *If* $\Delta \overset{x}{\Longrightarrow} C$ *then* $\Gamma; \Delta \xrightarrow{x} C$ *for any* Γ.

Proof. The proof proceeds by induction over the derivation $\Delta \overset{x}{\Longrightarrow} C$:

- *Case* init*:*

$$\dfrac{\dfrac{}{a \overset{x}{\Longrightarrow} a}\ \text{init}}{\Gamma; a \xrightarrow{x} a}\ \text{(to show)}$$

We reason as follows:

$\Gamma; a \xrightarrow{x} a$ by init.

- *Case* $\top$R*:*

$$\dfrac{\dfrac{}{\Delta \overset{x}{\Longrightarrow} \top}\ \top\text{R}}{\Gamma; \Delta \xrightarrow{x} \top}\ \text{(to show)}$$

We reason as follows:

$\Gamma; \Delta \xrightarrow{x} \top$ by $\top$R.

- *Case* **0**L*:*

$$\dfrac{\dfrac{}{\Delta, \mathbf{0} \overset{x}{\Longrightarrow} C}\ \mathbf{0}\text{L}}{\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C}\ \text{(to show)}$$

We reason as follows:

$\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C$ by **0**L.

- *Case* **1**R*:*

$$\dfrac{\dfrac{}{\cdot \overset{x}{\Longrightarrow} \mathbf{1}}\ \mathbf{1}\text{R}}{\Gamma; \cdot \xrightarrow{x} \mathbf{1}}\ \text{(to show)}$$

We reason as follows:

$\Gamma; \cdot \xrightarrow{x} \mathbf{1}$ by **1**R.

- *Case* $\&\text{L}_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Delta, A_i \overset{x}{\Longrightarrow} C\end{array}}{\Delta, A_1 \& A_2 \overset{x}{\Longrightarrow} C}\ \&\text{L}_i}{\Gamma; \Delta, A_1 \& A_2 \xrightarrow{x} C}\ \text{(to show)}$$

We reason as follows:

$\Gamma, A_1 \& A_2; \Delta, A_i \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma; \Delta, A_1 \& A_2 \xrightarrow{x} C$ by $\&\text{L}_i$.

- *Case* &R*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Delta \overset{x}{\Longrightarrow} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Delta \overset{y}{\Longrightarrow} C_2\end{array}}{\Delta \overset{x+y}{\Longrightarrow} C_1 \mathbin{\&} C_2}\,\&\mathrm{R}}{\Gamma; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\,(\text{to show})$$

We reason as follows:

$$\begin{array}{ll} \mathcal{D}_1' :: \Gamma; \Delta \xrightarrow{x} C_1 & \text{by i.h. on } \mathcal{D}_1. \\ \mathcal{D}_2' :: \Gamma; \Delta \xrightarrow{y} C_2 & \text{by i.h. on } \mathcal{D}_2. \\ \Gamma; \Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2 & \text{by } \&\text{R on } \mathcal{D}_1', \mathcal{D}_2'. \end{array}$$

- *Case* ***1***L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Delta \overset{x}{\Longrightarrow} C\end{array}}{\Delta, \mathbf{1} \overset{x}{\Longrightarrow} C}\,\mathbf{1}\mathrm{L}}{\Gamma; \Delta, \mathbf{1} \xrightarrow{x} C}\,(\text{to show})$$

We reason as follows:

$$\begin{array}{ll} \Gamma, \mathbf{1}; \Delta \xrightarrow{x} C & \text{by i.h. on } \mathcal{D}. \\ \Gamma; \Delta, \mathbf{1} \xrightarrow{x} C & \text{by } \mathbf{1}\text{L}. \end{array}$$

- *Case* $\otimes$L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Delta, A_1, A_2 \overset{x}{\Longrightarrow} C\end{array}}{\Delta, A_1 \otimes A_2 \overset{x}{\Longrightarrow} C}\,\otimes\mathrm{L}}{\Gamma; \Delta, A_1 \otimes A_2 \xrightarrow{x} C}\,(\text{to show})$$

We reason as follows:

$$\begin{array}{ll} \Gamma, A_1 \otimes A_2; \Delta, A_1, A_2 \xrightarrow{x} C & \text{by i.h. on } \mathcal{D}. \\ \Gamma; \Delta, A_1 \otimes A_2 \xrightarrow{x} C & \text{by } \otimes\text{L}. \end{array}$$

- *Case* $\otimes$R*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Delta_1 \overset{x}{\Longrightarrow} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Delta_2 \overset{y}{\Longrightarrow} C_2\end{array}}{\Delta_1, \Delta_2 \overset{x+y}{\Longrightarrow} C_1 \otimes C_2}\,\otimes\mathrm{R}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2}\,(\text{to show})$$

We reason as follows:

$$\begin{array}{ll} \mathcal{D}_1' :: \Gamma, \Delta_2; \Delta_1 \xrightarrow{x} C_1 & \text{by i.h. on } \mathcal{D}_1. \\ \mathcal{D}_2' :: \Gamma, \Delta_1; \Delta_2 \xrightarrow{y} C_2 & \text{by i.h. on } \mathcal{D}_2. \\ \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2 & \text{by } \otimes\text{R on } \mathcal{D}_1', \mathcal{D}_2'. \end{array}$$

- *Case* $\oplus$L*:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1\\ \vdots \\ \Delta, A_1 \overset{x}{\Longrightarrow} C\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots \\ \Delta, A_2 \overset{y}{\Longrightarrow} C\end{matrix}}{\Delta, A_1 \oplus A_2 \overset{x+y}{\Longrightarrow} C}\oplus\mathrm{L}}{\Gamma; \Delta, A_1 \oplus A_2 \overset{x+y}{\longrightarrow} C}\text{(to show)}$$

We reason as follows:

$$\begin{array}{lr} \mathcal{D}_1' :: \Gamma, A_1 \oplus A_2; \Delta, A_1 \overset{x}{\longrightarrow} C & \text{by i.h. on } \mathcal{D}_1. \\ \mathcal{D}_2' :: \Gamma, A_1 \oplus A_2; \Delta, A_2 \overset{y}{\longrightarrow} C & \text{by i.h. on } \mathcal{D}_2. \\ \Gamma; \Delta, A_1 \oplus A_2 \overset{x+y}{\longrightarrow} C & \text{by } \oplus\text{L on } \mathcal{D}_1', \mathcal{D}_2'. \end{array}$$

- *Case* $\oplus\mathrm{R}_i$, $i \in \{1,2\}$*:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Delta \overset{x}{\Longrightarrow} C_i\end{matrix}}{\Delta \overset{x}{\Longrightarrow} C_1 \oplus C_2}\oplus\mathrm{R}_i}{\Gamma; \Delta \overset{x}{\longrightarrow} C_1 \oplus C_2}\text{(to show)}$$

We reason as follows:

$$\begin{array}{lr} \Gamma; \Delta \overset{x}{\longrightarrow} C_i & \text{by i.h. on } \mathcal{D}. \\ \Gamma; \Delta \overset{x}{\longrightarrow} C_1 \oplus C_2 & \text{by } \oplus\mathrm{R}_i. \end{array}$$

- *Case* $\multimap$ L*:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1\\ \vdots \\ \Delta_1 \overset{x}{\Longrightarrow} A_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots \\ \Delta_2, A_2 \overset{y}{\Longrightarrow} C\end{matrix}}{\Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\Longrightarrow} C}\multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\longrightarrow} C}\text{(to show)}$$

We reason as follows:

$$\begin{array}{lr} \mathcal{D}_1' :: \Gamma, \Delta_2, A_1 \multimap A_2; \Delta_1 \overset{x}{\longrightarrow} A_1 & \text{by i.h. on } \mathcal{D}_1. \\ \mathcal{D}_2' :: \Gamma, \Delta_1, A_1 \multimap A_2; \Delta_2, A_2 \overset{y}{\longrightarrow} C & \text{by i.h. on } \mathcal{D}_2. \\ \Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\longrightarrow} C & \text{by } \multimap\text{L on } \mathcal{D}_1', \mathcal{D}_2'. \end{array}$$

- *Case* $\multimap$ R*:*

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \Delta, C_1 \overset{x}{\Longrightarrow} C_2\end{matrix}}{\Delta \overset{x}{\Longrightarrow} C_1 \multimap C_2}\multimap\mathrm{R}}{\Gamma; \Delta \overset{x}{\longrightarrow} C_1 \multimap C_2}\text{(to show)}$$

We reason as follows:

$$\begin{array}{lr} \Gamma; \Delta, C_1 \overset{x}{\longrightarrow} C_2 & \text{by i.h. on } \mathcal{D}. \\ \Gamma; \Delta \overset{x}{\longrightarrow} C_1 \multimap C_2 & \text{by } \multimap\text{R}. \end{array}$$

- *Case* contract*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Delta, A, A \overset{x}{\Longrightarrow} C\end{array}}{\Delta, A \overset{x+\mathbf{c}}{\Longrightarrow} C}\text{contract}}{\Gamma; \Delta, A \overset{x+\mathbf{c}}{\longrightarrow} C}\text{(to show)}$$

We reason as follows:

$\Gamma; \Delta, A, A \overset{x}{\longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Gamma; \Delta, A \overset{x+\mathbf{c}}{\longrightarrow} C$ by pcontract.

- *Case* weaken*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Delta \overset{x}{\Longrightarrow} C\end{array}}{\Delta, A \overset{x+\mathbf{w}}{\Longrightarrow} C}\text{weaken}}{\Gamma; \Delta, A \overset{x+\mathbf{w}}{\longrightarrow} C}\text{(to show)}$$

We reason as follows:

$\Gamma, A; \Delta \overset{x}{\longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Gamma; \Delta, A \overset{x+\mathbf{w}}{\longrightarrow} C$ by demote.

This completes the proof. □

Theorem 16 (Soundness)**.** *If* $\Gamma; \Delta \overset{x}{\longrightarrow} C$*, and this derivation does not use the* $\oplus$L *or* &R *rules, then* $\Gamma^?, \Delta \overset{x}{\Longrightarrow} C$ *for some* $\Gamma^? \subseteq \Gamma$*. If* $\mathbf{c} \nleq x$ *i.e. there are no occurrences of the* promote *rule in the given derivation, then* Γ' *may be chosen to be empty.*

Proof. To make the proof more intuitive, we will use $(-)^?$ to indicate that the given formula may occur 0 or 1 times. This is extended to contexts in such a way that e.g. $(A, B)^?$ is either (A, B), (A), (B) or $(\cdot)$. When we apply the contraction rule in the following, we will allow the contraction of $A^?$ and A into just A. If $A^? = A$ this is just contraction, and if $A^? = \cdot$, the contraction rule simply disappears.

Similarly, given $\Gamma^?, \Gamma^?$, we can always contract this together into $\Gamma^?$ by repeated uses of the contract rule.

The proof proceeds by induction on the derivation of $\Gamma; \Delta \overset{x}{\longrightarrow} C$:

- *Case* init*:*

$$\dfrac{\dfrac{}{\Gamma; a \overset{x}{\longrightarrow} a}\text{init}}{\Gamma^?, a \overset{x}{\Longrightarrow} a}\text{(to show)}$$

We reason as follows:

$a \overset{x}{\Longrightarrow} a$ by init.

- *Case* $\top$R*:*

$$\dfrac{\dfrac{}{\Gamma; \Delta \overset{x}{\longrightarrow} \top}\top\text{R}}{\Gamma^?, \Delta \overset{x}{\Longrightarrow} \top}\text{(to show)}$$

We reason as follows:

$\Delta \overset{x}{\Longrightarrow} \top$ by $\top$R.

- *Case* $\mathbf{0}$L*:*

$$\dfrac{\dfrac{}{\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C}\ \mathbf{0}\mathrm{L}}{\Gamma^?, \Delta, \mathbf{0} \overset{x}{\Longrightarrow} C}\ (\text{to show})$$

We reason as follows:

$\Delta, \mathbf{0} \overset{x}{\Longrightarrow} C$ by $\mathbf{0}$L.

- *Case* $\mathbf{1}$R*:*

$$\dfrac{\dfrac{}{\Gamma; \cdot \xrightarrow{x} \mathbf{1}}\ \mathbf{1}\mathrm{R}}{\Gamma^?, \cdot \overset{x}{\Longrightarrow} \mathbf{1}}\ (\text{to show})$$

We reason as follows:

$\cdot \overset{x}{\Longrightarrow} \mathbf{1}$ by $\mathbf{1}$R.

- *Case* $\&\mathrm{L}_i$, $i \in \{1, 2\}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A_1 \,\&\, A_2; \Delta, A_i \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A_1 \,\&\, A_2 \xrightarrow{x} C}\ \&\mathrm{L}_i}{\Gamma^?, \Delta, A_1 \,\&\, A_2 \overset{x}{\Longrightarrow} C}\ (\text{to show})$$

If $\mathbf{c} \not\leq x$ we reason as follows:

$\Delta, A_i \overset{x}{\Longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Delta, A_1 \,\&\, A_2 \overset{x}{\Longrightarrow} C$ by $\&\mathrm{L}_i$.

If $\mathbf{c} \leq x$, then $x = x + \mathbf{c}$ and we reason as follows:

$\Gamma^?, (A_1 \,\&\, A_2)^?, \Delta, A_i \overset{x}{\Longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Gamma^?, (A_2 \,\&\, A_2)^?, \Delta, A_1 \,\&\, A_2 \overset{x}{\Longrightarrow} C$ by $\&\mathrm{L}_i$.
$\Gamma^?, \Delta, A_1 \,\&\, A_2 \overset{x+\mathbf{c}}{\Longrightarrow} C$ by contract.

- *Case* $\mathbf{1}$L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, \mathbf{1}; \Delta \xrightarrow{x} C\end{array}}{\Gamma; \Delta, \mathbf{1} \xrightarrow{x} C}\ \mathbf{1}\mathrm{L}}{\Gamma^?, \Delta, \mathbf{1} \overset{x}{\Longrightarrow} C}\ (\text{to show})$$

If $\mathbf{c} \not\leq x$ we reason as follows:

$\Delta \overset{x}{\Longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Delta, \mathbf{1} \overset{x}{\Longrightarrow} C$ by $\mathbf{1}$L.

If $\mathbf{c} \leq x$, then $x = x + \mathbf{c}$ and we reason as follows:

$\Gamma^?, \mathbf{1}^?, \Delta \overset{x}{\Longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Gamma^?, \mathbf{1}^?, \Delta, \mathbf{1} \overset{x}{\Longrightarrow} C$ by $\mathbf{1}$L.
$\Gamma^?, \Delta, \mathbf{1} \overset{x+\mathbf{c}}{\Longrightarrow} C$ by contract.

- *Case* $\otimes$L:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A_1 \otimes A_2; \Delta, A_1, A_2 \xrightarrow{x} C\end{array}}{\Gamma; \Delta, A_1 \otimes A_2 \xrightarrow{x} C} \otimes\text{L}}{\Gamma^?, \Delta, A_1 \otimes A_2 \overset{x}{\Longrightarrow} C} \text{(to show)}$$

If $\mathbf{c} \not\leq x$ we reason as follows:

$\Delta, A_1, A_2 \overset{x}{\Longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Delta, A_1 \otimes A_2 \overset{x}{\Longrightarrow} C$ by $\otimes$L.

If $\mathbf{c} \leq x$, then $x = x + \mathbf{c}$ and we reason as follows:

$\Gamma^?, (A_1 \otimes A_2)^?, \Delta, A_1, A_2 \overset{x}{\Longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Gamma^?, (A_2 \otimes A_2)^?, \Delta, A_1 \otimes A_2 \overset{x}{\Longrightarrow} C$ by $\otimes$L.
$\Gamma^?, \Delta, A_1 \otimes A_2 \overset{x+\mathbf{c}}{\Longrightarrow} C$ by contract.

- *Case* $\oplus$R$_i$, $i \in \{1, 2\}$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta \xrightarrow{x} C_i\end{array}}{\Gamma; \Delta \xrightarrow{x} C_1 \oplus C_2} \oplus\text{R}_i}{\Gamma^?, \Delta \overset{x}{\Longrightarrow} C_1 \oplus C_2} \text{(to show)}$$

We reason as follows:

$\Gamma^?, \Delta \overset{x}{\Longrightarrow} C_i$ by i.h. on $\mathcal{D}$.
$\Gamma^?, \Delta \overset{x}{\Longrightarrow} C_1 \oplus C_2$ by $\oplus$R$_i$.

- *Case* $\multimap$ R:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta, C_1 \xrightarrow{x} C_2\end{array}}{\Gamma; \Delta \xrightarrow{x} C_1 \multimap C_2} \multimap \text{R}}{\Gamma^?, \Delta \overset{x}{\Longrightarrow} C_1 \multimap C_2} \text{(to show)}$$

We reason as follows:

$\Gamma^?, \Delta, C_1 \overset{x}{\Longrightarrow} C_2$ by i.h. on $\mathcal{D}$.
$\Gamma^?, \Delta \overset{x}{\Longrightarrow} C_1 \multimap C_2$ by $\multimap$ R.

- *Case* $\otimes$R:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, \Delta_2; \Delta_1 \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, \Delta_1; \Delta_2 \xrightarrow{y} C_2\end{array}}{\Gamma; \Delta_1, \Delta_2 \overset{x+y}{\longrightarrow} C_1 \otimes C_2} \otimes\text{R}}{\Gamma^?, \Delta_1, \Delta_2 \overset{x+y}{\Longrightarrow} C_1 \otimes C_2} \text{(to show)}$$

If $\mathbf{c} \not\leq x + y$ then $\mathbf{c} \not\leq x$ and $\mathbf{c} \not\leq y$, and we reason as follows:

$\mathcal{D}_1' :: \Delta_1 \overset{x}{\Longrightarrow} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Delta_2 \overset{y}{\Longrightarrow} C_2$ by i.h. on $\mathcal{D}_2$.
$\Delta_1, \Delta_2 \overset{x+y}{\Longrightarrow} C_1 \otimes C_2$ by $\otimes$R.

If $\mathbf{c} \leq x + y$, then $x + y = x + y + \mathbf{c}$ and we reason as follows:

$$\begin{array}{llr}
\mathcal{D}'_1 :: & \Gamma^?, \Delta_2^?, \Delta_1 \overset{x}{\Longrightarrow} C_1 & \text{by i.h. on } \mathcal{D}_1. \\
\mathcal{D}'_2 :: & \Gamma^?, \Delta_1^?, \Delta_2 \overset{y}{\Longrightarrow} C_2 & \text{by i.h. on } \mathcal{D}_2. \\
& \Gamma^?, \Gamma^?, \Delta_1^?, \Delta_2^?, \Delta_1, \Delta_2 \overset{x+y}{\Longrightarrow} C_1 \otimes C_2 & \text{by } \otimes\text{R}. \\
& \Gamma^?, \Delta_1, \Delta_2 \overset{x+y+\mathbf{c}}{\Longrightarrow} C_1 \otimes C_2 & \text{by contract.}
\end{array}$$

- *Case* $\multimap$ L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, \Delta_2, A_1 \multimap A_2; \Delta_1 \overset{x}{\longrightarrow} A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, \Delta_1, A_1 \multimap A_2; \Delta_2, A_2 \overset{y}{\longrightarrow} C\end{array}}{\Gamma; \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\longrightarrow} C} \multimap \text{L}}{\Gamma^?, \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\Longrightarrow} C} \text{(to show)}$$

If $\mathbf{c} \not\leq x + y$ then $\mathbf{c} \not\leq x$ and $\mathbf{c} \not\leq y$, and we reason as follows:

$$\begin{array}{llr}
\mathcal{D}'_1 :: & \Delta_1 \overset{x}{\Longrightarrow} A_1 & \text{by i.h. on } \mathcal{D}_1. \\
\mathcal{D}'_2 :: & \Delta_2, A_2 \overset{y}{\Longrightarrow} C & \text{by i.h. on } \mathcal{D}_2. \\
& \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\Longrightarrow} C & \text{by } \multimap \text{L.}
\end{array}$$

If $\mathbf{c} \leq x + y$, then $x + y = x + y + \mathbf{c}$ and we reason as follows:

$$\begin{array}{llr}
\mathcal{D}'_1 :: & \Gamma^?, \Delta_2^?, (A_1 \multimap A_2)^?, \Delta_1 \overset{x}{\Longrightarrow} A_1 & \text{by i.h. on } \mathcal{D}_1. \\
\mathcal{D}'_2 :: & \Gamma^?, \Delta_1^?, (A_1 \multimap A_2)^?, \Delta_2, A_2 \overset{y}{\Longrightarrow} C & \text{by i.h. on } \mathcal{D}_2. \\
& \Gamma^?, \Gamma^?, \Delta_1^?, \Delta_2^?, (A_1 \multimap A_2)^?, (A_1 \multimap A_2)^?, \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y}{\Longrightarrow} C & \\
& & \text{by } \multimap \text{L.} \\
& \Gamma^?, \Delta_1, \Delta_2, A_1 \multimap A_2 \overset{x+y+\mathbf{c}}{\Longrightarrow} C & \text{by contract.}
\end{array}$$

- *Case* promote*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma; \Delta, A \overset{x}{\longrightarrow} C\end{array}}{\Gamma, A; \Delta \overset{x+\mathbf{c}}{\longrightarrow} C} \text{promote}}{\Gamma^?, A^?, \Delta \overset{x+\mathbf{c}}{\Longrightarrow} C} \text{(to show)}$$

We reason as follows:

$$\begin{array}{lr}
\Gamma^?, A^?, \Delta, A \overset{x+\mathbf{c}}{\Longrightarrow} C & \text{by i.h. on } \mathcal{D}. \\
\Gamma^?, \Delta, A \overset{x+\mathbf{c}}{\Longrightarrow} C & \text{by contract.}
\end{array}$$

- *Case* demote*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A; \Delta \overset{x}{\longrightarrow} C\end{array}}{\Gamma; \Delta, A \overset{x+\mathbf{w}}{\longrightarrow} C} \text{promote}}{\Gamma^?, \Delta, A \overset{x+\mathbf{w}}{\Longrightarrow} C} \text{(to show)}$$

We reason as follows:

$$\begin{array}{lr}
\Gamma^?, A^?, \Delta \overset{x}{\Longrightarrow} C & \text{by i.h. on } \mathcal{D}. \\
\Gamma^?, \Delta, A \overset{x+\mathbf{w}}{\Longrightarrow} C & \text{by weaken.}
\end{array}$$

□

The preceding soundness theorem has a few extra restrictions that were not present in the paper. The reason for this is that without these restrictions, the theorem is not true in general. In the remainder of this appendix, we will discuss how soundness fails, and how to fix the problem.

First of all, where does the proof of soundness break down? To see this, let us consider the case for &R:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma;\Delta \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma;\Delta \xrightarrow{y} C_2\end{array}}{\Gamma;\Delta \xrightarrow{x+y} C_1 \,\&\, C_2}\,\&\mathrm{R}}{\Gamma^?,\Delta \stackrel{x+y}{\Longrightarrow} C_1 \,\&\, C_2}\,\text{(to show)}$$

If $\mathbf{c} \nleq x$ we reason as follows:

$\mathcal{D}_1' :: \Delta \stackrel{x}{\Longrightarrow} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Delta \stackrel{y}{\Longrightarrow} C_2$ by i.h. on $\mathcal{D}_2$.
$\Delta \stackrel{x+y}{\Longrightarrow} C_1 \,\&\, C_2$ by &R.

If $\mathbf{c} \leq x$, then $x = x + \mathbf{c}$ and we reason as follows:

$\mathcal{D}_1' :: \Gamma^?,\Delta \stackrel{x}{\Longrightarrow} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma^?,\Delta \stackrel{y}{\Longrightarrow} C_2$ by i.h. on $\mathcal{D}_2$.

At this point, it would be tempting to attempt to reapply the &R rule in the hopes of deriving $\Gamma^?,\Delta \stackrel{x+y}{\Longrightarrow} C_1 \,\&\, C_2$. Unfortunately, to apply the rule we need the context in both premises to be exactly the same, and this is not necessarily the case. Recall that the notation $\Gamma^?$ indicates *some* subset of Γ. In particular, there is no guarantee that the two occurrences of $\Gamma^?$ in $\mathcal{D}_1'$ and $\mathcal{D}_2'$ in fact denote the same subset.

If we have $\mathbf{w} \leq x+y$, then we can simply apply weakening to both contexts, turning $\Gamma^?$ into Γ, and then reapply the &R rule to get the desired result. Thus, the only case in which the above proof does not go through is when the derivation is of the form $\Gamma;\Delta \xrightarrow{\mathbf{c}} C$, where there is an occurrence of &R or ⊕L somewhere in the derivation. For this reason, we will in the following focus on the case where the sequent arrow annotation is $\mathbf{c}$, i.e. the strict logic fragment.

Before we show how to fix this problem, we note that it is not simply a matter of suitably reformulating the soundness theorem to get the proof to go through. To see this, we note that we have the following derivation in our system:

$$\dfrac{\dfrac{}{\cdot;a \longrightarrow a}\,\text{init} \quad \dfrac{\dfrac{}{a;b \longrightarrow b}\,\text{init} \quad \dfrac{\dfrac{\dfrac{}{b;a \longrightarrow a}\,\text{init}}{a,b;\cdot \longrightarrow a}\,\text{promote} \quad \dfrac{\dfrac{}{a;b \longrightarrow b}\,\text{init}}{a,b;\cdot \longrightarrow b}\,\text{promote}}{a,b;\cdot \longrightarrow a \,\&\, b}\,\&\mathrm{R}}{a;b \longrightarrow b \otimes (a \,\&\, b)}\,\otimes\mathrm{R}}{\cdot;a,b \longrightarrow a \otimes (b \otimes (a \,\&\, b))}\,\otimes\mathrm{R}$$

but there is clearly no way to derive $a,b \Longrightarrow a \otimes (b \otimes (a \,\&\, b))$ no matter how contraction is applied.

The problem in this case is that by allowing the context Γ to be copied additively to each premise, we allow the above type of "asymmetric contraction" which is unsound with regard to ordinary contraction.

A similar problem makes itself apparent with the $\oplus$L rule. To fix the above problem, we replace the &R and $\oplus$L rules with the following rules:

$$\frac{\cdot;\Delta \xrightarrow{x} A \qquad \cdot;\Delta \xrightarrow{y} B}{\Gamma;\Delta \xrightarrow{x+y} A \mathbin{\&} B}\,\&\mathrm{R} \qquad \frac{\cdot;\Delta, A \xrightarrow{x} C \qquad \cdot;\Delta, B \xrightarrow{y} C}{\Gamma;\Delta, A\oplus B \xrightarrow{x+y} C}\,\oplus\mathrm{L}$$

$$\frac{\cdot;\Delta, A\oplus B, A \xrightarrow{x} C \qquad \cdot;\Delta, A\oplus B, B \xrightarrow{y} C}{\Gamma;\Delta, A\oplus B \xrightarrow{x+y+\mathbf{c}} C}\,\oplus\mathrm{L}^{\mathbf{c}}$$

Intuitively, these rules force any contraction (i.e. uses of the promote rule) to take place before the next additive connective is decomposed. By doing this, we effectively prevent the problem that caused the previous unsoundness.

The need for two $\oplus$L rules is unfortunate, but seems unavoidable in the current presentation. In effect it is due to the fact that the $\oplus$L rule is both additive — which implies the "may use" context must be emptied — but also constitutes a use of the hypothesis $A \oplus B$; which we might want to decompose twice. For this reason we add the $\oplus\mathrm{L}^{\mathbf{c}}$ rule to allow for this contraction.

Remark Morally, the $\oplus\mathrm{L}^{\mathbf{c}}$ rule should not be needed. To see this, we first note that the $\oplus$L rule in MALL is invertible, as e.g. the following derivation shows:

$$\frac{\dfrac{\overline{A_i \Longrightarrow A_i}\,\mathrm{id}_{A_i}}{A_i \Longrightarrow A_1\oplus A_2}\,\oplus\mathrm{R}_i \qquad \begin{array}{c}\vdots\\ \Delta, A_1\oplus A_2 \Longrightarrow C\end{array}}{\Delta, A_i \Longrightarrow C}\,\mathrm{cut}$$

Since both identity expansion and cut admissibility hold in strict logic, we get the desired inversion property. Now, consider a derivation that applies contraction to a disjunction in the context:

$$\frac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Delta, A_1\oplus A_2, A_1\oplus A_2 \Longrightarrow C\end{array}}{\Delta, A_1\oplus A_2 \Longrightarrow C}\,\mathrm{contract}$$

By appealing to the invertibility of $A_1 \oplus A_2$ repeatedly, we get the following derivations

$$\begin{array}{c}\mathcal{D}_{11} :: \Delta, A_1, A_1 \Longrightarrow C\\ \mathcal{D}_{12} :: \Delta, A_1, A_2 \Longrightarrow C\\ \mathcal{D}_{21} :: \Delta, A_2, A_1 \Longrightarrow C\\ \mathcal{D}_{22} :: \Delta, A_2, A_2 \Longrightarrow C\end{array}$$

From these, we may now construct the following derivation:

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_{11}\\ \vdots\\ \Delta, A_1, A_1 \Longrightarrow C\end{array}}{\Delta, A_1 \Longrightarrow C}\,\mathrm{contract} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{22}\\ \vdots\\ \Delta, A_2, A_2 \Longrightarrow C\end{array}}{\Delta, A_2 \Longrightarrow C}\,\mathrm{contract}}{\Delta, A_1\oplus A_2 \Longrightarrow C}\,\oplus\mathrm{L}$$

By repeating the above procedure, we can ensure that contraction is not applied to disjunctions, hence the $\oplus\mathrm{L}^{\mathbf{c}}$ rule should morally be superfluous. Unfortunately, the argument above relies on the fact that cut is admissible — the very fact we want to prove — and hence cannot be used directly.

That aside, let us now show that with the altered &R and ⊕L rules above, the system is sound with regard to MALL. The situation in the &R case is now as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \cdot;\Delta \xrightarrow{x} C_1\end{array}\quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \cdot;\Delta \xrightarrow{y} C_2\end{array}}{\Gamma;\Delta \xrightarrow{x+y} C_1 \mathbin{\&} C_2}\,\&\mathrm{R}}{\Gamma^?,\Delta \xRightarrow{x+y} C_1 \mathbin{\&} C_2}\,\text{(to show)}$$

We reason as follows:

$\mathcal{D}_1' :: (\cdot)^?,\Delta \xRightarrow{x} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: (\cdot)^?,\Delta \xRightarrow{y} C_2$ by i.h. on $\mathcal{D}_2$.

Because $(\cdot)^? = \cdot$ the two contexts are equal, and we may now apply the &R rule to get the desired derivation.

- *Case* ⊕L*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \cdot;\Delta,A_1 \xrightarrow{x} C\end{array}\quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \cdot;\Delta,A_2 \xrightarrow{y} C\end{array}}{\Gamma;\Delta,A_1\oplus A_2 \xrightarrow{x+y} C}\,\oplus\mathrm{L}}{\Gamma^?,\Delta,A_1\oplus A_2 \xRightarrow{x+y} C}\,\text{(to show)}$$

We reason as follows:

$\mathcal{D}_1' :: (\cdot)^?,\Delta,A_1 \xRightarrow{x} C$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: (\cdot)^?,\Delta,A_2 \xRightarrow{y} C$ by i.h. on $\mathcal{D}_2$.
$\Delta,A_1\oplus A_2 \xRightarrow{x+y} C$ by ⊕L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus\mathrm{L}^{\mathbf{c}}$*:*

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \cdot;\Delta,A_1\oplus A_2,A_1 \xrightarrow{x} C\end{array}\quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \cdot;\Delta,A_2\oplus A_2,A_2 \xrightarrow{y} C_2\end{array}}{\Gamma;\Delta,A_1\oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\,\oplus\mathrm{L}^{\mathbf{c}}}{\Gamma^?,\Delta,A_1\oplus A_2 \xRightarrow{x+y+\mathbf{c}} C}\,\text{(to show)}$$

We reason as follows:

$\mathcal{D}_1' :: (\cdot)^?,\Delta,A_1\oplus A_2,A_1 \xRightarrow{x}$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: (\cdot)^?,\Delta,A_1\oplus A_2,A_2 \xRightarrow{y} C$ by i.h. on $\mathcal{D}_2$.
$\Delta,A_1\oplus A_2,A_1\oplus A_2 \xRightarrow{x+y}$ by ⊕L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.
$\Delta,A_1\oplus A_2 \xRightarrow{x+y+\mathbf{c}} C$ by contract.

With the above change, we have thus regained soundness, but of course it is no longer clear that the system is internally sound, i.e. has the cut elimination property. To show this, we must go through the weakening, contraction and cut theorems and show that these properties continue to hold in this new system.

Before we do that, however, we need the following useful lemma

Lemma 4 (Strengthening)**.** *If* $\Gamma, A; \Delta \xrightarrow{x} C$ *then either* $\Gamma; \Delta \xrightarrow{x} C$ *or* $\Gamma; \Delta, A \xrightarrow{x} C$ *and* $\mathbf{c} \leq x$*. In either case, the height of the resulting derivation is no greater than the height of the given derivation.*

Proof. By induction on the given derivation of $\Gamma, A; \Delta \xrightarrow{x} C$. We use the shorthand $\Gamma; \Delta, A^? \xrightarrow{x} C$ to represent the result of appealing to the induction hypothesis in cases where it does not matter whether A was strengthened away or not.

- *Case* init*:*

$$\mathcal{D} :: \frac{}{\Gamma, A; a \xrightarrow{x} a}\,\text{init}$$

We reason as follows:

$\Gamma; a \xrightarrow{x} a$ by init.

- *Case* $\top$R*:*

$$\mathcal{D} :: \frac{}{\Gamma, A; \Delta \xrightarrow{x} \top}\,\top\text{R}$$

We reason as follows:

$\Gamma; \Delta \xrightarrow{x} \top$ by $\top$R.

- *Case* **0**L*:*

$$\mathcal{D} :: \frac{}{\Gamma, A; \Delta, \mathbf{0} \xrightarrow{x} C}\,\mathbf{0}\text{L}$$

We reason as follows:

$\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C$ by $\mathbf{0}$L.

- *Case* **1**R*:*

$$\mathcal{D} :: \frac{}{\Gamma, A; \cdot \xrightarrow{x} \mathbf{1}}\,\mathbf{1}\text{R}$$

We reason as follows:

$\Gamma; \cdot \xrightarrow{x} \mathbf{1}$ by $\mathbf{1}$R.

- *Case* &L$_i$, $i \in \{1, 2\}$*:*

$$\frac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A, A_1 \,\&\, A_2; \Delta, A_i \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A_1 \,\&\, A_2 \xrightarrow{x} C}\,\&\text{L}_i$$

We reason as follows:

$\Gamma, A_1 \,\&\, A_2; \Delta, A^?, A_i \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma; \Delta, A^?, A_1 \,\&\, A_2 \xrightarrow{x} C$ by &L$_i$.

- *Case* &R*:*

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \cdot; \Delta \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \cdot; \Delta \xrightarrow{y} C_2\end{array}}{\Gamma, A; \Delta \xrightarrow{x+y} C_1 \,\&\, C_2}\,\&\text{R}$$

We reason as follows:

$\Gamma; \Delta \xrightarrow{x+y} C_1 \,\&\, C_2$ by &R on $\mathcal{D}_1$ and $\mathcal{D}_2$.

- *Case* **1**L*:*

$$\cfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A, \mathbf{1}; \Delta \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, \mathbf{1} \xrightarrow{x} C}\ \mathbf{1}\mathrm{L}$$

We reason as follows:

$\Gamma, \mathbf{1}; \Delta, A^? \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma; \Delta, A^?, \mathbf{1} \xrightarrow{x} C$ by $\mathbf{1}$L.

- *Case* $\otimes$L*:*

$$\cfrac{\begin{array}{c}\mathcal{D}\\ \vdots \\ \Gamma, A, A_1 \otimes A_2; \Delta, A_1, A_2 \xrightarrow{x} C\end{array}}{\Gamma, A; \Delta, A_1 \otimes A_2 \xrightarrow{x} C}\ \otimes\mathrm{L}$$

We reason as follows:

$\Gamma, A_1 \otimes A_2; \Delta, A^?, A_1, A_2 \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.
$\Gamma; \Delta, A^?, A_1 \otimes A_2 \xrightarrow{x} C$ by $\otimes$L.

- *Case* $\otimes$R*:*

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \Gamma, A, \Delta_2; \Delta_1 \xrightarrow{x} C_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \Gamma, A, \Delta_1; \Delta_2 \xrightarrow{y} C_2\end{array}}{\Gamma, A; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2}\ \otimes\mathrm{R}$$

If applying the induction hypothesis to $\mathcal{D}_1$ and $\mathcal{D}_2$ yields either $\mathcal{D}_1' :: \Gamma, \Delta_2; \Delta_1, A \xrightarrow{x} C_1$ or $\mathcal{D}_2' :: \Gamma, \Delta_1; \Delta_2, A \xrightarrow{y} C_2$, we reason as follows:

$\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2$ or $\mathcal{D}_1$ and $\mathcal{D}_2'$.

Otherwise, A must be strengthened away from each subderivation, and we reason as follows:

$\mathcal{D}_1' :: \Gamma, \Delta_2; \Delta_1 \xrightarrow{x} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma, \Delta_1; \Delta_2 \xrightarrow{x} C_2$ by i.h. on $\mathcal{D}_2$.
$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C_1 \otimes C_2$ by $\otimes$R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L*:*

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \cdot; \Delta, A_1 \xrightarrow{x} C\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \cdot; \Delta, A_2 \xrightarrow{y} C\end{array}}{\Gamma, A; \Delta, A_1 \oplus A_2 \xrightarrow{x+y} C}\ \oplus\mathrm{L}$$

We reason as follows:

$\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y} C$ by $\oplus$L.

- *Case* $\oplus\mathrm{L}^{\mathbf{c}}$*:*

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots \\ \cdot; \Delta, A_1 \oplus A_2, A_1 \xrightarrow{x} C\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots \\ \cdot; \Delta, A_1 \oplus A_2, A_2 \xrightarrow{y} C\end{array}}{\Gamma, A; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\ \oplus\mathrm{L}^{\mathbf{c}}$$

We reason as follows:

$\Gamma;\Delta,A_1\oplus A_2 \overset{x+y+\mathbf{c}}{\longrightarrow} C$ by $\oplus\mathrm{L}^{\mathbf{c}}$.

- *Case* $\oplus\mathrm{R}_i$, $i\in\{1,2\}$:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma,A;\Delta \overset{x}{\longrightarrow} C_i\end{array}}{\Gamma,A;\Delta \overset{x}{\longrightarrow} C_1\oplus C_2}\ \oplus\mathrm{R}_i$$

We reason as follows:

$\Gamma;\Delta,A^? \overset{x}{\longrightarrow} C_i$ by i.h. on $\mathcal{D}$.
$\Gamma;\Delta,A^? \overset{x}{\longrightarrow} C_1\oplus C_2$ by $\oplus\mathrm{R}_i$.

- *Case* $\multimap\mathrm{L}$:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma,A,\Delta_2,A_1\multimap A_2;\Delta_1 \overset{x}{\longrightarrow} A_1\end{array}\quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \Gamma,A,\Delta_1,A_1\multimap A_2;\Delta_2,A_2 \overset{y}{\longrightarrow} C\end{array}}{\Gamma,A;\Delta_1,\Delta_2,A_1\multimap A_2 \overset{x+y}{\longrightarrow} C}\ \multimap\mathrm{L}$$

If applying the induction hypothesis to $\mathcal{D}_1$ and $\mathcal{D}_2$ yields either $\mathcal{D}_1' :: \Gamma,\Delta_2,A_1\multimap A_2;\Delta_1,A \overset{x}{\longrightarrow} A_1$ or $\mathcal{D}_2' :: \Gamma,\Delta_1,A_1\multimap A_2;\Delta_2,A_2,A \overset{y}{\longrightarrow} C$, we reason as follows:

$\Gamma;\Delta_1,\Delta_2,A_1\multimap A_2,A \overset{x+y}{\longrightarrow} C$
by $\multimap\mathrm{L}$ on $\mathcal{D}_1'$ and $\mathcal{D}_2$ or $\mathcal{D}_1$ and $\mathcal{D}_2'$.

Otherwise, A must be strengthened away from each subderivation, and we reason as follows:

$\mathcal{D}_1' :: \Gamma,\Delta_2,A_1\multimap A_2;\Delta_1 \overset{x}{\longrightarrow} A_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \Gamma,\Delta_1,A_1\multimap A_2;\Delta_2,A_2 \overset{x}{\longrightarrow} C$ by i.h. on $\mathcal{D}_2$.
$\Gamma;\Delta_1,\Delta_2,A_1\multimap A_2 \overset{x+y}{\longrightarrow} C$ by $\multimap\mathrm{L}$ on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\multimap\mathrm{R}$:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma,A;\Delta,C_1 \overset{x}{\longrightarrow} C_2\end{array}}{\Gamma,A;\Delta \overset{x}{\longrightarrow} C_1\multimap C_2}\ \multimap\mathrm{R}$$

We reason as follows:

$\Gamma;\Delta,A^?,C_1 \overset{x}{\longrightarrow} C_2$ by i.h. on $\mathcal{D}$.
$\Gamma;\Delta,A^? \overset{x}{\longrightarrow} C_1\multimap C_2$ by $\multimap\mathrm{R}$.

- *Case* promote, *A not principal:*

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \Gamma,A;\Delta,B \overset{x}{\longrightarrow} C\end{array}}{\Gamma,A,B;\Delta \overset{x+\mathbf{c}}{\longrightarrow} C}\ \text{promote}$$

We reason as follows:

$\Gamma;\Delta,A^?,B \overset{x}{\longrightarrow} C$ by i.h. on $\mathcal{D}$.
$\Gamma,B;\Delta,A^? \overset{x+\mathbf{c}}{\longrightarrow} C$ by promote.

- *Case* promote, *A principal:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma; \Delta, A \xrightarrow{x} C\end{matrix}}{\Gamma, A; \Delta \xrightarrow{x+\mathbf{c}} C}\text{ promote}$$

We reason as follows:

$\Gamma; \Delta, A \xrightarrow{x} C$ by $\mathcal{D}$.

- *Case* demote*:*

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma, A, B; \Delta \xrightarrow{x} C\end{matrix}}{\Gamma, A; \Delta, B \xrightarrow{x+\mathbf{w}} C}\text{ demote}$$

We reason as follows:

$\Gamma, B; \Delta, A^? \xrightarrow{x} C$ by i.h. on $\mathcal{D}$.

$\Gamma; \Delta, A^?, B \xrightarrow{x+\mathbf{w}} C$ by demote.

This completes the proof. □

By repeatedly applying the preceding lemma, we get the following useful corollary:

Corollary 1. *If* $\Gamma; \Delta \xrightarrow{x} C$ *then* $\cdot; \Gamma', \Delta \xrightarrow{x} C$ *for some* $\Gamma' \subseteq \Gamma$*. Furthermore, if* $\mathbf{c} \not\leq x$ *then* Γ' *is empty.*

We are now in a position to reprove the remaining theorems of the paper for this new system. As there is a substantial overlap between the inference rules, we will only prove the new cases, i.e. the cases where the &R and ⊕L rules were involved. The remaining cases are unchanged.

Theorem 17 (Weakening)**.** *The following rule is admissible:*

$$\dfrac{\Gamma; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\text{ weaken}$$

Furthermore, it is strongly *admissible, in the sense that it does not change the shape of the resulting derivation.*

Proof. • *Case* &R*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \cdot; \Delta \xrightarrow{x} C_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \cdot; \Delta \xrightarrow{y} C_2\end{matrix}}{\Gamma; \Delta \xrightarrow{x+y} C_1 \,\&\, C_2}\text{ \&R}}{\Gamma, A; \Delta \xrightarrow{x+y} C_1 \,\&\, C_2}\text{ (to show)}$$

We reason as follows:

$\Gamma, A; \Delta \xrightarrow{x+y} C_1 \,\&\, C_2$ by &R on $\mathcal{D}_1$ and $\mathcal{D}_2$.

- *Case* ⊕L*:*

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \cdot; \Delta, A_1 \xrightarrow{x} C\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \cdot; \Delta, A_2 \xrightarrow{y} C\end{matrix}}{\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y} C}\text{ }\oplus\text{L}}{\Gamma, A; \Delta, A_1 \oplus A_2 \xrightarrow{x+y} C}\text{ (to show)}$$

We reason as follows:

$$\Gamma, A; \Delta, A_1 \oplus A_2 \xrightarrow{x+y} C \qquad \text{by } \oplus\text{L on } \mathcal{D}_1 \text{ and } \mathcal{D}_2.$$

- *Case* $\oplus\mathrm{L}^{\mathbf{c}}$*:*

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\cdot; \Delta, A_1 \oplus A_2, A_1 \xrightarrow{x} C} \qquad \dfrac{\mathcal{D}_2}{\cdot; \Delta, A_1 \oplus A_2, A_2 \xrightarrow{y} C}}{\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\oplus\mathrm{L}^{\mathbf{c}}}{\Gamma, A; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\text{(to show)}$$

We reason as follows:

$$\Gamma, A; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C \qquad \text{by } \oplus\mathrm{L}^{\mathbf{c}} \text{ on } \mathcal{D}_1 \text{ and } \mathcal{D}_2.$$

This completes the proof. □

Theorem 18 (Contraction)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma, A, A; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\text{ucontract} \qquad \dfrac{\Gamma, A; \Delta, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x} C}\text{contract} \qquad \dfrac{\Gamma; \Delta, A, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{c}} C}\text{pcontract}$$

Proof. First, note that by applying the strengthening lemma to

$$\Gamma, A, A; \Delta \xrightarrow{x} C$$

we get either

$$\Gamma, A; \Delta \xrightarrow{x} C$$

in which case we have shown ucontract admissible, or

$$\Gamma, A; \Delta, A \xrightarrow{x} C, \qquad \mathbf{c} \leq x$$

in which case we can appeal to the admissibility of contract to get

$$\dfrac{\dfrac{\vdots}{\Gamma, A; \Delta, A \xrightarrow{x} C}}{\dfrac{\Gamma; \Delta, A \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C}\text{promote}}\text{contract}$$

as $\mathbf{c} \leq x$ implies $x = x + \mathbf{c}$. Thus, the admissibility of the ucontract rule follows directly from the admissibility of the contract rule. Similarly, to prove the admissibility of the contract rule, we can apply the strengthening lemma to

$$\Gamma, A; \Delta, A \xrightarrow{x} C$$

to get either

$$\Gamma; \Delta, A \xrightarrow{x} C$$

in which case we are done, or

$$\Gamma; \Delta, A, A \xrightarrow{x} C, \qquad \mathbf{c} \leq x$$

in which case we may appeal directly to the pcontract rule. Thus, we only need to consider cases for the pcontract rule in the following. We proceed by induction on the height of the given derivation.

- *Case* &R*:*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdots}\quad\quad \overset{\mathcal{D}_2}{\vdots}}{\dfrac{\cdot; \Delta, A, A \xrightarrow{x} C_1 \quad \cdot; \Delta, A, A \xrightarrow{y} C_2}{\Gamma; \Delta, A, A \xrightarrow{x+y} C_1 \,\&\, C_2}\,\&\mathrm{R}}}{\Gamma; \Delta, A \xrightarrow{x+y+\mathbf{c}} C_1 \,\&\, C_2}\,(\text{to show})$$

We reason as follows:

$\mathcal{D}_1' :: \cdot; \Delta, A \xrightarrow{x+\mathbf{c}} C_1$ by pcontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \cdot; \Delta, A \xrightarrow{y+\mathbf{c}} C_2$ by pcontract on $\mathcal{D}_2$.
$\Gamma; \Delta, A \xrightarrow{x+y+\mathbf{c}} C_1 \,\&\, C_2$ by &R on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus$L, $A = A_1 \oplus A_2$ *principal:*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdots}\quad\quad \overset{\mathcal{D}_2}{\vdots}}{\dfrac{\cdot; \Delta, A_1 \oplus A_2, A_1 \xrightarrow{x} C \quad \cdot; \Delta, A_1 \oplus A_2, A_2 \xrightarrow{y} C}{\Gamma; \Delta, A_1 \oplus A_2, A_1 \oplus A_2 \xrightarrow{x+y} C}\,\oplus\mathrm{L}}}{\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\,(\text{to show})$$

We reason as follows:

$\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C$ by $\oplus\mathrm{L}^{\mathbf{c}}$ on $\mathcal{D}_1$ and $\mathcal{D}_2$.

- *Case* $\oplus$L, A *not principal:*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdots}\quad\quad \overset{\mathcal{D}_2}{\vdots}}{\dfrac{\cdot; \Delta, A, A, A_1 \xrightarrow{x} C \quad \cdot; \Delta, A, A, A_2 \xrightarrow{y} C}{\Gamma; \Delta, A, A, A_1 \oplus A_2 \xrightarrow{x+y} C}\,\oplus\mathrm{L}}}{\Gamma; \Delta, A, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\,(\text{to show})$$

We reason as follows:

$\mathcal{D}_1' :: \cdot; \Delta, A, A_1 \xrightarrow{x+\mathbf{c}} C$ by pcontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \cdot; \Delta, A, A_2 \xrightarrow{y+\mathbf{c}} C$ by pcontract on $\mathcal{D}_2$.
$\Gamma; \Delta, A, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C$ by $\oplus$L on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case* $\oplus\mathrm{L}^{\mathbf{c}}$, $A = A_1 \oplus A_2$ *principal:*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\vdots}\quad\quad \overset{\mathcal{D}_2}{\vdots}}{\dfrac{\cdot; \Delta, A_1 \oplus A_2, A_1 \oplus A_2, A_1 \xrightarrow{x} C \quad \cdot; \Delta, A_1 \oplus A_2, A_1 \oplus A_2, A_2 \xrightarrow{y} C}{\Gamma; \Delta, A_1 \oplus A_2, A_1 \oplus A_2 \xrightarrow{x+y} C}\,\oplus\mathrm{L}^{\mathbf{c}}}}{\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\,(\text{to show})$$

We reason as follows:

$\mathcal{D}_1' :: \cdot; \Delta, A_1 \oplus A_2, A_1 \xrightarrow{x+\mathbf{c}} C$ by pcontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \cdot; \Delta, A_1 \oplus A_2, A_2 \xrightarrow{y+\mathbf{c}} C$ by pcontract on $\mathcal{D}_2$.
$\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C$ by $\oplus\mathrm{L}^{\mathbf{c}}$ on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

- *Case $\oplus L^{\mathbf{c}}$, A not principal:*

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \cdot; \Delta, A, A, A_1 \oplus A_2, A_1 \xrightarrow{x} C\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \cdot; \Delta, A, A, A_1 \oplus A_2, A_2 \xrightarrow{y} C\end{array}}{\Gamma; \Delta, A, A, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\oplus L^{\mathbf{c}}}{\Gamma; \Delta, A, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C}\text{(to show)}$$

We reason as follows:

$\mathcal{D}_1' :: \cdot; \Delta, A, A_1 \oplus A_2, A_1 \xrightarrow{x+\mathbf{c}} C$ by pcontract on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \cdot; \Delta, A, A_1 \oplus A_2, A_2 \xrightarrow{y+\mathbf{c}} C$ by pcontract on $\mathcal{D}_2$.
$\Gamma; \Delta, A, A_1 \oplus A_2 \xrightarrow{x+y+\mathbf{c}} C$ by $\oplus L^{\mathbf{c}}$ on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

This completes the proof. □

Theorem 19 (Cut admissibility)**.** *The following inference rules are admissible*

$$\cfrac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \Gamma; A, \Delta_2 \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C}\text{cut} \qquad \cfrac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \Gamma, A; \Delta_2 \xrightarrow{y} C}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C}\text{ucut}$$

Proof. In addition to the cases we proved in the previous cut admissibility theorem, we need to revisit the following cases for cut:

1. Cases where the first premise ends in $\oplus L$ or $\oplus L^{\mathbf{c}}$.
2. Cases where the second premise ends in &R.
3. Cases where the second premise ends in $\oplus L$ or $\oplus L^{\mathbf{c}}$, and finally
4. The principal cases for $\oplus$ and &.

For the ucut rule, we can appeal to the strengthening lemma, which we will show at the end of the proof. We will now consider these cases in the order indicated above. First, the cases where the first premise ends in an $\oplus L$ or $\oplus L^{\mathbf{c}}$ rule:

- *Case $\oplus L$:*

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \cdot; \Delta_1, B_1 \xrightarrow{x} A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \cdot; \Delta_1, B_2 \xrightarrow{y} A\end{array}}{\Gamma; \Delta_1, B_1 \oplus B_2 \xrightarrow{x+y} A}\oplus L \qquad \begin{array}{c}\mathcal{E}\\ \vdots\\ \Gamma; A, \Delta_2 \xrightarrow{z} C\end{array}}{\Gamma; \Delta_1, B_1 \oplus B_2, \Delta_2 \xrightarrow{x+y+z} C}\text{cut (to show)}$$

We reason as follows:

$\mathcal{E}' :: \cdot; A, \Delta_2, \Gamma' \xrightarrow{y} C, \quad \Gamma' \subseteq \Gamma$ by strengthening on $\mathcal{E}$.
$\mathcal{D}_1' :: \cdot; \Delta_1, B_1, \Delta_2, \Gamma' \xrightarrow{x+z} C$ by cut on $(A, \mathcal{D}_1, \mathcal{E}')$.
$\mathcal{D}_2' :: \cdot; \Delta_1, B_2, \Delta_2, \Gamma' \xrightarrow{y+z} C$ by cut on $(A, \mathcal{D}_2, \mathcal{E}')$.
$\Gamma; \Delta_1, B_1 \oplus B_2, \Delta_2, \Gamma' \xrightarrow{x+y+z} C$ by $\oplus L$ on $\mathcal{D}_1'$ and $\mathcal{D}_2'$.

Now, either $\Gamma' = \cdot$, in which case we are done, or $\mathbf{c} \leq z$ in which case we can promote all items in Γ' into the other context to get

$$\Gamma, \Gamma'; \Delta_1, B_1 \oplus B_2, \Delta_2 \xrightarrow{x+y+z} C$$

from which the desired result follows by applying ucontract repeatedly.

- *Case* $\oplus\mathrm{L}^{\mathbf{c}}$*:*

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ \cdot;\Delta_1, B_1\oplus B_2, B_1 \xrightarrow{x} A\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \cdot;\Delta_1, B_1\oplus B_2, B_2 \xrightarrow{y} A\end{matrix}}{\Gamma;\Delta_1, B_1\oplus B_2 \xrightarrow{x+y+\mathbf{c}} A}\oplus\mathrm{L}^{\mathbf{c}} \quad \begin{matrix}\mathcal{E}\\ \vdots\\ \Gamma; A, \Delta_2 \xrightarrow{z} C\end{matrix}}{\Gamma;\Delta_1, B_1\oplus B_2, \Delta_2 \xrightarrow{x+y+z+\mathbf{c}} C}\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{lll}
\mathcal{E}' :: \cdot; A, \Delta_2, \Gamma' \xrightarrow{z} C, \qquad \Gamma' \subseteq \Gamma & & \text{by strengthening on } \mathcal{E}.\\
\mathcal{D}_1' :: \cdot;\Delta_1, B_1\oplus B_2, B_1, \Delta_2, \Gamma' \xrightarrow{x+z} C & & \text{by cut on } (A, \mathcal{D}_1, \mathcal{E}').\\
\mathcal{D}_2' :: \cdot;\Delta_1, B_1\oplus B_2, B_2, \Delta_2, \Gamma' \xrightarrow{y+z} C & & \text{by cut on } (A, \mathcal{D}_2, \mathcal{E}').\\
\quad\;\; \Gamma;\Delta_1, B_1\oplus B_2, \Delta_2, \Gamma' \xrightarrow{x+y+z+\mathbf{c}} C & & \text{by } \oplus\mathrm{L}^{\mathbf{c}} \text{ on } \mathcal{D}_1' \text{ and } \mathcal{D}_2'.\\
\quad\;\; \Gamma, \Gamma';\Delta_1, B_1\oplus B_2, \Delta_2 \xrightarrow{x+y+z+\mathbf{c}} C & & \text{by promote (repeated).}\\
\quad\;\; \Gamma;\Delta_1, B_1\oplus B_2, \Delta_2 \xrightarrow{x+y+z+\mathbf{c}} C & & \text{by ucontract (repeated).}
\end{array}
$$

Next, we have the single case where the second premise ends in a &R rule:

- *Case* &R*:*

$$
\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma;\Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}_1\\ \vdots\\ \cdot; A, \Delta_2 \xrightarrow{y} C_1\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ \vdots\\ \cdot; A, \Delta_2 \xrightarrow{z} C_2\end{matrix}}{\Gamma; A, \Delta_2 \xrightarrow{y+z} C_1 \,\&\, C_2}\&\mathrm{R}}{\Gamma;\Delta_1, \Delta_2 \xrightarrow{x+y+z} C_1 \,\&\, C_2}\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{lll}
\mathcal{D}' :: \cdot;\Delta_1, \Gamma' \xrightarrow{x} A, \qquad \Gamma' \subseteq \Gamma & & \text{by strengthening on } \mathcal{D}.\\
\mathcal{E}_1' :: \cdot;\Delta_1, \Gamma', \Delta_2 \xrightarrow{x+y} C_1 & & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_1).\\
\mathcal{E}_2' :: \cdot;\Delta_1, \Gamma', \Delta_2 \xrightarrow{x+z} C_2 & & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_2).\\
\quad\;\; \Gamma;\Delta_1, \Gamma', \Delta_2 \xrightarrow{x+y+z} C_1 \,\&\, C_2 & & \text{by } \&\mathrm{R} \text{ on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'.
\end{array}
$$

Now, if $\Gamma' = \cdot$, we are done. If not, we must have $\mathbf{c} \leq x$, in which case we can promote all items in Γ' into the other context to get

$$
\Gamma, \Gamma';\Delta_1, \Delta_2 \xrightarrow{x+y+z} C_1 \,\&\, C_2
$$

from which the desired result follows by repeated use of the admissible ucontract rule.

Next, we consider the cases where the second derivation ends in either the $\oplus$L or $\oplus\mathrm{L}^{\mathbf{c}}$ rule, and for which the cut formula is *not* the principal formula:

- *Case* $\oplus$L*:*

$$
\dfrac{\begin{matrix}\mathcal{D}\\ \vdots\\ \Gamma;\Delta_1 \xrightarrow{x} A\end{matrix} \quad \dfrac{\begin{matrix}\mathcal{E}_1\\ \vdots\\ \cdot; A, \Delta_2, B_1 \xrightarrow{y} C\end{matrix} \quad \begin{matrix}\mathcal{E}_2\\ \vdots\\ \cdot; A, \Delta_2, B_2 \xrightarrow{z} C\end{matrix}}{\Gamma; A, \Delta_2, B_1\oplus B_2 \xrightarrow{y+z} C}\oplus\mathrm{L}}{\Gamma;\Delta_1, \Delta_2, B_1\oplus B_2 \xrightarrow{x+y+z} C}\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{ll}
\mathcal{D}' :: \cdot; \Delta_1, \Gamma' \xrightarrow{x} A, \quad \Gamma' \subseteq \Gamma & \text{by strengthening on } \mathcal{D}. \\
\mathcal{E}_1' :: \cdot; \Delta_1, \Gamma', \Delta_2, B_1 \xrightarrow{x+z} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_1). \\
\mathcal{E}_2' :: \cdot; \Delta_1, \Gamma', \Delta_2, B_2 \xrightarrow{y+z} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_2). \\
\quad \Gamma; \Delta_1, \Gamma', \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z} C & \text{by } \oplus\text{L on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'.
\end{array}
$$

Now, if $\Gamma' = \cdot$, we are done. If not, we must have $\mathbf{c} \leq x$, in which case we can promote all items in Γ' into the other context to get

$$\Gamma, \Gamma'; \Delta_1, \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z} C$$

from which the desired result follows by repeated use of the admissible ucontract rule.

- *Case* $\oplus\mathrm{L}^{\mathbf{c}}$*:*

$$
\dfrac{\Gamma; \Delta_1 \xrightarrow{x} A \qquad \dfrac{\cdot; A, \Delta_2, B_1 \oplus B_2, B_1 \xrightarrow{y} C \qquad \cdot; A, \Delta_2, B_1 \oplus B_2, B_2 \xrightarrow{z} C}{\Gamma; A, \Delta_2, B_1 \oplus B_2 \xrightarrow{y+z+\mathbf{c}} C}\oplus\mathrm{L}^{\mathbf{c}}}{\Gamma; \Delta_1, \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z+\mathbf{c}} C}\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{ll}
\mathcal{D}' :: \cdot; \Delta_1, \Gamma' \xrightarrow{x} A, \quad \Gamma' \subseteq \Gamma & \text{by strengthening on } \mathcal{D}. \\
\mathcal{E}_1' :: \cdot; \Delta_1, \Gamma', \Delta_2, B_1 \oplus B_2, B_1 \xrightarrow{x+z} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_1). \\
\mathcal{E}_2' :: \cdot; \Delta_1, \Gamma', \Delta_2, B_1 \oplus B_2, B_2 \xrightarrow{y+z} C & \text{by cut on } (A, \mathcal{D}', \mathcal{E}_2). \\
\quad \Gamma; \Delta_1, \Gamma', \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z+\mathbf{c}} C & \text{by } \oplus\mathrm{L}^{\mathbf{c}} \text{ on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'. \\
\quad \Gamma, \Gamma'; \Delta_1, \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z+\mathbf{c}} C & \text{by promote (repeated).} \\
\quad \Gamma; \Delta_1, \Delta_2, B_1 \oplus B_2 \xrightarrow{x+y+z+\mathbf{c}} C & \text{by ucontract (repeated).}
\end{array}
$$

Finally, we have the principal cases for cut. Note that there are two cases for $\oplus$, depending on which left rule is used:

- *Case* $A = A_1 \mathbin{\&} A_2$, $i \in \{1, 2\}$*:*

$$
\dfrac{\dfrac{\cdot; \Delta_1 \xrightarrow{x} A_1 \qquad \cdot; \Delta_1 \xrightarrow{y} A_2}{\mathcal{D} :: \Gamma; \Delta_1 \xrightarrow{x+y} A_1 \mathbin{\&} A_2}\&\mathrm{R} \qquad \dfrac{\Gamma, A_1 \mathbin{\&} A_2; \Delta_2, A_i \xrightarrow{z} C}{\Gamma; \Delta_2, A_1 \mathbin{\&} A_2 \xrightarrow{z} C}\&\mathrm{L}_i}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C}\text{cut (to show)}
$$

We reason as follows:

$$
\begin{array}{ll}
\mathcal{E}' :: \Gamma, \Delta_1; \Delta_2, A_i \xrightarrow{x+y+z} C & \text{by ucut on } (A_1 \mathbin{\&} A_2, \mathcal{D}, \mathcal{E}). \\
\mathcal{D}_i' :: \Gamma, \Delta_1; \Delta_1 \xrightarrow{x+y} A_i & \text{by weakening on } \mathcal{D}_i. \\
\quad \Gamma, \Delta_1; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by cut on } (A_i, \mathcal{D}_i', \mathcal{E}'). \\
\quad \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by contract (repeated).}
\end{array}
$$

- *Case* $A = A_1 \oplus A_2$, $i \in \{1, 2\}$*:*

$$
\dfrac{\dfrac{\Gamma; \Delta_1 \xrightarrow{x} A_i}{\mathcal{D} :: \Gamma; \Delta_1 \xrightarrow{x} A_1 \oplus A_2}\oplus\mathrm{R}_i \qquad \dfrac{\cdot; \Delta_2, A_1 \xrightarrow{y} C \qquad \cdot; \Delta_2, A_2 \xrightarrow{z} C}{\Gamma; \Delta_2, A_1 \oplus A_2 \xrightarrow{y+z} C}\oplus\mathrm{L}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C}\text{cut (to show)}
$$

We reason as follows:

$$\begin{array}{lll} \mathcal{E}_i' :: & \Gamma; \Delta_2, A_i \xrightarrow{x+y} C & \text{by weakening on } \mathcal{E}_i. \\ & \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by cut on } (A_i, \mathcal{D}, \mathcal{E}_i'). \end{array}$$

- *Case* $A = A_1 \oplus A_2$, $i \in \{1, 2\}$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}' \\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A_i\end{array}}{\mathcal{D} :: \Gamma; \Delta_1 \xrightarrow{x} A_1 \oplus A_2}\oplus\mathrm{R}_i \qquad \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \vdots \\ \cdot; \Delta_2, A_1 \oplus A_2, A_1 \xrightarrow{y} C\end{array} \quad \begin{array}{c}\mathcal{E}_2 \\ \vdots \\ \cdot; \Delta_2, A_1 \oplus A_2, A_2 \xrightarrow{z} C\end{array}}{\Gamma; \Delta_2, A_1 \oplus A_2 \xrightarrow{y+z+\mathbf{c}} C}\oplus\mathrm{L}^{\mathbf{c}}}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z+\mathbf{c}} C}\text{cut (to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{E}_i' :: & \Gamma; \Delta_2, A_1 \oplus A_2, A_i \xrightarrow{y+z} C & \text{by weakening on } \mathcal{E}_i. \\ \mathcal{E}_i'' :: & \Gamma; \Delta_2, \Delta_1, A_i \xrightarrow{x+y+z} C & \text{by cut on } (A_1 \oplus A_2, \mathcal{D}, \mathcal{E}_i'). \\ & \Gamma; \Delta_1, \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by cut on } (A_i, \mathcal{D}', \mathcal{E}_i''). \\ & \Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y+z+\mathbf{c}} C & \text{by pcontract (repeated).} \end{array}$$

This concludes the cases for the cut rule.

For the ucut rule, we reason as follows. Given

$$\dfrac{\begin{array}{c}\mathcal{D} \\ \vdots \\ \Gamma; \Delta_1 \xrightarrow{x} A\end{array} \quad \begin{array}{c}\mathcal{E} \\ \vdots \\ \Gamma, A; \Delta_2 \xrightarrow{y} C\end{array}}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C}\text{ucut (to show)}$$

we first apply the strengthening lemma to $\mathcal{E}$. If this results in a derivation of

$$\Gamma; \Delta_2 \xrightarrow{y} C$$

then the result is immediate by the admissibility of weakening. If not, we have a derivation

$$\mathcal{E}' :: \Gamma; \Delta_2, A \xrightarrow{y} C$$

and $\mathbf{c} \leq y$. Applying the cut rule to $(A, \mathcal{D}, \mathcal{E}')$ yields a derivation of the sequent

$$\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C$$

and as $\mathbf{c} \leq y$ we have $x + y = x + y + \mathbf{c}$ and we can therefore apply the promote rule repeatedly to get a derivation of the desired sequent.

This concludes the proof. □

Finally, we show that our system is still complete with regard to MALL:

Theorem 20 (Completeness)**.** *If* $\Delta \stackrel{x}{\Longrightarrow} C$ *then* $\Gamma; \Delta \xrightarrow{x} C$ *for any* Γ.

Proof. The cases where the last rule in the given derivation is not &R or $\oplus$L carry over directly from the previous proof of completeness. The remaining two cases we handle as follows:

- *Case* &R*:*

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \vdots & \vdots \\ \Delta \xRightarrow{x} C_1 & \Delta \xRightarrow{y} C_2\end{array}}{\Delta \xRightarrow{x+y} C_1 \,\&\, C_2}\,\&\mathrm{R}}{\Gamma; \Delta \xrightarrow{x+y} C_1 \,\&\, C_2}\,\text{(to show)}$$

We reason as follows:

$\mathcal{D}_1' :: \cdot; \Delta \xrightarrow{x} C_1$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \cdot; \Delta \xrightarrow{y} C_2$ by i.h. on $\mathcal{D}_2$.
$\Gamma; \Delta \xrightarrow{x+y} C_1 \,\&\, C_2$ by &R on $\mathcal{D}_1'$, $\mathcal{D}_2'$.

- *Case* $\oplus$L*:*

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \vdots & \vdots \\ \Delta, A_1 \xRightarrow{x} C & \Delta, A_2 \xRightarrow{y} C\end{array}}{\Delta, A_1 \oplus A_2 \xRightarrow{x+y} C}\,\oplus\mathrm{L}}{\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y} C}\,\text{(to show)}$$

We reason as follows:

$\mathcal{D}_1' :: \cdot; \Delta, A_1 \xrightarrow{x} C$ by i.h. on $\mathcal{D}_1$.
$\mathcal{D}_2' :: \cdot; \Delta, A_2 \xrightarrow{y} C$ by i.h. on $\mathcal{D}_2$.
$\Gamma; \Delta, A_1 \oplus A_2 \xrightarrow{x+y} C$ by $\oplus$L on $\mathcal{D}_1'$, $\mathcal{D}_2'$.

This completes the proof. □

A.2 Proofs concerning the focused calculus

Lemma 5 (Strengthening)**.** *The following properties hold:*

1. *If* $\Gamma, \pi; \Delta, [N] \longrightarrow \nu$ *then either* $\Gamma; \Delta, [N] \longrightarrow \nu$ *or* $\Gamma; \Delta, \pi, [N] \longrightarrow \nu$.
2. *If* $\Gamma, \pi; \Delta \longrightarrow [P]$ *then either* $\Gamma; \Delta \longrightarrow [P]$ *or* $\Gamma; \Delta, \pi \longrightarrow [P]$.

Furthermore, the resulting output derivations have the same structure as the input derivations.

Proof. By induction over the structure of $\Gamma, \pi; \Delta, [N] \longrightarrow \nu$ and $\Gamma, \pi; \Delta \longrightarrow [P]$. We use the notation $\pi^?$ to represent either π or $\cdot$, in cases where our reasoning is the same for both cases.

- Case init$^-$:

$$\dfrac{\dfrac{}{\Gamma, \pi; [n] \longrightarrow n}\,\mathrm{init}^-}{\Gamma; \pi^?, [n] \longrightarrow n}\,\text{(to show)}$$

We reason as follows:

$\Gamma; [n] \longrightarrow n$ by init$^-$.

- Case $\&\,\mathrm{L}_i$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \pi; \Delta, [N_i] \longrightarrow \nu\end{array}}{\Gamma, \pi; \Delta, [N_1 \,\&\, N_2] \longrightarrow \nu}\,\&\,\mathrm{L}_i}{\Gamma; \Delta, \pi^?, [N_1 \,\&\, N_2] \longrightarrow \nu}\,\text{(to show)}$$

We reason as follows:

$\mathcal{D}'$ :: $\Gamma; \Delta, \pi^?, [N_i] \longrightarrow \nu$ by the induction hypothesis on $\mathcal{D}$.
$\Gamma; \Delta, \pi^?, [N_1 \,\&\, N_2] \longrightarrow \nu$ by $\&\,\mathrm{L}_i$.

- Case $\multimap$L:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \pi, \Delta_2; \Delta_1 \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \pi, \Delta_1; \Delta_2, [N] \longrightarrow \nu}}{\Gamma, \pi; \Delta_1, \Delta_2, [P \multimap N] \longrightarrow \nu} \multimap\mathrm{L}}{\Gamma; \Delta_1, \Delta_2, \pi^?, [P \multimap N] \longrightarrow \nu} \text{(to show)}$$

We reason as follows: if applying the induction hypothesis to $\mathcal{D}$ results in a derivation $\mathcal{D}' :: \Gamma, \Delta_2; \Delta_1, \pi \longrightarrow [P]$, we have

$\Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu$ by $\multimap$L on $\mathcal{D}'$ and $\mathcal{E}$.

Likewise, if applying the induction hypothesis to $\mathcal{E}$ results in a derivation $\mathcal{E}' :: \Gamma, \Delta_1; \Delta_2, \pi, [N] \longrightarrow \nu$, we have

$\Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu$ by $\multimap$L on $\mathcal{D}$ and $\mathcal{E}'$.

Otherwise, π must be strengthened away from both derivations, and we reason as follows:

$\mathcal{D}'$ :: $\Gamma, \Delta_2; \Delta_1 \longrightarrow [P]$ by the induction hypothesis on $\mathcal{D}$.
$\mathcal{E}'$ :: $\Gamma, \Delta_1; \Delta_2, [N] \longrightarrow \nu$ by the induction hypothesis on $\mathcal{E}$.
$\Gamma; \Delta_1, \Delta_2, [P \multimap N] \longrightarrow \nu$ by $\multimap$L on $\mathcal{D}', \mathcal{E}'$.

- Case $\uparrow$L: Here, we have two cases, depending on what happens to π:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta, \langle P \rangle \longrightarrow \langle \nu \rangle}}{\Gamma, (\Gamma', \pi); \Delta, [\uparrow P] \longrightarrow \nu} \uparrow\mathrm{L}}{\Gamma, \Gamma'; \Delta, \pi^?, [\uparrow P] \longrightarrow \nu} \text{(to show)}$$

In this case we reason as follows:

$\Gamma, \Gamma'; \Delta, [\uparrow P] \longrightarrow \nu$ by $\uparrow$L on $\mathcal{D}$.

In the other case, we have

$$\dfrac{\dfrac{\overset{\mathcal{D}}{(\Gamma, \pi), \Delta, \langle P \rangle \longrightarrow \langle \nu \rangle}}{(\Gamma, \pi), \Gamma'; \Delta, [\uparrow P] \longrightarrow \nu} \uparrow\mathrm{L}}{\Gamma, \Gamma'; \Delta, \pi^?, [\uparrow P] \longrightarrow \nu} \text{(to show)}$$

and we reason as follows:

$\Gamma, \Gamma'; \Delta, \pi, [\uparrow P] \longrightarrow \nu$ by $\uparrow$L on $\mathcal{D}$.

- Case init^+:

$$\dfrac{\dfrac{}{\Gamma, \pi; p \longrightarrow [p]} \mathrm{init}^+}{\Gamma; \pi^?, p \longrightarrow [p]} \text{(to show)}$$

We reason as follows:

$\Gamma; p \longrightarrow [p]$ by init^-.

- Case $\mathbf{1}$R:

$$\dfrac{\dfrac{}{\Gamma, \pi; \cdot \longrightarrow [\mathbf{1}]}\ \mathbf{1}\mathrm{R}}{\Gamma; \pi^{?} \longrightarrow [\mathbf{1}]}\ \text{(to show)}$$

We reason as follows:

$\Gamma; \cdot \longrightarrow [\mathbf{1}]$ by $\mathbf{1}$R.

- Case $\otimes$R:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \pi, \Delta_2; \Delta_1 \longrightarrow [P]\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Gamma, \pi, \Delta_1; \Delta_2 \longrightarrow [Q]\end{array}}{\Gamma, \pi; \Delta_1, \Delta_2 \longrightarrow [P \otimes Q]}\ \otimes\mathrm{R}}{\Gamma; \Delta_1, \Delta_2, \pi^{?} \longrightarrow [P \otimes Q]}\ \text{(to show)}$$

We reason as follows: if applying the induction hypothesis to $\mathcal{D}$ results in a derivation $\mathcal{D}' :: \Gamma, \Delta_2; \Delta_1, \pi \longrightarrow [P]$, we have

$\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]$ by $\otimes$R on $\mathcal{D}'$ and $\mathcal{E}$.

Likewise, if applying the induction hypothesis to $\mathcal{E}$ results in a derivation $\mathcal{E}' :: \Gamma, \Delta_1; \Delta_2, \pi \longrightarrow [Q]$, we have

$\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]$ by $\otimes$R on $\mathcal{D}$ and $\mathcal{E}'$.

Otherwise, π must be strengthened away from both derivations, and we reason as follows:

$\mathcal{D}' :: \Gamma, \Delta_2; \Delta_1 \longrightarrow [P]$ by the induction hypothesis on $\mathcal{D}$.
$\mathcal{E}' :: \Gamma, \Delta_1; \Delta_2 \longrightarrow [Q]$ by the induction hypothesis on $\mathcal{E}$.
$\Gamma; \Delta_1, \Delta_2 \longrightarrow [P \otimes Q]$ by $\otimes$R on $\mathcal{D}', \mathcal{E}'$.

- Case $\oplus\mathrm{R}_i$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \pi; \Delta \longrightarrow [P_i]\end{array}}{\Gamma, \pi; \Delta \longrightarrow [P_1 \oplus P_2]}\ \oplus\mathrm{R}_i}{\Gamma; \Delta, \pi^{?} \longrightarrow [P_1 \oplus P_2]}\ \text{(to show)}$$

We reason as follows:

$\Gamma; \Delta, \pi^{?} \longrightarrow [P_i]$ by the induction hypothesis on $\mathcal{D}$.
$\Gamma; \Delta, \pi^{?} \longrightarrow [P_1 \oplus P_2]$ by $\oplus\mathrm{R}_i$.

- Case $\downarrow$L: Here, we have two cases, depending on what happens to π:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \Delta \longrightarrow \langle N \rangle\end{array}}{\Gamma, (\Gamma', \pi); \Delta \longrightarrow [{\downarrow} N]}\ {\downarrow}\mathrm{R}}{\Gamma, \Gamma'; \Delta, \pi^{?} \longrightarrow [{\downarrow} N]}\ \text{(to show)}$$

In this case we reason as follows:

$\Gamma, \Gamma'; \Delta \longrightarrow [{\downarrow} N]$ by $\downarrow$R on $\mathcal{D}$.

In the other case, we have

$$\dfrac{\dfrac{\overset{\mathcal{D}}{(\Gamma,\pi),\Delta \longrightarrow \langle N\rangle}}{(\Gamma,\pi),\Gamma';\Delta \longrightarrow [{\downarrow}N]}\,{\downarrow}\mathrm{R}}{\Gamma,\Gamma';\Delta,\pi^{?} \longrightarrow [{\downarrow}N]}\,\text{(to show)}$$

and we reason as follows:

$\Gamma,\Gamma';\Delta,\pi \longrightarrow [{\downarrow}N]$ by ${\downarrow}$R on $\mathcal{D}$.

This completes the proof. □

Next, admissibilty of promotion:

Lemma 6 (Admissibility of promotion)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma;\Delta,\pi,[N] \longrightarrow \nu}{\Gamma,\pi;\Delta,[N] \longrightarrow \nu}\,\text{promote}^- \qquad \dfrac{\Gamma;\Delta,\pi \longrightarrow [P]}{\Gamma,\pi;\Delta \longrightarrow [P]}\,\text{promote}^+$$

Proof. By induction on the given derivations. We begin with the cases for promote$^-$:

- Case init$^-$: Impossible
- Case $\&\,\mathrm{L}_i$:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta,\pi,[N_i] \longrightarrow \nu}}{\Gamma;\Delta,\pi,[N_1 \,\&\, N_2] \longrightarrow \nu}\,\&\,\mathrm{L}_i}{\Gamma,\pi;\Delta,[N_1 \,\&\, N_2] \longrightarrow \nu}\,\text{(to show)}$$

  We reason as follows:

  $\mathcal{D}' :: \Gamma,\pi;\Delta,[N_i] \longrightarrow \nu$ by promote$^-$ on $\mathcal{D}$.
  $\Gamma,\pi;\Delta,[N_1 \,\&\, N_2] \longrightarrow \nu$ by $\&\,\mathrm{L}_i$.

- Case $\multimap$L, π in linear context of first premiss:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma,\Delta_2;\Delta_1,\pi \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma,(\Delta_1,\pi);\Delta_2,[N] \longrightarrow \nu}}{\Gamma;(\Delta_1,\pi),\Delta_2,[P \multimap N] \longrightarrow \nu}\,\multimap\mathrm{L}}{\Gamma,\pi;\Delta_1,\Delta_2,[P \multimap N] \longrightarrow \nu}\,\text{(to show)}$$

  We reason as follows:

  $\mathcal{D}' :: \Gamma,\pi,\Delta_2;\Delta_1 \longrightarrow [P]$ by promote$^+$ on $\mathcal{D}$.
  $\Gamma,\pi;\Delta_1,\Delta_2,[P \multimap N] \longrightarrow \nu$ by $\multimap$L on $\mathcal{D}'$ and $\mathcal{E}$.

- Case $\multimap$L, π in linear context of second premiss:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma,(\Delta_2,\pi);\Delta_1 \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma,\Delta_1;\Delta_2,\pi,[N] \longrightarrow \nu}}{\Gamma;\Delta_1,(\Delta_2,\pi),[P \multimap N] \longrightarrow \nu}\,\multimap\mathrm{L}}{\Gamma,\pi;\Delta_1,\Delta_2,[P \multimap N] \longrightarrow \nu}\,\text{(to show)}$$

  We reason as follows:

$\mathcal{E}'$ :: $\Gamma, \pi, \Delta_1; \Delta_2, [N] \longrightarrow \nu$ by promote$^-$ on $\mathcal{E}$.
$\Gamma, \pi; \Delta_1, \Delta_2, [P \multimap N] \longrightarrow \nu$ by $\multimap$L on $\mathcal{D}$ and $\mathcal{E}'$.

- Case $\uparrow$L:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta, \pi, \langle P \rangle \longrightarrow \langle \nu \rangle}}{\Gamma, \Gamma'; \Delta, \pi, [\uparrow P] \longrightarrow \nu}\uparrow\mathrm{L}}{\Gamma, \Gamma', \pi; \Delta, [\uparrow P] \longrightarrow \nu}\text{(to show)}$$

In this case we reason as follows:

$\Gamma, \Gamma', \pi; \Delta, [\uparrow P] \longrightarrow \nu$ by $\uparrow$L on $\mathcal{D}$.

Next, the cases for promote$^+$:

- Case init$^+$: Impossible.
- Case **1**R: Impossible.
- Case $\otimes$R, π in linear context of first premiss:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2; \Delta_1, \pi \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1, \pi; \Delta_2 \longrightarrow [Q]}}{\Gamma; (\Delta_1, \pi), \Delta_2 \longrightarrow [P \otimes Q]}\otimes\mathrm{R}}{\Gamma, \pi; \Delta_1, \Delta_2 \longrightarrow [P \otimes Q]}\text{(to show)}$$

We reason as follows:

$\mathcal{D}'$ :: $\Gamma, \pi, \Delta_2; \Delta_1 \longrightarrow [P]$ by promote$^+$ on $\mathcal{D}$.
$\Gamma, \pi; \Delta_1, \Delta_2 \longrightarrow [P \otimes Q]$ by $\otimes$R on $\mathcal{D}'$ on $\mathcal{E}$.

- Case $\otimes$R, π in linear context of second premiss:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2, \pi; \Delta_1 \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1; \Delta_2, \pi \longrightarrow [Q]}}{\Gamma; \Delta_1, (\Delta_2, \pi) \longrightarrow [P \otimes Q]}\otimes\mathrm{R}}{\Gamma, \pi; \Delta_1, \Delta_2 \longrightarrow [P \otimes Q]}\text{(to show)}$$

We reason as follows:

$\mathcal{E}'$ :: $\Gamma, \pi, \Delta_1; \Delta_2 \longrightarrow [Q]$ by promote$^+$ on $\mathcal{E}$.
$\Gamma, \pi; \Delta_1, \Delta_2 \longrightarrow [P \otimes Q]$ by $\otimes$R on $\mathcal{D}$ on $\mathcal{E}'$.

- Case $\oplus$R$_i$:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta, \pi \longrightarrow [P_i]}}{\Gamma; \Delta, \pi \longrightarrow [P_1 \oplus P_2]}\oplus\mathrm{R}_i}{\Gamma, \pi; \Delta \longrightarrow [P_1 \oplus P_2]}\text{(to show)}$$

We reason as follows:

$\Gamma, \pi; \Delta \longrightarrow [P_i]$ by promote$^+$ on $\mathcal{D}$.
$\Gamma, \pi; \Delta \longrightarrow [P_1 \oplus P_2]$ by $\oplus$R$_i$.

- Case $\downarrow$L:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \Delta, \pi \longrightarrow \langle N \rangle\end{array}}{\Gamma, \Gamma'; \Delta, \pi \longrightarrow [\downarrow N]}\downarrow\mathrm{R}}{\Gamma, \Gamma', \pi; \Delta \longrightarrow [\downarrow N]}\text{(to show)}$$

In this case we reason as follows:

$\Gamma, \pi, \Gamma'; \Delta \longrightarrow [\downarrow N]$ by $\downarrow$R on $\mathcal{D}$.

This completes the proof. □

Next, contraction properties galore.

Theorem 21 (Neutral contraction)**.** *The following inference rules are admissible*

$$\dfrac{\Gamma, \pi, \pi; \Delta, [N] \longrightarrow \nu}{\Gamma, \pi; \Delta, [N] \longrightarrow \nu}\text{ucontract}^- \quad \dfrac{\Gamma, \pi; \Delta, \pi, [N] \longrightarrow \nu}{\Gamma; \Delta, \pi, [N] \longrightarrow \nu}\text{contract}^- \quad \dfrac{\Gamma; \Delta, \pi, \pi, [N] \longrightarrow \nu}{\Gamma; \Delta, \pi, [N] \longrightarrow \nu}\text{pcontract}^-$$

$$\dfrac{\Gamma, \pi, \pi; \Delta \longrightarrow [P]}{\Gamma, \pi; \Delta \longrightarrow [P]}\text{ucontract}^+ \quad \dfrac{\Gamma, \pi; \Delta, \pi \longrightarrow [P]}{\Gamma; \Delta, \pi \longrightarrow [P]}\text{contract}^+ \quad \dfrac{\Gamma; \Delta, \pi, \pi \longrightarrow [P]}{\Gamma; \Delta, \pi \longrightarrow [P]}\text{pcontract}^+$$

$$\dfrac{\Sigma, \pi, \pi \longrightarrow \langle N \rangle}{\Sigma, \pi \longrightarrow \langle N \rangle}\text{icontract} \quad \dfrac{\Gamma, \pi, \pi \longrightarrow \nu}{\Gamma, \pi \longrightarrow \nu}\text{contract}$$

Proof. By repeated use of the preceding lemmas, it suffices to show that the pcontract$^-$ and pcontract$^+$ rules are admissible. Thus, for instance, given

$$\Gamma, \pi, \pi; \Delta \longrightarrow [P]$$

we apply the strengthening lemma to get either

$$\Gamma, \pi; \Delta \longrightarrow [P] \qquad \text{or} \qquad \Gamma, \pi; \Delta, \pi \longrightarrow [P]$$

In the former case, we are done, and in the latter case, we appeal to the contract$^+$ rule followed by the promote$^+$ rule. For the contract$^+$ rule, we again apply the strengthening lemma, resulting in either

$$\Gamma; \Delta, \pi \longrightarrow [P] \qquad \text{or} \qquad \Gamma; \Delta, \pi, \pi \longrightarrow [P]$$

Again, in the former case, we are done, and in the latter case we appeal to the pcontract$^+$ rule. A similar argument reduces ucontract$^-$ to contract$^-$ to pcontract$^-$.

To show the admissibility of these rules, we proceed by induction on the given derivations. First, the cases for pcontract$^-$:

- Case init$^-$: Impossible.
- Case $\&$L$_i$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \Delta, \pi, \pi, [N_i] \longrightarrow \nu\end{array}}{\Gamma; \Delta, \pi, \pi, [N_1 \,\&\, N_2] \longrightarrow \nu}\&\mathrm{L}_i}{\Gamma; \Delta, \pi, [N_1 \,\&\, N_2] \longrightarrow \nu}\text{(to show)}$$

We reason as follows:

$\Gamma; \Delta, \pi, [N_i] \longrightarrow \nu$ by the induction hypothesis on $\mathcal{D}$.
$\Gamma; \Delta, \pi, [N_1 \,\&\, N_2] \longrightarrow \nu$ by $\&$L$_i$.

- Case $\multimap$L, π, π in linear context of first premiss:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2; \Delta_1, \pi, \pi \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1, \pi, \pi; \Delta_2, [N] \longrightarrow \nu}}{\Gamma; (\Delta_1, \pi, \pi), \Delta_2, [P \multimap N] \longrightarrow \nu} \multimap\text{L}}{\Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu} \text{(to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Gamma, \Delta_2; \Delta_1, \pi \longrightarrow [P] & \text{by pcontract}^+ \text{ on } \mathcal{D}. \\ \mathcal{E}' :: & \Gamma, \Delta_1, \pi; \Delta_2, [N] \longrightarrow \nu & \text{by ucontract}^- \text{ on } \mathcal{E}. \\ & \Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu & \text{by } \multimap\text{L on } \mathcal{D}' \text{ and } \mathcal{E}'. \end{array}$$

- Case $\multimap$L, π, π in linear context of second premiss:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2, \pi, \pi; \Delta_1 \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1; \Delta_2, \pi, \pi, [N] \longrightarrow \nu}}{\Gamma; \Delta_1, (\Delta_2, \pi, \pi), [P \multimap N] \longrightarrow \nu} \multimap\text{L}}{\Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu} \text{(to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Gamma, \Delta_2, \pi; \Delta_1 \longrightarrow [P] & \text{by ucontract}^+ \text{ on } \mathcal{D}. \\ \mathcal{E}' :: & \Gamma, \Delta_1; \Delta_2, \pi, [N] \longrightarrow \nu & \text{by pcontract}^- \text{ on } \mathcal{E}. \\ & \Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu & \text{by } \multimap\text{L on } \mathcal{D}' \text{ and } \mathcal{E}'. \end{array}$$

- Case $\multimap$L, π and π in separate linear contexts:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2, \pi; \Delta_1, \pi \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1, \pi; \Delta_2, \pi, [N] \longrightarrow \nu}}{\Gamma; (\Delta_1, \pi), (\Delta_2, \pi), [P \multimap N] \longrightarrow \nu} \multimap\text{L}}{\Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu} \text{(to show)}$$

We reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: & \Gamma, \Delta_2; \Delta_1, \pi \longrightarrow [P] & \text{by contract}^+ \text{ on } \mathcal{D}. \\ & \Gamma, \Delta_1; \Delta_2, \pi, [N] \longrightarrow \nu & \text{by contract}^- \text{ on } \mathcal{E}. \\ \mathcal{E}' :: & \Gamma, \Delta_1, \pi; \Delta_2, [N] \longrightarrow \nu & \text{by promote}^-. \\ & \Gamma; \Delta_1, \Delta_2, \pi, [P \multimap N] \longrightarrow \nu & \text{by } \multimap\text{L on } \mathcal{D}' \text{ and } \mathcal{E}'. \end{array}$$

- Case $\uparrow$L:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta, \pi, \pi, \langle P \rangle \longrightarrow \langle \nu \rangle}}{\Gamma, \Gamma'; \Delta, \pi, \pi, [\uparrow P] \longrightarrow \nu} \uparrow\text{L}}{\Gamma, \Gamma'; \Delta, \pi, [\uparrow P] \longrightarrow \nu} \text{(to show)}$$

We reason as follows:

$$\begin{array}{ll} \Gamma, \Delta, \pi, \langle P \rangle \longrightarrow \langle \nu \rangle & \text{by icontract on } \mathcal{D}. \\ \Gamma, \Gamma'; \Delta, \pi, [\uparrow P] \longrightarrow \nu & \text{by } \uparrow\text{L}. \end{array}$$

Next, the cases for pcontract$^+$:

- Case init$^+$: Impossible.

- Case $\mathbf{1}$R: Impossible.
- Case $\oplus\mathrm{R}_i$:

$$\cfrac{\cfrac{\overset{\mathcal{D}}{\Gamma; \Delta, \pi, \pi \longrightarrow [P_i]}}{\Gamma; \Delta, \pi, \pi \longrightarrow [P_1 \oplus P_2]}\ \oplus\mathrm{R}_i}{\Gamma; \Delta, \pi \longrightarrow [P_1 \oplus P_2]}\ \text{(to show)}$$

We reason as follows:

$\Gamma; \Delta, \pi \longrightarrow [P_i]$	by the induction hypothesis on $\mathcal{D}$.
$\Gamma; \Delta, \pi \longrightarrow [P_1 \oplus P_2]$	by $\oplus\mathrm{R}_i$.

- Case $\otimes$R, π, π in linear context of first premiss:

$$\cfrac{\cfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2; \Delta_1, \pi, \pi \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1, \pi, \pi; \Delta_2 \longrightarrow [Q]}}{\Gamma; (\Delta_1, \pi, \pi), \Delta_2 \longrightarrow [P \otimes Q]}\ \otimes\mathrm{R}}{\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]}\ \text{(to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, \Delta_2; \Delta_1, \pi \longrightarrow [P]$	by pcontract$^+$ on $\mathcal{D}$.
$\mathcal{E}' :: \Gamma, \Delta_1, \pi; \Delta_2 \longrightarrow [Q]$	by ucontract$^+$ on $\mathcal{E}$.
$\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]$	by $\otimes$R on $\mathcal{D}'$ and $\mathcal{E}'$.

- Case $\otimes$R, π, π in linear context of second premiss:

$$\cfrac{\cfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2, \pi, \pi; \Delta_1 \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1; \Delta_2, \pi, \pi \longrightarrow [Q]}}{\Gamma; \Delta_1, (\Delta_2, \pi, \pi) \longrightarrow [P \otimes Q]}\ \otimes\mathrm{R}}{\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]}\ \text{(to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, \Delta_2, \pi; \Delta_1 \longrightarrow [P]$	by ucontract$^+$ on $\mathcal{D}$.
$\mathcal{E}' :: \Gamma, \Delta_1; \Delta_2, \pi \longrightarrow [Q]$	by pcontract$^+$ on $\mathcal{E}$.
$\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]$	by $\otimes$R on $\mathcal{D}'$ and $\mathcal{E}'$.

- Case $\otimes$R, π and π in separate linear contexts:

$$\cfrac{\cfrac{\overset{\mathcal{D}}{\Gamma, \Delta_2, \pi; \Delta_1, \pi \longrightarrow [P]} \quad \overset{\mathcal{E}}{\Gamma, \Delta_1, \pi; \Delta_2, \pi \longrightarrow [Q]}}{\Gamma; (\Delta_1, \pi), (\Delta_2, \pi) \longrightarrow [P \otimes Q]}\ \otimes\mathrm{R}}{\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]}\ \text{(to show)}$$

We reason as follows:

$\mathcal{D}' :: \Gamma, \Delta_2; \Delta_1, \pi \longrightarrow [P]$	by contract$^+$ on $\mathcal{D}$.
$\Gamma, \Delta_1; \Delta_2, \pi \longrightarrow [Q]$	by contract$^+$ on $\mathcal{E}$.
$\mathcal{E}' :: \Gamma, \Delta_1, \pi; \Delta_2 \longrightarrow [Q]$	by promote$^+$.
$\Gamma; \Delta_1, \Delta_2, \pi \longrightarrow [P \otimes Q]$	by $\otimes$R on $\mathcal{D}'$ and $\mathcal{E}'$.

- Case $\downarrow$R:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, \Delta, \pi, \pi \longrightarrow \langle N \rangle}}{\Gamma, \Gamma'; \Delta, \pi, \pi \longrightarrow [\downarrow N]} \downarrow\mathrm{R}}{\Gamma, \Gamma'; \Delta, \pi \longrightarrow [\downarrow N]} \text{(to show)}$$

We reason as follows:

$\Gamma, \Delta, \pi \longrightarrow \langle N \rangle$	by icontract on $\mathcal{D}$.
$\Gamma, \Gamma'; \Delta, \pi \longrightarrow [\downarrow N]$	by $\downarrow$R.

Next, the cases for icontract:

- Case $\top$R:

$$\dfrac{\dfrac{}{\Sigma, \pi, \pi \longrightarrow \langle \top \rangle} \top\mathrm{R}}{\Sigma, \pi \longrightarrow \langle \top \rangle} \text{(to show)}$$

We reason as follows:

$\Sigma, \pi \longrightarrow \langle \top \rangle$	by $\top$R.

- Case **0**L:

$$\dfrac{\dfrac{}{\Sigma, \pi, \pi, \langle \mathbf{0} \rangle \longrightarrow \langle N \rangle} \mathbf{0}\mathrm{L}}{\Sigma, \pi, \langle \mathbf{0} \rangle \longrightarrow \langle N \rangle} \text{(to show)}$$

We reason as follows:

$\Sigma, \pi, \langle \mathbf{0} \rangle \longrightarrow \langle N \rangle$	by **0**L.

- Case & R:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Sigma, \pi, \pi \longrightarrow \langle N \rangle} \quad \overset{\mathcal{E}}{\Sigma, \pi, \pi \longrightarrow \langle M \rangle}}{\Sigma, \pi, \pi \longrightarrow \langle N \,\&\, M \rangle} \&\,\mathrm{R}}{\Sigma, \pi \longrightarrow \langle N \,\&\, M \rangle} \text{(to show)}$$

We reason as follows:

$\mathcal{D}' :: \Sigma, \pi \longrightarrow \langle N \rangle$	by icontract on $\mathcal{D}$.
$\mathcal{E}' :: \Sigma, \pi \longrightarrow \langle M \rangle$	by icontract on $\mathcal{E}$.
$\Sigma, \pi \longrightarrow \langle N \,\&\, M \rangle$	by & R on $\mathcal{D}'$ and $\mathcal{E}'$.

- Case **1**L:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Sigma, \pi, \pi \longrightarrow \langle N \rangle}}{\Sigma, \pi, \pi, \langle \mathbf{1} \rangle \longrightarrow \langle N \rangle} \mathbf{1}\mathrm{L}}{\Sigma, \pi, \langle \mathbf{1} \rangle \longrightarrow \langle N \rangle} \text{(to show)}$$

We reason as follows:

$\Sigma, \pi \longrightarrow \langle N \rangle$	by icontract on $\mathcal{D}$.
$\Sigma, \pi, \langle \mathbf{1} \rangle \longrightarrow \langle N \rangle$	by **1**L.

- Case $\otimes$L:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Sigma, \pi, \pi, \langle P\rangle, \langle Q\rangle \longrightarrow \langle N\rangle\end{array}}{\Sigma, \pi, \pi, \langle P \otimes Q\rangle \longrightarrow \langle N\rangle}\otimes\text{L}}{\Sigma, \pi, \langle P \otimes Q\rangle \longrightarrow \langle N\rangle}\text{(to show)}$$

  We reason as follows:

  $\Sigma, \pi, \langle P\rangle, \langle Q\rangle \longrightarrow \langle N\rangle$ by icontract on $\mathcal{D}$.
  $\Sigma, \pi, \langle P \otimes Q\rangle \longrightarrow \langle N\rangle$ by $\otimes$L.

- Case $\oplus$L:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Sigma, \pi, \pi, \langle P\rangle \longrightarrow \langle N\rangle\end{array} \quad \begin{array}{c}\mathcal{E}\\ \Sigma, \pi, \pi, \langle Q\rangle \longrightarrow \langle N\rangle\end{array}}{\Sigma, \pi, \pi, \langle P \oplus Q\rangle \longrightarrow \langle N\rangle}\oplus\text{L}}{\Sigma, \pi, \langle P \oplus Q\rangle \longrightarrow \langle N\rangle}\text{(to show)}$$

  We reason as follows:

  $\mathcal{D}' :: \Sigma, \pi, \langle P\rangle \longrightarrow \langle N\rangle$ by icontract on $\mathcal{D}$.
  $\mathcal{E}' :: \Sigma, \pi, \langle Q\rangle \longrightarrow \langle N\rangle$ by icontract on $\mathcal{E}$.
  $\Sigma, \pi, \langle P \oplus Q\rangle \longrightarrow \langle N\rangle$ by $\oplus$L on $\mathcal{D}'$ and $\mathcal{E}'$.

- Case $\multimap$R:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Sigma, \pi, \pi, \langle P\rangle \longrightarrow \langle N\rangle\end{array}}{\Sigma, \pi, \pi \longrightarrow \langle P \multimap N\rangle}\multimap\text{R}}{\Sigma, \pi \longrightarrow \langle P \multimap N\rangle}\text{(to show)}$$

  We reason as follows:

  $\Sigma, \pi, \langle P\rangle \longrightarrow \langle N\rangle$ by icontract on $\mathcal{D}$.
  $\Sigma, \pi \longrightarrow \langle P \multimap N\rangle$ by $\multimap$R.

- Case rel:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \pi, \pi, \Delta \longrightarrow \nu\end{array}}{\Gamma, \pi, \pi, \langle\Delta\rangle \longrightarrow \langle\nu\rangle}\text{rel}}{\Gamma, \pi, \langle\Delta\rangle \longrightarrow \langle\nu\rangle}\text{(to show)}$$

  We reason as follows:

  $\Gamma, \pi, \Delta \longrightarrow \nu$ by contract on $\mathcal{D}$.
  $\Gamma, \pi, \langle\Delta\rangle \longrightarrow \langle\nu\rangle$ by rel.

Finally, the cases for contract:

- Case $\downarrow$L, π non-principal:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ {\downarrow}N; \Gamma, \pi, \pi, [N] \longrightarrow \nu\end{array}}{\Gamma, \pi, \pi, {\downarrow}N \longrightarrow \nu}{\downarrow}\text{L}}{\Gamma, \pi, {\downarrow}N \longrightarrow \nu}\text{(to show)}$$

  We reason as follows:

$\downarrow N; \Gamma, \pi, [N] \longrightarrow \nu$ by pcontract$^-$ on $\mathcal{D}$.
$\Gamma, \pi, \downarrow N \longrightarrow \nu$ by $\downarrow$L.

- Case $\downarrow$L, $\pi = \downarrow N$ principal:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \downarrow N; \Gamma, \downarrow N, [N] \longrightarrow \nu\end{matrix}}{\Gamma, \downarrow N, \downarrow N \longrightarrow \nu}\downarrow\mathrm{L}}{\Gamma, \downarrow N \longrightarrow \nu}\text{(to show)}$$

We reason as follows:

$\cdot; \Gamma, \downarrow N, [N] \longrightarrow \nu$ by contract$^-$ on $\mathcal{D}$.
$\downarrow N; \Gamma, [N] \longrightarrow \nu$ by promote$^-$.
$\Gamma, \downarrow N \longrightarrow \nu$ by $\downarrow$L.

- Case $\uparrow$R:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \cdot; \Gamma, \pi, \pi \longrightarrow [P]\end{matrix}}{\Gamma, \pi, \pi \longrightarrow \uparrow P}\uparrow\mathrm{R}}{\Gamma, \pi \longrightarrow \uparrow P}\text{(to show)}$$

We reason as follows:

$\cdot; \Gamma, \pi \longrightarrow [P]$ by pcontract$^+$ on $\mathcal{D}$.
$\Gamma, \pi \longrightarrow \uparrow P$ by $\uparrow$R.

□