$$\frac{}{\Gamma, \phi \vdash \phi}\ \mathsf{ax} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \bot_{\mathrm{E}} \qquad \frac{}{\Gamma \vdash \top}\ \top_{\mathrm{I}}$$

$$\frac{\Gamma \vdash \phi_1 \quad \Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \wedge \phi_2}\ \wedge_{\mathrm{I}} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\ \wedge_{\mathrm{E}}^{i}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee_{\mathrm{I}}^{i} \qquad \frac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi}\ \vee_{\mathrm{E}}$$

$$\frac{\Gamma \vdash \phi\{x \mapsto t\}}{\Gamma \vdash \exists x.\phi}\ \exists_{\mathrm{I}} \qquad \frac{\Gamma \vdash \exists x.\phi \quad \Gamma, \phi \vdash \psi}{\Gamma \vdash \psi}\ \exists_{\mathrm{E}}\ (x \notin \mathsf{fv}(\Gamma, \psi))$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \forall x.\phi}\ \forall_{\mathrm{I}}\ (x \notin \mathsf{fv}(\Gamma)) \qquad \frac{\Gamma \vdash \forall x.\phi}{\Gamma \vdash \phi\{x \mapsto t\}}\ \forall_{\mathrm{E}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\ \Rightarrow_{\mathrm{I}} \qquad \frac{\Gamma \vdash \phi \Rightarrow \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}\ \Rightarrow_{\mathrm{E}}$$

$$\frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi}\ \neg_{\mathrm{I}} \qquad \frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}\ \mathrm{RAA} \qquad \frac{\Gamma \vdash \neg\phi \quad \Gamma \vdash \phi}{\Gamma \vdash \bot}\ \neg_{\mathrm{E}}$$

Rules $\wedge_{\mathrm{E}}^{i}$ and $\vee_{\mathrm{I}}^{i}$ are for $i \in \{1, 2\}$. Implicit α-renaming is allowed, which is useful to satisfy the side conditions of rules $\exists_{\mathrm{E}}$ and $\forall_{\mathrm{I}}$.

Figure 1: Natural deduction rules for first-order classical logic.