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Mixing Time of the Asymmetric Simple Exclusion Problem on a Ring with Two Particles

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Abstract. In this paper, we are interested in the behaviour of the Asymmetric Simple Exclusion Problem (ASEP) on a ring in the non-stationary mode. More precisely, we study the time of convergence towards the stationary distribution (mixing time). We model the ASEP as a Markov chain and prove, using elementary techniques of coupling and martingale theory that the mixing time of ASEP is quadratic in the case of two particles.

1 Introduction

The asymmetric simple exclusion process (ASEP) has attracted a lot of interest because of many applications in the context of traffic flow theory, the kinetics of biopolymerization, and, more generally, because it is a paradigmatic model for nonequilibrium systems (see, e.g., [5]). Here, we consider a discrete version of the ASEP on a *ring* topology with *sequential (anticlockwise) update*.

The model can be defined as follows. Consider N sites, numbered clockwise from 1 to N on a ring. (Site 1 is immediately to the right of N.) Each site i ($1 \leq i \leq N$) can be occupied by a particle ($x_i = 1$) or may be empty ($x_i = 0$). A configuration $X = (x_1, \cdots, x_N)$ is a tuple of $\Omega = \{0,1\}^N$. Given a configuration X, if we find a particle at site $i = N$ and no particle at site 1, we move the particle one site to the right (i.e., at site 1) with probability p. Otherwise, the particle stays at site N. We continue the update with site $i = N-1$, and so forth until site $i = 1$ is updated. A *round* is thus completed. The process continues by another round starting at site $i = N$ iteratively. One thus defines a Markov chain $\mathcal{M}(N, p, m)$ where N is the number of sites, p the probability of "hopping" to the right, and m the number of particles. It is easy to show that $\mathcal{M}(N, p, m)$ is ergodic, and that the stationary distribution is uniform.

As noted by Schütz [15], exact results for the dynamics of the ASEP model are scarce, even in the case of the two-particle system ($m = 2$). Although the two-particle system may seem trivial, it exhibits already some of the characteristic behaviour of ASEP at finite density, such as the apparition of a phenomenon

of superdiffusion. (Superdiffusion is absent in the symmetric exclusion process or with non-interactive particles.) While the stationary phase of ASEP has been well studied in statistical physics literature (see, e.g., [5]), the speed of convergence to stationarity for ASEP remains an important problem. One common measure of the speed of convergence is the "mixing time", i.e., the number of steps after which the current distribution is ε-close to the stationary distribution (see definitions below).

We prove in this paper, using elementary techniques of coupling and martingales, that ASEP is "rapidly mixing". More precisely, we prove that the mixing time is bounded by $\frac{3eN^2}{pq} \ln \varepsilon^{-1}$ (with $q = 1 - p$) for the case of $m = 2$ particles.

Related work

Schütz studies some quantitative results for the dynamics of the two-particle ASEP system with sequential update, but in the continuous-time framework, on an infinite one-dimensional lattice [15]. This study has been adapted to the ring topology by Priezzhev [11]. These authors do not treat the problem of mixing time.

The mixing time of the ASEP has been previously treated in [9] and [4]. Both works treat the case of an arbitrary number m of particles, but in a different framework (*totally* asymmetric exclusion process): the update is random (rather than sequential) and the hopping is nonprobabilistic (i.e., $p = 1$). In [9], Fill, generalizing the geometric bounds of Diaconis and Strook [7], shows that the mixing time is polynomial in the number of sites N and exponential in the number of particles m. In [4], Coppersmith and Pak, using the geometric bounds of Diaconis and Saloff-Coste on symmetric exclusion process [6], obtain the bound $CN^3 \ln(N) \ln \varepsilon^{-1}$, and conjecture the bound $CN^2 \ln \varepsilon^{-1}$.

To our knowledge, our work treats the mixing time of the ASEP with sequential (rather than random) update for the first time. Note that a comparison of update procedures is studied in [12].

Plan of the paper

After some preliminaries on Markov chains (Sec. 2), we introduce a coupling process on ASEP, and interpret it as an iterated game of gambler's ruin (Sec. 3). We give our result on the mixing time of ASEP in Sec. 4, before giving some final remarks (Sec. 5).

2 Preliminaries

2.1 ASEP as a Markov chain

The ASEP, on a ring of length N with sequential update, can be seen as a Markov chain $(X^t)_t$ defined on the space $\Omega = \{0,1\}^N$, where: $X^{t+1} = (x_1^{t+1}, ...x_N^{t+1})$ is computed from $X^t = (x_1^t, ...x_N^t)$ as follows:

For i from N to 1:

$$(x_i^{t+1}, x_{i+1}^{t+1}) = \begin{cases} \begin{cases} (0,1) \text{ with probability } p \\ (1,0) \text{ with probability } q = 1-p \end{cases} & \text{if } (x_i^t, x_{i+1}^t) = (1,0) \\ (x_i^t, x_{i+1}^t) & \text{otherwise} \end{cases}$$

Here t represents the current number of *rounds*. Each round is made of N elementary *steps*. The evolution of a configuration X with two non-contiguous particles is represented in Fig. 1: the moves of the two particles are independent. The evolution of a configuration X with two contiguous particles is represented

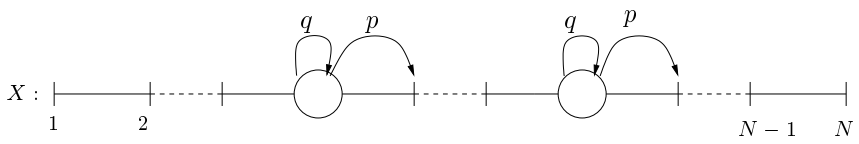

Fig. 1. Evolution during a round of a configuration with two non-contiguous particles.

in Fig. 2. Note that there are only three possibilities of evolution due to the fact that the left particle is blocked when the right particle does not move.

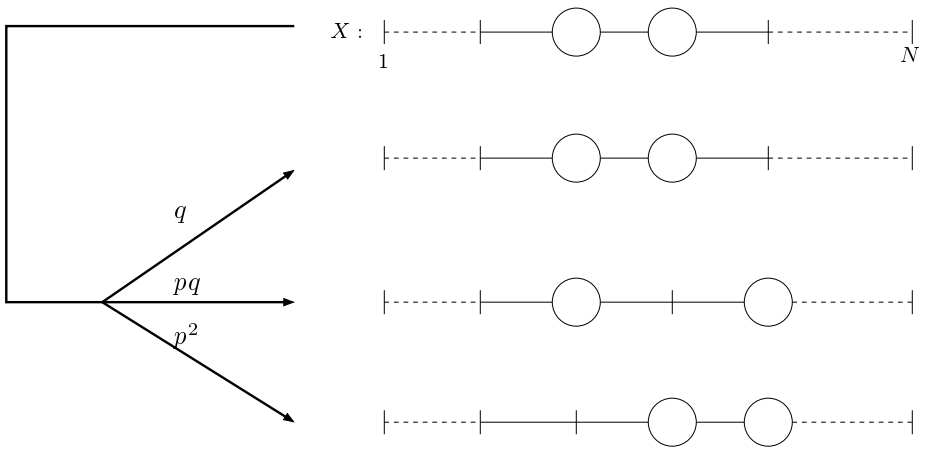

Fig. 2. Evolution in a case of two contiguous particles.

2.2 Mixing

The following material on mixing and coupling is borrowed from [13] (*cf.* [1, 10, 16]).

Definition 1. *The* variation distance *between two probability distributions μ, ν on Ω is defined by* $\|\mu - \nu\| = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)| = \max_{S\subseteq\Omega} |\mu(S) - \nu(S)|$.

Let $p^t(x, .)$ denote the distribution of the Markov chain at time t, given that it starts in configuration $X_0 = x$. Following standard practice, we will measure the distance of the chain from stationarity by the quantity $\Delta_x(t) = \|p^t(x, .) - \pi\|$.

Convergence of the chain means that $\Delta_x(t) \to 0$ as $t \to \infty$, for all x. To measure the *rate* of convergence, we introduce the quantity

Definition 2. *For $\varepsilon > 0$, the* mixing time $\tau(\varepsilon)$ *of the chain (from initial state x) is* $\tau_x(\varepsilon) = \min\{t : \Delta_x(t') \le \varepsilon \quad \text{for all } t' \ge t\}$,

Our goal will be to calculate *a priori* upper bounds on the mixing time.

We say a Markov chain is *rapidly mixing* if the mixing time is bounded above by a polynomial in n and $\ln \varepsilon^{-1}$, where n is the size of each configuration in the state space.

One of the most popular methods for bounding mixing time is coupling, both because of its elegance and its simplicity. Coupling is an elementary probabilistic method for bounding the mixing time of a Markov chain $\mathcal{M} = (X^t)_t$ by relating it to the stopping time of an associated stochastic process.

Definition 3. *A* coupling *is a Markov chain on $\Omega \times \Omega$ defining a stochastic process $(X^t, Y^t)_{t=0}^{\infty}$ with the properties:*

1. *Each of the processes X^t and Y^t is a faithful copy of $\mathcal{M}$ (given initial states $X^0 = x$ and $Y^0 = y$).*
2. *If $X^t = Y^t$, then $X^{t+1} = Y^{t+1}$.*

Condition 1 ensures that each process, viewed in isolation, is just simulating the original chain — yet the coupling updates them simultaneously so that they will tend to coalesce, or move closer together, according to some notion of distance. Once the pair of configurations agree, condition 2 guarantees they agree from that time forward. The coupling (or expected coalescence) time can provide a good bound on the mixing time of $\mathcal{M}$ if it is a carefully chosen coupling.

Definition 4. *For initial states x, y, the* expected coupling time *is* $\mathcal{T} = \max_{x,y} E(T^{x,y})$, *where* $T^{x,y} = \min\{t : \ X_t = Y_t \mid X_0 = x, Y_0 = y\}$, *and $E(.)$ denotes expectation.*

The following result relates the mixing time and the coupling time (see, e.g., [2]).

Theorem 1. *The mixing time of an ergodic Markov chain satisfies:*

$$\tau(\varepsilon) \le \lceil e\mathcal{T} \ln \varepsilon^{-1} \rceil.$$

3 ASEP As Iterated Gambler's Ruin

Our aim is to compute the expected coupling time of a pair (X^t, Y^t) for an appropriate coupling. We say that a particle of $X = (x_1, ..., x_N)$ and a particle of $Y = (y_1, ...y_N)$ *match*, when they are located at the same position, say i (i.e: $x_i = y_i = 1$).
Our goal is to find a coupling of (X^t, Y^t), which, starting from an arbitrary couple (X^0, Y^0) (where no particles match), ends with a final couple (X^f, Y^f) where each particle of X matches with a particle of Y, and to compute the expected time for reaching (X^f, Y^f) in terms of rounds.

During a *first phase*, no particle matches (see Fig. 3).

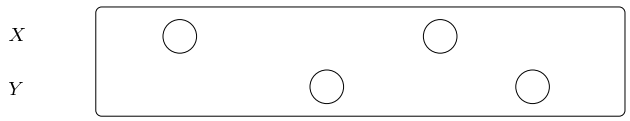

Fig. 3. A couple of configurations in phase 1.

After at most N^2/pq rounds on average, there is a first matching: the position of a particle of X happens to coincide with a particle of Y.[1]

Then the *second phase* starts. During this phase, one goes from one matching particle to two matching particles. During this phase, there is always (at least) one matching particle. In the rest of this paper we focus on this second phase.

We assume that one particle of X (resp. Y) matches, and the other does not. The matched particle is represented in black, the other in white. The black particle of X (resp. Y) is denoted by b_X (resp. b_Y), or sometimes simply b. Without loss of generality, we will suppose, in order to fix the ideas, that the white particle of X, denoted by w_X, is located on the left of b_X, and the white particle of Y, denoted by w_Y is on the right of b_Y.

Let ℓ be the distance (from left to right) from the black particle b to the white particle, viz. w_Y, to its right.

Let d be the distance from the left white particle, viz. w_X, to the right white particule, viz. w_Y. Such a couple is depicted in Fig. 4.

Our coupling consists, at each round, in:

- making the white particles move in unison when possible (i.e., when b is *not* immediately to the right of w_X)

- making the black particles move in unison when possible (i.e., when w_Y is *not* immediately to the right of b).

The aim of this coupling is to keep the matching of the black particles while preserving the distance d between the white particles. We explain hereafter that

[1] It is easy to see that the problem of duration of phase 1 is bounded by the expected time for a symmetric random walk with two absorbing states, similar to the random walk of Fig. 8.

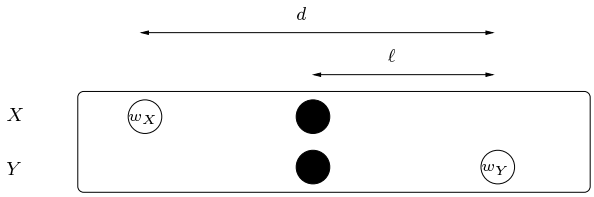

Fig. 4. A couple of configurations in phase 2.

such a process can be seen as a *Gambler's ruin game*, where d is the objective of gain and ℓ the current fortune of the gambler. At each round of the ASEP, the fortune ℓ evolves by $-1, 0$ or $+1$ with probability $pq, p^2 + q^2$, or pq respectively. A typical evolution of a couple (X, Y) is depicted in Fig. 5 (cf. Fig. 11).

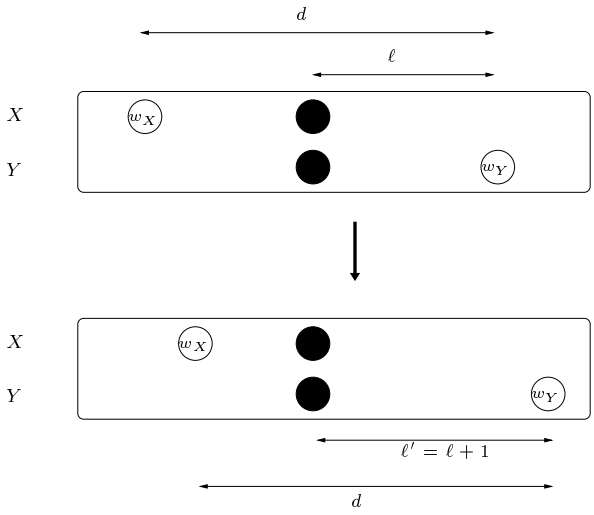

Fig. 5. Typical evolution in phase 2

During a game, d remains constant. There are however two "stopping cases" of the game, which correspond to the cases where d is not preserved:

- Stopping case 1 (Reaching the objective: $\ell' = d$): b is currently immediately to the right of w_X (i.e., $\ell = d - 1$), in which case w_X stays at the same position while w_Y moves with probability p, which yields: $\ell' = d$. This is depicted on Fig. 6. A new game starts with: $d_{new} = d + 1, \ell_{new} = d$.
- Stopping case 2 (Ruin: $\ell' = 0$): w_Y is currently immediately to the right of b_Y (i.e., $\ell = 1$), in which case b_Y stays at the same position while b_Y moves with probability p, which yields: $\ell' = 0$. After repainting black w_Y (and

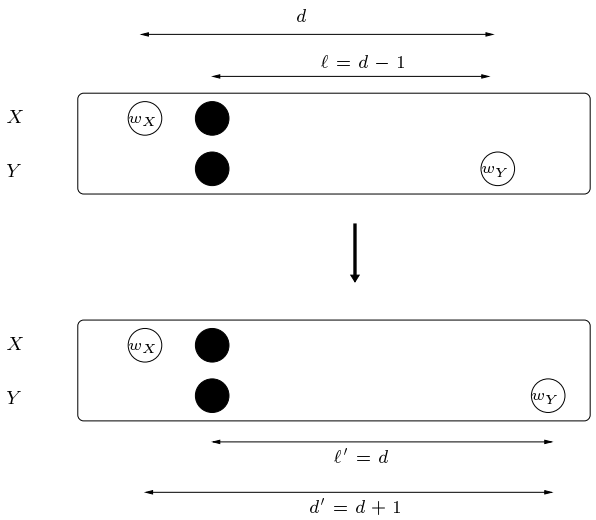

Fig. 6. Stopping case 1.

white b_Y), in order to recover the matching of one particle, a new game starts with: $d_{new} = N - d + 1 \wedge \ell_{new} = N - d$. This is depicted on Fig. 7. Note that, in the new couple of configurations, w_Y (resp. w_X) is now at the left (resp. right) of b.

The details are given in Appendix A. It should be clear that our coupling satisfies the statements of Definition 3. With such a coupling, the evolution of ℓ can be seen as the evolution of the fortune of a gambler, which evolves, at each round by $+1$ (resp. $0, -1$) with probability pq (resp. $p^2 + q^2, pq$) until he reaches his objective (case 1: $\ell := d$), or is ruined (case 2: $\ell := 0$). This is depicted on Fig. 8.

After the end of a game on case 1 (resp. case 2), a new game restarts with $d_{new} = d+1, \ell_{new} = d$ (resp. $d_{new} = N-d+1, \ell_{new} = N-d$). Note that, at the beginning of each new game, we have always: $\ell = d-1$. (This corresponds to the fact that b is initially always located immediately to the right of a white particle.) The *last game* of the iteration is the game which starts with an objective $d = N - 1$, and stops on case 1 (reaching the objective). In this case, d increases, thus becoming equal to N. This means that the white particles matched.

Let us denote by $game(d, \ell)$, the process of playing a gambler's ruin game with objective d and initial fortune ℓ. The second phase of the coupling process can be described as the iterated loop:

DO
play $game(d, \ell)$

1. if the game stops with $\ell = d$ and $d < N - 1$, then:

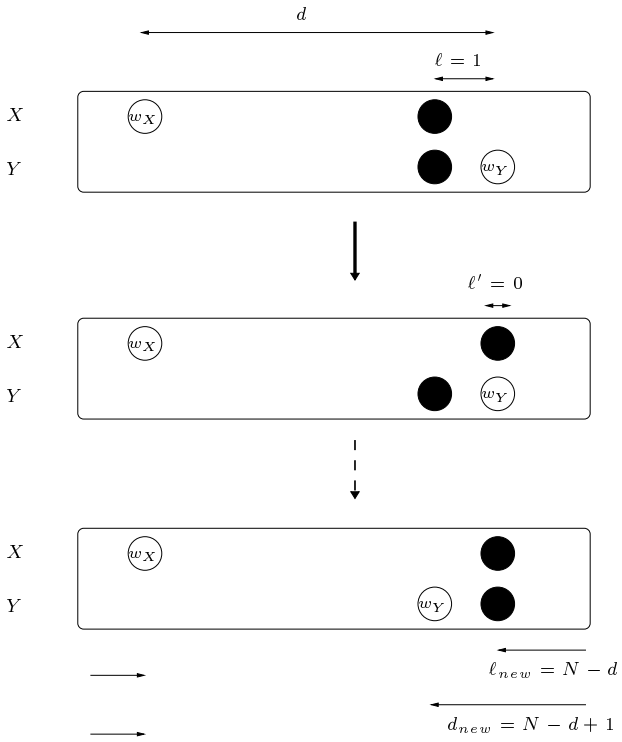

Fig. 7. Stopping case 2.

$d := d + 1; \ell := d - 1$

2. if the game stops with $\ell = 0$, then
   $d := N - d + 1; \ell := d - 1$
3. if the game stops with $\ell = d$ and $d = N - 1$, then exit.

OD

The expected time and probabilities of stopping cases of a $game(d, \ell)$ can be found in the literature on the gambler's ruin problem (see, e.g., [8]): The expected time is $\ell(d - \ell)/pq$ and the probability of gain (resp. ruin) is $(d - \ell)/d$ (resp. ℓ/d). Recall that, every game (except perhaps the very first one) starts with $\ell = d - 1$. So we have:

Proposition 1. *Given d, we have for $game(d, d - 1)$:*

- *the expected time of termination is: $(d - 1)/pq$ rounds ($\leq N/pq$);*
- *the probability of termination with $\ell = d$ (case 1) is: $(d - 1)/d$,*
- *the probability of termination with $\ell = 0$ (case 2) is: $1/d$.*

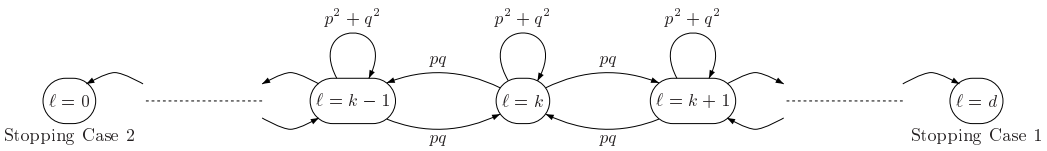

Fig. 8. One game of gambler's ruin.

We can represent the evolution of d from one game to the following one as depicted in Fig. 9.

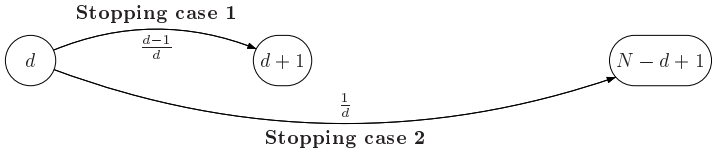

Fig. 9. Evolution of d at the end of each game.

The evolution of d during the iterated game is then depicted in Fig. 10.

4 Results

4.1 Mixing time

Each transition occurs at each end of a game (which takes $(d-1)/pq$ rounds on average). The second phase, which corresponds to the matching of the second particle, thus takes on average at most: $\Sigma_{i=1}^{M} E(T_i)$, where M is a random variable corresponding to the number of games, and T_i the duration of the i-th game. Using (a slight generalization of) Wald's lemma (see, e.g., [3]), we have:

$\Sigma_{i=1}^{M} E(T_i) \leq E(M) \times \max_i(E(T_i)) = E(M)N/pq$.

A linear upper bound (equal to 2N) of the expected number of games $E(M)$ is given in Section 4.2 (see Proposition 2). It follows that the second phase takes at most $2N^2/pq$ rounds on average. Therefore the expected coupling time $\mathcal{T}$ of the full process (including phase 1) is at most $3N^2/pq$ rounds. By theorem 1, we have:

Theorem 2. *The mixing time of the ASEP on a ring with 2 particles is (at most)* $\frac{3eN^2}{pq} \ln \varepsilon^{-1}$.

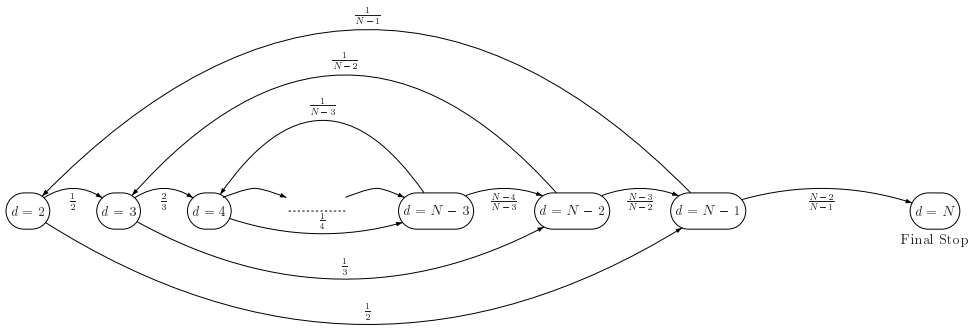

Fig. 10. Evolution of d during the iterated game.

4.2 Expected number of games

Our goal is to compute an upper bound on the expected time of absorption (state $d = N$) of the Markov chain described in Fig. 10.

Let us now consider the $(N-2) \times (N-2)$ matrix $A = (a_{i,j})$ corresponding to the Markov chain restricted to the transient states (therefore $2 \leqslant i \leqslant N-1$ and $2 \leqslant j \leqslant N-1$). We have,

- $a_{k,k+1} = \frac{k-1}{k}$ for $k = 2, ..., N-2$,
- $a_{k,N-k+1} = \frac{1}{k}$ for $k = 2, ..., N-1$,
- $a_{i,j} = 0$, otherwise.

Note that, for even N and $k = \frac{N-2}{2}$: $a_{k,k+1} = a_{k,N-k+1} = 1$. The matrix A writes:

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{2}{3} & 0 & \dots & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{3}{4} & \dots & 0 & \frac{1}{4} & 0 & 0 \\ \vdots & & & \ddots & \ddots & \iddots & & & \vdots \\ \vdots & & & & \iddots & \ddots & & & \vdots \\ \vdots & & & \iddots & & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \frac{1}{N-3} & 0 & \dots & 0 & 0 & \frac{N-4}{N-3} & 0 \\ 0 & \frac{1}{N-2} & 0 & 0 & \dots & 0 & 0 & 0 & \frac{N-3}{N-2} \\ \frac{1}{N-1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have to compute $\max(E_i)_{2\leqslant i\leqslant N-1}$ where E_i is the expecting time of the random walk described by matrix A for going from state i i to the absorbing state N.

Proposition 2. $\forall i \in \{2, ..., N-1\},\ E_i \leq 2N$.

Proof. The proof is done in three steps:

1. Computation of $(E_i)_{2\leqslant i\leqslant N-1}$. The vector $(E_i)_{2\leqslant i\leqslant N-1}$ can be computed as $(I-A)^{-1}\mathbf{1}$, where I is the identity matrix and $\mathbf{1}$ denotes the vector with all coordinates equal to 1 (see [3]). Let $B = (b_{i,j})$ defined by :
   - $b_{i,j} = \frac{j}{2(N-j)}$ if $2 \leqslant j < i \leqslant N-j+1 \leqslant N-1$,
   - $b_{i,j} = \frac{(i-1)(2N-i)}{2(j-1)(N-j)}$ if $2 \leqslant i \leqslant j \leqslant N-i+1 \leqslant N-1$,
   - $b_{i,j} = \frac{(N+j-1)}{2(j-1)}$ if $1 \leqslant N-j+1 \leqslant i \leqslant j \leqslant N-1$,
   - $b_{i,j} = \frac{(N-i)(N-i+1)}{2(j-1)(N-j)}$ if $N-i+1 \leqslant j < i \leqslant N-1$.

   We can check that $B = (I-A)^{-1}$. So $(E_i)_{1\leqslant 2\leqslant N-1} = B\mathbf{1}$, hence:
   $E_i = \sum_{j=2}^{N-1} b_{i,j},\ \forall i \in \{1, ..., N-1\}$,

   $$E_i = \ N\sum_{j=N-i+1}^{N-2}\frac{1}{j} + \frac{(i-1)(2N-i)}{N-1}\sum_{j=i-1}^{N-i}\frac{1}{j},\ \text{ if } \ 2 \leqslant i \leqslant \tfrac{N+1}{2},$$

   $$E_i = N\sum_{j=i-1}^{N-2}\frac{1}{j} + \frac{(N-i)(N-i+1)}{N-1}\sum_{j=N-i+1}^{i-2}\frac{1}{j}, \text{ if } \ \tfrac{N+1}{2} < i \leqslant N-1.$$
2. Lower and upper bounds for i_N. The value of index i_N for which the maximum of E_i $(2 \leqslant i \leqslant N-1)$ is reached, is such that: $\frac{N+1}{4} \leq i_N \leq \frac{N+1}{2}$ (see Lemma 3, Appendix B).
3. Upper bound on E_{i_N}. Lemma 4 (see Appendix B) gives us a constant $C \leqslant 2$ such that, for all $\frac{N+1}{4} \leq i \leq \frac{N+1}{2}$, we have: $E_i \leq CN$.

The combination of the two last results achieves the proof. □

5 Final Remarks

We have proved, for the first time to our knowledge, that the mixing time of the ASEP on a ring is quadratic. The update considered here is the sequential (anticlockwise) update, but we can find, at the price of little complications, a similar quadratic bound for the *synchronous* update, where all the particles are moved in parallel at each round. Note that such a kind of update (sequential or synchronous) is used in the trafic flow model of Nagel-Schreckenberg (see, e.g., [14]) rather than random update. The main limitation of our work lies of course in the small number of particles considered ($m = 2$). As pointed out in the introduction, the two-particle ASEP exhibits already some non-trivial behaviour. Priezzhev explains, also in the ring framework with sequential update, how to pass from the two-particle system to the general m-particle system for computing the non-stationary probabilities, via a geometrical treatment of the Bethe Ansatz method [11]. It would remain to understand if such a passage to a general number of particles is possible in our case.

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Appendix A: Detailed explanation of the coupling

During a round, there are 3 kinds of elementary step:

– **Case 1:** if b is isolated (i.e., $1 < \ell < d-1$), the white particles move in unison as well as the black particles. See Fig. 11.

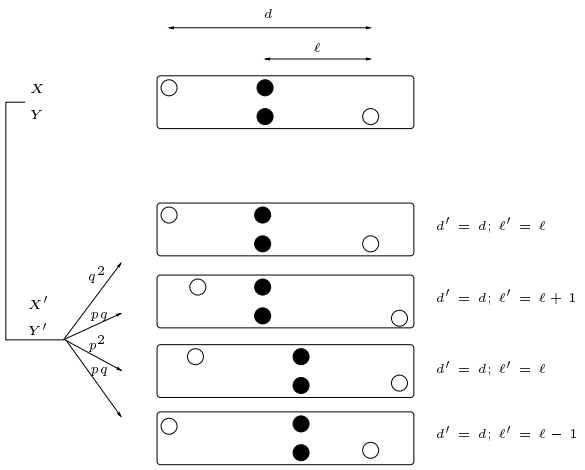

Fig. 11. Typical evolution of a couple

– **Case 2:** b is immediately to the right of w_X ($\ell = d-1$; $\ell > 1$). The black particles move in unison. The right white particle w_Y moves independently from the black ones. There is one subcase where w_X stays at the same position while w_Y moves, which yields $d' = d+1$. See Fig. 12.
– **Case 3:** w_Y is immediately to the right of b ($\ell = 1$, $\ell < d-1$). White particles move in unison. The black particle b_X of X moves independently. There is one subcase where b_Y stays at the same position while b_X moves, which yields $d' = N - d + 1$ (after repainting). See Fig. 13.

Actually, there is an additional case corresponding to the limit case where $\ell = 1$ and $d = \ell + 1 = 2$. This is a simple combination of Cases 2 and 3.

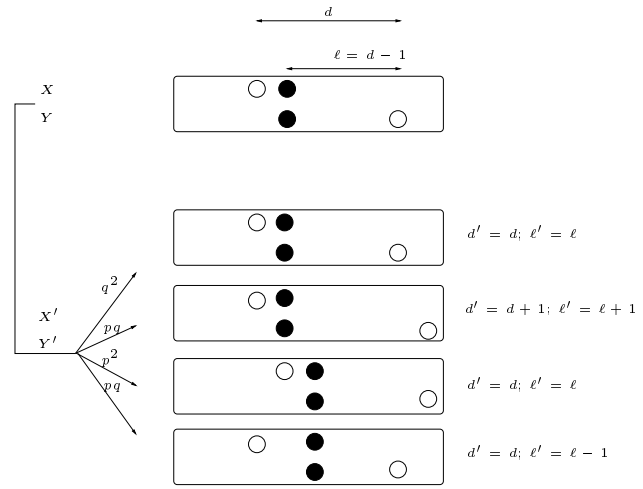

Fig. 12. Evolution when b and w_X are contiguous.

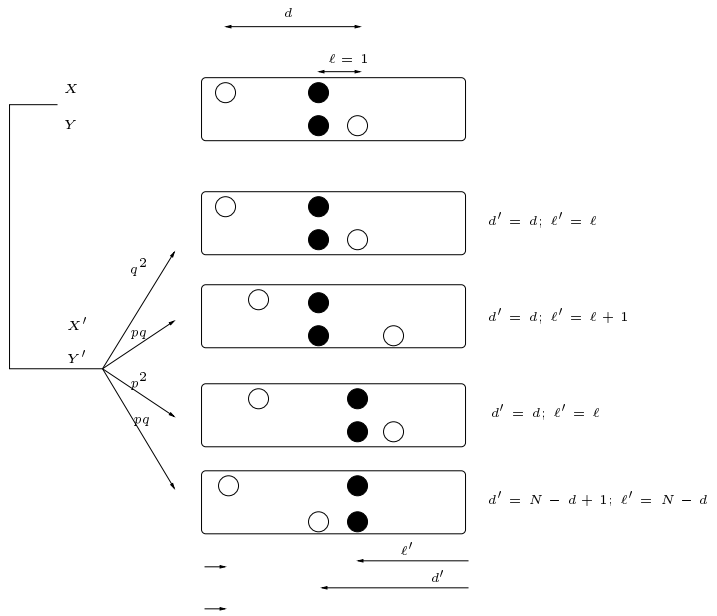

Fig. 13. Evolution when b and w_Y are contiguous.

Appendix B: Lemmas used in the proof of Proposition 2

Proposition 2 relies on Lemmas 3 and 4, which themselves rely on Lemmas 1 and 2.

Lemma 1. *If* $2 \leqslant i < \frac{N+1}{2}$, *we have :* $E_i - E_{N-i+1} > 0$.

Proof. Let $2 \leqslant i < \frac{N+1}{2}$. If $j < i$ or $j \geqslant N-i+1$,we have $b_{i,j} - b_{N-i+1,j} = 0$. If $i \leqslant j < N-i+1$, we have $b_{i,j} - b_{N-i+1,j} = \frac{(i-1)(N-i)}{(j-1)(N-j)} > 0$. □

Lemma 2. *If* $3 \leqslant i \leqslant \frac{N+1}{2}$, *we have :*

$$E_i - E_{i-1} = -1 + \frac{2(N-i+1)}{N-1} \sum_{i-1}^{N-i} \frac{1}{j}.$$

Lemma 3. *Let* i_N *be in* $\{2, ..., N-1\}$ *such that* $E_{i_N} = \max\{E_i \ ; \ i \in \{2, ..., N-1\}\}$. *Then :*

$$\frac{N+1}{4} \leqslant i_N \leqslant \frac{N+1}{2}.$$

Proof. Thanks to Lemma 1, we have $i_N \leqslant \frac{N+1}{2}$. For the other inequality, we may suppose $N \geqslant 8$.

- Let i be such that $3 \leqslant i \leqslant \frac{N+1}{2}$; then, $E_i > E_{i-1}$ if and only if $\sum_{i-1}^{N-i} \frac{1}{j} > \frac{N-1}{2(N-i+1)}$ (see Lemma 2). As $i \mapsto \sum_{i-1}^{N-i} \frac{1}{j}$ is a decreasing sequence going from $\sum_{1}^{N-2} \frac{1}{j}$ to either $\frac{4(N-1)}{N(N-2)}$ or $\frac{2}{N+1}$, according to the parity of N, and $i \mapsto \frac{N-1}{2(N-i+1)}$ is an increasing sequence going from $\frac{1}{2}$ to $\frac{N-1}{N+2}$ or $\frac{N-1}{N+1}$ respectively, there exists some index l such that $E_2 ... E_l$ is increasing and the sequence $E_{l+1}, ...$ is decreasing. Therefore, We must have $l = i_N$ and i_N is caracterized by the following :
$$\begin{cases} \forall i \leqslant i_N, \ E_i > E_{i-1}, \\ \forall i > i_N, \ E_i < E_{i-1}. \end{cases}.$$
- Let i be the smallest integer greater or equal to $\frac{N+1}{4}$ ($\frac{N+1}{4} \leqslant i \leqslant \frac{N+4}{4}$). In order to prove $\frac{N+1}{4} \leqslant i_N$, we only have to prove $E_i > E_{i-1}$.
$E_i - E_{i-1} = -1 + \frac{2(N-i+1)}{N-1} \sum_{i-1}^{N-i} \frac{1}{j} \geqslant -1 + \frac{3N}{2(N-1)} \sum_{i-1}^{N-i} \frac{1}{j} \geqslant -1 + \frac{3}{2} \sum_{i-1}^{N-i} \frac{1}{j}$.
Recall the inequality $\ln \frac{N-i+1}{i-1} \leqslant \sum_{i-1}^{N-i} \frac{1}{j}$ (using $\frac{1}{j} \geqslant \ln \frac{j+1}{j}$). Therefore:
$E_i - E_{i-1} \geqslant -1 + \frac{3}{2} \ln \frac{N-i+1}{i-1} \geqslant -1 + \frac{3}{2} \ln \frac{3N}{N} = -1 + \frac{3}{2} \ln 3 > 0$.

□

Lemma 4. *There exists* $C > 0$ *with* $0 < C \leq 2$ *such that, for all* $N \in \mathbb{N}$ *and all* i *with* $\frac{N+1}{4} \leqslant i \leqslant \frac{N+2}{2}$, *we have :*

$$E_i \leqslant CN.$$

Proof. Let $N \geqslant 8$ and i be such that $\frac{N+1}{4} \leqslant i \leqslant \frac{N+1}{2}$. Then :
$E_i = N \sum_{j=N-i+1}^{N-2} \frac{1}{j} + \frac{i(2N-i)}{N-1} \sum_{j=i-1}^{j=N-i} \frac{1}{j}$.
Recall the inequality $\sum_{l}^{m} \frac{1}{j} \leqslant \ln \frac{m}{l-1}$ (using $\frac{1}{j} \leqslant \ln \frac{j}{j-1}$). Thus:

$$\begin{aligned} E_i &\leqslant N \ln \tfrac{N-2}{N-i} + \tfrac{(i-1)(2N-i)}{N-1} \ln \tfrac{N-i}{i-2} \\ &\leqslant N \ln \tfrac{2(N-2)}{N-1} + (\tfrac{N-1}{2} \tfrac{7N}{4} \tfrac{1}{N-1}) \ln \tfrac{3N-1}{N-7} \\ &\leqslant N(\ln 2 + \tfrac{7}{8} \ln 3). \end{aligned}$$

□